Mark Scheme:

Each part of Question 1 is worth 4 marks which are awarded solely for the correct answer.

Each of Questions 2-7 is worth 15 marks

QUESTION 1:

A. Considering the sequence, \( a_2 = l, a_3 = l^2, a_4 = l^3 \), each additional term multiplies the previous term by \( l \). The product of the first 15 terms is equal to \( l^{1+2+...+14} = l^{14\times15} = l^{105} \). The answer is (d).

B. Call the length of one of the sides of the hexagon \( p \), then the side of the square is equal to \( p + (1 - p) = 1 \). Then as the hexagon side forms a triangle in each corner of the square, using Pythagoras, \( p^2 = (1 - p)^2 + (1 - p)^2 \). Solving this quadratic results in \( p = 2 \pm \sqrt{2} \), but as the length must be less than 1 the answer is (b).

C. We can rewrite the given equation as \( (x + \frac{a}{2})^2 + (y + \frac{b}{2})^2 = c + \frac{a^2}{4} + \frac{b^2}{4} \). For the circle to contain the origin, the distance from the centre to the origin must be less than the radius, so \( \frac{a^2}{4} + \frac{b^2}{4} < c + \frac{a^2}{4} + \frac{b^2}{4} \). The answer is (a).

D. \( \cos^n(x) + \cos^{2n}(x) = \cos^n(x)(1 + \cos^n(x)) = 0 \). For this to be true, if \( n \) is even, \( \cos(x) = 0 \) has two roots, but when \( n \) is odd either \( \cos(x) = 0 \) or \( \cos(x) = -1 \), which is three roots. Hence the answer is (d).

E. When \( x = 0, y = 1 - 1 = 0 \), so we can rule out (d) and (e). To work out the number of \( x \)-axis intersection points, consider \( (x - 1)^2 = \cos(\pi x) \). The shape of these graphs means they cannot intersect 6 times (eliminating (b)). The answer cannot be (c), because we know there is a crossing point \( x = 2 \), but that \( y \) is positive when \( x = 1 \). So the answer is (a).

F. Using the factor theorem, for \( (x^2 + 1) \) to be a factor, \( (x^2 + 1) = 0 \), so \( x^2 = -1 \). Then the equation given becomes \( (4)^n - (2)^n(-2)^n = 0 \). This only holds when \( (-2)^n \) is positive, so the answer is (b).

G. Considering the first few terms \( x_0 = 1, x_1 = x_0 = 1, x_2 = 2, x_3 = 4, x_4 = 8 \), and so on. By observation, \( x_n = 2^{n-1} \) for \( n \geq 1 \). As this is a geometric progression, we can evaluate the sum of the sequence as

\[
\sum_{k=0}^{\infty} \frac{1}{x_k} = \frac{1}{1} + \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 1 + \frac{1}{1 - \frac{1}{2}} = 3
\]

The answer is (d).
H. The area bounded by the $x$-axis and the curve $y = f(x)$, $A_1$ is equal to

$$A_1 = \int_{-\sqrt{a}}^{\sqrt{a}} f(x) \, dx = \frac{4}{3} a^{\frac{3}{2}},$$

whilst the area bounded by the $x$-axis and the curve $y = \frac{1}{2} g(x)$, $A_2$ is equal to

$$A_2 = \left| \int_{-\sqrt{\frac{3}{5}}}^{\sqrt{\frac{3}{5}}} g(x) \, dx \right| = \frac{8}{5} a^{\frac{5}{4}}.$$

We require an $a$ such that $A_1 > A_2$, so

$$\frac{4}{3} a^{\frac{3}{2}} > \frac{8}{5} a^{\frac{5}{4}} \quad \Rightarrow \quad 20a^{\frac{6}{5}} > 24 a^{\frac{5}{4}} \quad \Rightarrow \quad a^{\frac{1}{4}} > \frac{6}{5},$$

and so the answer is (e).

I. Let $ax + by = c$, which rearranges to $y = \frac{a}{b} x + \frac{c}{b}$. Given that $b$ is positive we can interpret this as achieving the maximum $c$ when the line $y = \frac{a}{b} x + \frac{c}{b}$ is moved up the $y$-axis whilst still intersecting the disc formed by $x^2 + y^2 \leq 1$. Hence the line should be tangent to the unit circle.

By Pythagoras,

$$\left(\frac{c}{b}\right)^2 = 1 + \left(\frac{a}{b}\right)^2,$$

and so the answer is (c).

J. We are trying to construct counter-examples for each of the statements. Note that $0 \leq x(n) \leq 9$. (a) is true since, for example, $\Pi(4) = 1$, but 4 is not prime. (b) is false - we don’t need to consider even $n$ beyond $x_n = 4$; for this case we know no primes end in a 4, but for example $\Pi(64) = 1$ as $64 = 2^6$. For odd $n$, $x(n) = 1, \Pi(n) = 1$, counterexample $n = 121 = 11^2$; $x(n) = 3, \Pi(n) = 1$, counterexample $n = 243 = 3^5$; $x(n) = 5, \Pi(n) = 1$, counterexample $n = 25 = 5^2; x(n) = 7, \Pi(n) = 1$, counterexample $n = 16807 = 7^5; x(n) = 9, \Pi(n) = 1$, counterexample $n = 9 = 3^2$. (c), (d), and (e) are all true. The answer is (b).
2. (i) [1 mark] We have
\[
A(B(x)) = 2(3x + 2) + 1 = 6x + 5; \\
B(A(x)) = 3(2x + 1) + 2 = 6x + 5.
\]

(ii) [3 marks] We note
\[
A^2(x) = 2(2x + 1) + 1 = 4x + 2 + 1, \\
A^3(x) = 2(4x + 2 + 1) + 1 = 8x + 4 + 2 + 1,
\]
and so more generally
\[
A^n(x) = 2^n x + 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 = 2^n x + (2^n - 1)
\]
using the geometric series formula (pattern spotting sufficient).

(iii) [4 marks] As \(108 = 2^23^3\) then \(F\) can be achieved using two applications of \(A\) and three applications of \(B\). As \(AB = BA\) then only one such \(F\) can be achieved but the number of different orders in which \(A, A, B, B, B\) might be performed is \(5C_2 = 10\).

(iv) [3 marks] Note that in each case the constant coefficient is one less than the coefficient of \(x\). We can prove this by noting
\[
A(ax + (a - 1)) = 2(ax + (a - 1)) + 1 = 2ax + 2a - 1; \\
B(ax + (a - 1)) = 3(ax + (a - 1)) + 2 = 3ax + 3a - 1.
\]
So \(c = 107\). [Alternatively to find \(c\) a student might just determine \(A^2B^3\).]

[Alternative: Commuting argument:
By part (i) \(A\) and \(B\) commute. Therefore we only need to check 1 of the possible configurations. From this calculation we find that \(c = 107\).]

(v) [4 marks] As each \(A^mB^n(x)\) will have a constant coefficient one less than its \(x\) coefficient it follows that \(k = 214 - 92 = 122\). However the \(x\) coefficient of \(A^mB^n(x)\) can never be less than 2 so the sum of 122 such functions cannot have an \(x\) coefficient less than 244.

[Alternative: Divisible by 6 argument:
Each term contains at least an \(A\) and at least a \(B\), and so each \(x\) coefficient is a multiple of 6. However 214 is not divisible by 6 and hence there exist no positive integers.]
3. 3. Solution:

(i) [1 mark] Note that
\[ f(2\alpha - x) = (2\alpha - x - \alpha)^2 = (\alpha - x)^2 = (x - \alpha)^2 = f(x) \]
for all \( x \) and hence \( f \) is bilateral.

(ii) [2 marks] Consider, for example, \( x = \alpha + 1 \) where
\[ f(\alpha + 1) = 1 \neq -1 = f(\alpha - 1) = f(2\alpha - (\alpha + 1)). \]
It follows that \( f \) is not bilateral.

(iii) [2 marks] Note that
\[
\int_a^b x^n \, dx = \left[ \frac{x^{n+1}}{n+1} \right]_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}
= -\left( \frac{a^{n+1} - b^{n+1}}{n+1} \right) = -\left[ \frac{x^{n+1}}{n+1} \right]_a^b = -\int_a^b x^n \, dx
\]
as required.

[Alternatively: Some students may show this graphically and argue that area is preserved under reflection]

(iv) [3 marks] Since \( f \) is a polynomial there is a non-negative integer \( d \) and reals \( c_0, \ldots, c_d \) such that
\[ f(x) = c_0 + c_1 x + \cdots + c_d x^d \]
for all \( x \). Integration is linear so by the previous part we have
\[
\int_a^b f(x) \, dx = \sum_{i=0}^d c_i \int_a^b x^i \, dx
= -\sum_{i=0}^d c_i \int_b^a x^i \, dx = -\int_b^a f(x) \, dx
\]
as required.

(v) [2 marks] The first integral is just the signed area under the graph of \( y = f(x) \) between \( \alpha \) and \( t \) and the second integral is the signed area under the graph of \( y = f(x) \) between \( 2\alpha - t \) and \( \alpha \). The second signed area is a reflection of the first, and area is preserved under reflection. Hence the integrals are equal.

(vi) [3 marks] For \( t \geq \alpha \) we have by the previous two parts that
\[
G(t) = \int_\alpha^t f(x) \, dx = \int_\alpha^{2\alpha-t} f(x) \, dx = -\int_\alpha^{2\alpha-t} f(x) \, dx = -G(2\alpha - t).
\]
If \( t \leq \alpha \) then put \( u = 2\alpha - t \geq \alpha \) and note that by what we have just shown
\[
G(2\alpha - t) = G(u) = -G(2\alpha - u) = -G(t).
\]
The result follows.

(vii) [2 marks] Since \( f \) is a bilateral polynomial we see \( G(2\alpha - t) = -G(t) \) for all \( t \). On the other hand since \( G \) is bilateral we have \( G(2\alpha - t) = G(t) \) for all \( t \), so \( G(t) = 0 \) for all \( t \) as required.
4. (i) [3 marks] Let \( d_1 \) be the distance from \((0, 0)\) to the point where \( C_1 \) touches the \(x\)-axis. Note that the \(x\)-axis is tangent to \( C_1 \) and hence perpendicular to the radius at this point. So \( C_1 \) has centre \((d_1, 1)\). We have a right-angled triangle, with \( \frac{1}{d_1} = \tan(\alpha) \), so \( d_1 = \frac{1}{\tan(\alpha)} \). So the centre of \( C_1 \) is \((\frac{1}{\tan(\alpha)}, 1)\).

(ii) [1 mark] \( C_1 \) has centre \((\frac{1}{\tan(\alpha)}, 1)\) and radius 1, so has equation
\[
(x - \frac{1}{\tan(\alpha)})^2 + (y - 1)^2 = 1.
\]

(iii) [3 marks] Let \( d_2 \) be the distance between the points where \( C_1 \) and \( C_2 \) touch the \(x\)-axis. Then Pythagoras on the right-angled triangle gives \((1 + 3)^2 = 2^2 + d_2^2\), so \( d_2^2 = 12 \).

Also we have similar triangles (both have a right angle and share angle \(\alpha\)) so
\[
\frac{3}{1} = \frac{d_2 + d_1}{d_1}.
\]

so \( d_2 = 2d_1 \).

So \( 12d_2^2 = (2d_1)^2 = 4d_1^2 \), so \( d_1 = \sqrt{3} \) (must have \(d_1 > 0\)). So \( \tan(\alpha) = \frac{1}{d_1} = \frac{1}{\sqrt{3}} \) so \( \alpha = 30^\circ \) (or \( \frac{\pi}{6} \)).

(iv) [3 marks] Take \(\alpha = 30^\circ \). Let \( C_3 \) have radius \( r \). Let \( d_3 \) be the distance between the points where \( C_2 \) and \( C_3 \) touch the \(x\)-axis. Then by similar triangles we have
\[
\frac{r}{d_1 + d_2 + d_3} = \frac{1}{d_1} = \frac{1}{\sqrt{3}}.
\]

So \( r = \frac{d_1 + d_2 + d_3}{\sqrt{3}} = 3 + \frac{d_3}{\sqrt{3}} \). So \( d_3 = \sqrt{3}(r - 3) \).

Also since \( C_2 \) and \( C_3 \) touch Pythagoras gives
\[
(r + 3)^2 = (r - 3)^2 + d_2^2 = (r - 3)^2 + 3(r - 3)^2 = 4(r - 3)^2,
\]

so \( r^2 + 6r + 9 = 4r^2 - 24r + 36 \), which factorises to \((r - 1)(r - 9) = 0\).

We’re looking for \( C_3 \) larger than \( C_2 \) so \( r = 9 \).

(v) [5 marks] Centres of triangle \( C_1 \) and \( C_2 \) are \((\frac{1}{\tan(\alpha)}, 1)\) and \((\frac{3}{\tan(\alpha)}, 3)\) respectively. Area of trapezium is half (bottom plus top) times height, so:
\[
\frac{3 + 1}{2} \cdot \frac{2}{\tan(\alpha)} = \frac{4}{\tan(\alpha)}.
\]

Or break down as rectangle (area = \(\frac{2}{\tan(\alpha)}\)) plus triangle (area = \(\frac{2}{\tan(\alpha)}\)).

Now deduct \( C_1 \) sector and \( C_2 \) sector from trapezium area. Area of \( C_1 \) sector is \(\frac{1}{2}1^2(\frac{\pi}{2} + \alpha) = \frac{\pi}{4} + \frac{\alpha}{2} \).

Area of \( C_2 \) sector is \(\frac{1}{2}3^2(\frac{\pi}{2} - \alpha) = \frac{9\pi}{4} - \frac{9\alpha}{2} \).

So interstitial area is:
\[
\frac{4}{\tan(\alpha)} - \left(\frac{\pi}{4} + \frac{\alpha}{2}\right) - \left(\frac{9\pi}{4} - \frac{9\alpha}{2}\right) = \frac{4}{\tan(\alpha)} - \frac{5\pi}{2} + 4\alpha = 4\sqrt{3} - \frac{11\pi}{6}
\]
5. (i) [3 marks] We have
\[
\begin{align*}
  s_1 &= 2(A + B) + C = 2 \\
  s_2 &= 4(2A + B) + C = 10 \\
  s_3 &= 8(3A + B) + C = 34
\end{align*}
\]

(ii) [3 marks] Subtracting the first equation from the other two gives
\[
\begin{align*}
  6A + 2B &= 8 \\
  22A + 6B &= 32
\end{align*}
\]
whence $4A = 8$, so $A = 2$, $B = -2$, $C = 2$ and $f(n) = (n - 1)2^{n+1} + 2$.

(iii) [2 marks] We have
\[
\begin{align*}
  s_{k+1} &= f(k) + (k + 1)2^{k+1} \\
          &= (k - 1)2^{k+1} + 2 + (k + 1)2^{k+1} \\
          &= k2^{k+2} + 2 = f(k + 1)
\end{align*}
\]
as required.

(iv) [4 marks] We have
\[
\begin{align*}
  t_n &= (n + 2n + 4n + \cdots + 2^n.n) - (2 + 8 + 24 + \cdots + 2^n.n) \\
  &= n(2^{n+1} - 1) - f(n) \\
  &= n(2^{n+1} - 1) - (n - 1)2^{n+1} - 2 \\
  &= 2^{n+1} - n - 2.
\end{align*}
\]
Now $u_n = t_n/2^n$, so
\[
u_n = 2 - \frac{n + 2}{2^n}.
\]

(v) [3 marks]
\[
\begin{align*}
  \sum_{k=1}^{n} s_k &= \sum_{k=1}^{n} (2k2^k - 2^{k+1} + 2) \\
                   &= \sum_{k=1}^{n} (k2^{k+1}) - \sum_{k=1}^{n} (2^{k+1}) + 2n \\
                   &= 2 \sum_{k=1}^{n} (k2^k) - 2^{n+2} + 4 + 2n \\
                   &= 2f(n) - 2^{n+2} + 4 + 2n \\
                   &= 2^{n+2}n - 2^{n+3} + 2n + 8
\end{align*}
\]
6. (i) [3 marks] There are no possible arrangements - if A is a 1, then either B and D are both 1s or both 0s. However, if B and D are both 1s then C must also be a 1 - but that would require all the dancers to be 1s which is forbidden. If B and D are both 0s then C must also be a 0 otherwise D would not be off-beat. But if C is a 0 they cannot be off-beat.

(ii) [3 marks] Assume that A is a 1 and holds hands with F and B, then either F and B are both 1s or both 0s. If both F and B are 1s then this pattern must propagate around the circle, forcing everyone to be 1s, which is forbidden. If F and B are both 0s then C and E must also be 0s, to keep F and B off-beat. However to ensure C, D, and E are off-beat D must be a 1. Hence the only possible arrangements are those where precisely two dancers on opposite positions on the ring are 1, and there are 3 such arrangements.

(iii) [3 marks] Each person holding hands either requires one of the three dancers to be a 1 or all three to be a 1. If all three, then this propagates round resulting in all 1s, which is forbidden. Thus for each triplet of dancers one person is a 1. Then either spot the 1,0,0 pattern which only repeats when n is a multiple of three, or look at the sum of each local triplet of dancers which must be equal to n and also equal to 3k where k is the number of dancers who are 1s.

(iv) [2 marks] If n is even two separate rings form, however each ring can only be off-beat if the number of dancers are a multiple of 3, by previous argument. If n is odd, then n must be a multiple of 3 still because a ring is still formed (with displaced dancers).

(v) [1 mark] Either one dancer is a 1 or three dancers are 1s and one is a 0. There are 8 different ways in total.

(vi) [3 marks] There must be at least one dancer who is a 1. Holding hands with this dancer there must be either no dancers or precisely two dancers who are 1s. If none of the dancers are 1s, then the alternating 0, 1 pattern is very obvious. If two dancers are 1s, then this leads to a situation where all dancers are 1s, which is still forbidden. Hence there are 2 possible ways of arranging off-beat dances.
7. (Example taken from Graham, Knuth, Patashnik, *Concrete Mathematics.*

(i) [2 marks] Three 2-spans:

(ii) [4 marks] Eight 3-spans:

(iii) [3 marks] In a 4-span, the top group may have \( t = 1, 2, 3 \) or 4 elements, and may be connected to the hub by any of \( t \) line segments in each case. If \( t = 4 \), that is the end of the story, but if \( t < 4 \) then the remaining tips may form any \((4 - t)\)-span. Thus (using the notation of the next part),

\[
z_4 = 1 \cdot z_3 + 2 \cdot z_2 + 3 \cdot z_1 + 4 = 1 \times 8 + 2 \times 3 + 3 \times 1 + 4 = 21.
\]

(iv) [4 marks] More generally, we have

\[
z_n = 1 \cdot z_{n-1} + 2 \cdot z_{n-2} + \cdots + (n-1) \cdot z_1 + n.
\]

It follows that

\[
z_5 = 1 \times 21 + 2 \times 8 + 3 \times 3 + 4 \times 1 + 5 = 55.
\]

(v) [2 marks]

\[
z_6 = 1 \times 55 + 2 \times 21 + 3 \times 8 + 4 \times 3 + 5 \times 1 + 6 = 144.
\]