SOLUTIONS FOR EXTRA ADMISSIONS TEST IN
MATHEMATICS, COMPUTER SCIENCE AND JOINT SCHOOLS
NOVEMBER 2023

A Each of the given polynomials is of the form $y = x^3 - 3x^2 + kx$. The first derivative is $3x^2 - 6x + k$, and the second derivative is $6x - 6$. So if there is a point with zero derivative, then it will either be a local maximum or a local minimum, unless $x = 1$. At $x = 1$ the first derivative is $k - 3$, which is only zero if $k = 3$.

The answer is (c)

B Note that $\sqrt[10]{10^{11}}$ is $10^{11/10}$, so $\log_{10}\left(\sqrt[10]{10^{11}}\right) = \frac{11}{10}$. This is less than $\frac{11}{9}$.

To compare $\sqrt{\frac{3}{2}}$ with $\frac{11}{10}$, compare their squares; compare $\frac{3}{2}$ with $\frac{121}{100}$. The former is $\frac{150}{100}$ which is larger than $\frac{121}{100}$.

Now note that $\sqrt{3}\cos(44^\circ)$ is slightly larger than $\sqrt{3}\cos(45^\circ)$ which is $\sqrt{\frac{3}{2}}$.

Finally note that $\pi > 3$ so $\frac{\pi}{2} > \frac{3}{2} > \frac{11}{10}$.

So the smallest of the numbers is $\frac{11}{10}$.

The answer is (a)

C The required sum is the sum of all cubes up to $20^3$ minus the sum of the even cubes up to $20^3$.

$$1^3 + 3^3 + 5^3 + 7^3 + \ldots + 19^3 = (1^3 + 2^3 + 3^3 + 4^3 + \ldots + 20^3) - (2^3 + 4^3 + 6^3 + \ldots + 20^3)$$

$$= (1^3 + 2^3 + 3^3 + 4^3 + \ldots + 20^3) - 2^3 (1^3 + 2^3 + 3^3 + \ldots + 10^3)$$

$$= \frac{20^2 \times 21^2}{4} - 8 \times \frac{10^2 \times 11^2}{4}$$

$$= 10^2 \times (21^2 - 2 \times 11^2)$$

$$= 10^2 \times (441 - 2 \times 121)$$

$$= 19,900$$

The answer is (b)

D The numbers $x$ and $y$ are each either odd or even, so the squares $x^2$ and $y^2$ are each either multiples of $4$ or one more than a multiple of $4$. There are four cases to check, and after checking each, we find that the expression on the left-hand side could be a multiple of $4$ (if $x$ and $y$ are both even, or both odd), or could be one more than a multiple of $4$ (if $x$ is odd and $y$ is even), or could be three more than a multiple of $4$ (if $x$ is even and $y$ is odd). The number on the right-hand side is two more than a multiple of $4$. So there are no solutions.

The answer is (a)

E Starting with 1, there are nine one-digit numbers. Then from 11 to 99 there are 90 two-digit numbers, then there are 900 three-digit numbers, and so on. So the required sum is (grouping terms by the number of digits of $n$) equal to

$$9 \times 10^{-1} + 90 \times 10^{-2} + 900 \times 10^{-3} + \ldots$$

which is a geometric series with first term $\frac{9}{20}$, common ratio $\frac{1}{2}$ and sum to infinity equal to $\frac{9}{10}$.

The answer is (c)
If $a = c$ and $b = d$ then the expression we’re given for $f \left( \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right)$ becomes $\begin{pmatrix} a^2 + b^2 \\ 2ab + b^2 \end{pmatrix}$.

We’re looking for whole numbers (positive or negative or zero) for $a$ and $b$ such that $a^2 + b^2 = 2$ and $2ab + b^2 = 0$. From the second equation we conclude that $b = 0$ or $a + b = 0$. If $b = 0$ then $a^2 = 2$ but there are no whole numbers that square to 2. If $a + b = 0$ then we can eliminate $b$ from the first equation and conclude that $a = \pm 1$. We should check that $(1, -1)$ and $(-1, 1)$ both work.

The answer is (c)

G The triangular numbers described in the question as the sum of the first $k$ positive integers for some $k \geq 1$ have the form $\frac{1}{2}k(k+1)$, if we sum the arithmetic series.

Let’s write $N = x + y$. Then $0 < y \leq N$ and $f = \frac{1}{2}N(N+1) + y$. In words, first we choose a triangular number and then we add some positive quantity $y$. The number $y$ can never be quite large enough to make the “next” triangular number because the difference between $\frac{1}{2}N(N+1)$ and $\frac{1}{2}(N+1)(N+2)$ is $N+1$. But $f$ can take all other values.

The answer is (e)

H We’re told that $2x^4 - 3x^3 - 5x^2 + 2x + 2 = mx$ has exactly four real solutions $x_1, x_2, x_3, x_4$. Moving the $mx$ to the other side of the equation, the Factor Theorem implies that the resulting polynomial factorises like this;

$$2x^4 - 3x^3 - 5x^2 + (2 - m)x + 2 = 2(x - x_1)(x - x_2)(x - x_3)(x - x_4),$$

where we have been careful to include the factor of 2 in order to match the leading coefficient of $x$. Now substitute $x = 0$ in both sides of the equation for the result that $2 = 2x_1x_2x_3x_4$.

The answer is (b)

I The equations describe grid of parallel lines, with the line $y = 1 - 10x$ crossing opposite corners and the midpoint of the grid, but no other grid points.

There would normally be 16 points where the line $y = 1 - 10x$ crosses 16 given lines, but at three grid points the line $y = 1 - 10x$ crosses two of the given lines “simultaneously”. There are only 13 distinct points where the line $y = 1 - 10x$ crosses one or more of the other lines.

The answer is (b)

J The equation rearranges to

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = x^2$$

If we recognise the left-hand side as $(x + y)^4$ then we can simplify this down to $(x + y)^4 = x^2$. Now if $x$ is positive then $(x + y)^2 = x$ and so $y = -x \pm \sqrt{x}$. Since $x$ grows faster than $\sqrt{x}$, the value of $y$ is eventually negative for both of these solutions (once $x > 1$). Only one of the graphs behaves like this.

The answer is (d)