Examiners' Report: Preliminary Examination in Mathematics Trinity Term 2023

October 26, 2023

Part I

A. STATISTICS

178 candidates in Mathematics and Mathematics & Statistics were awarded an overall year outcome. Candidates on both degrees submit the same assessments and no distinction is made between the two groups in this document.

Table 1: Numbers in each outcome class

	Numbers					Percentages				
	2023	(2022)	(2021)	(2019)	(2018)	2023	(2022)	(2021)	(2019)	(2018)
Distinction	52	(53)	(60)	(54)	(58)	63.48	(29.78)	(30.61)	(29.19)	(29.44)
Pass	113	(116)	(124)	(120)	(126)	1.12	(65.17)	(63.27)	(64.86)	(63.96)
Partial Pass	11	(6)	(7)	(8)	(10)	6.18	(3.37)	(3.57)	(4.32)	(5.08)
Incomplete	0	(0)	(2)	(1)	(0)	(0.00)	(0.00)	(1.02)	(0.54)	(0.00)
Fail	2	(3)	(3)	(2)	(3)	(1.12)	(1.69)	(1.53)	(1.08)	(1.52)
Total	178	(178)	(196)	(185)	(197)	-	-	-	-	-

B. NEW EXAMINING METHODS AND PROCEDURES

None.

C. CHANGES IN EXAMINING METHODS AND PROCEDURES CURRENTLY UNDER DISCUSSION OR CONTEMPLATED FOR THE FUTURE

None.

D. NOTICE OF EXAMINATION CONVENTIONS FOR CANDIDATES

The Notice to Candidates, containing details of the examinations and assessments, was issued to all candidates at the beginning of Trinity term. The Examination Conventions in full were made available at

https://www.maths.ox.ac.uk/members/students/undergraduate-courses/examinations-assessments/examination-conventions.

Part II

A. GENERAL COMMENTS ON THE EXAMINATION

Acknowledgements

First, the Moderators should like to thank the Undergraduate Studies Administration Team.

We should also like to thank Charlotte Turner-Smith for her invaluable experience with the Mitigating Circumstances Panel, and Matthew Brechin and Waldemar Schlackow for maintaining and running the examination database and their assistance during the final examination board meeting.

We should like to thank the lecturers for their feedback on proposed exam questions; the assessors for their extraordinary assistance with marking; Kathryn Gillow for coordinating the Computational Mathematics assessments, and the team of graduate checkers for their rapid work checking the marks on the papers, and Imogen Harbinson-Frith for her work running the examinations process this year.

Timetable

The examinations began on Monday 19th June and ended on Friday 23rd June.

Setting and checking of papers

The Moderators set and checked the questions, model solutions, and mark schemes. Every question was carefully considered by at least two moderators, and feedback was sought from lecturers. In a small number of cases feedback from lecturers was not available, and those were discussed in more detail until the Board of Moderators was satisfied that all questions were appropriate.

The questions were then combined into papers which were considered by the Board of Moderators and small changes were made to satisfy the Board that the papers were appropriate. After this a final proof-reading of the papers was completed before the Camera Ready Copies (CRCs) were produced. The whole Board of Moderators signed off the CRCs which were submitted to Examination Schools.

Marking and marks processing

The Moderators and Assessors marked the scripts according to the mark schemes and entered the marks. Small adjustments to some mark schemes were made at this stage, and care was take to ensure these were consistently applied to all candidates.

A team of graduate checkers, supervised by Imogen Harbinson-Frith, Haleigh Bellamy, Anwen Amos, and Charlotte Turner-Smith sorted all the scripts for each paper and carefully cross checked these against the mark scheme to spot any unmarked parts of questions, addition errors, or wrongly recorded marks. A number of errors were corrected, with each change checked and signed off by a Moderator, at least one of whom was present throughout the process.

Determination of University Standardised Marks

Marks for each individual assessment are reported as a University Standard Mark (USM) which is an integer between 0 and 100 inclusive. The Moderators used their academic judgment to map the raw marks on individual assessments to USMs using a process similar to previous years. In coming to this judgement the board followed the advice from the Mathematics Teaching Committee that the percentages awarded for each USM range of the examination should be in line with recent years. This alignment can be seen in Table 1; in more detail, for Papers I–V, a piecewise linear map was constructed as follows:

1. Candidates' raw marks for a given paper were ranked in descending order.
2. The default percentages p_1 of Distinctions and p_2 of Nominal Upper Seconds were selected.
3. The candidate at the p_1 percentile from the top of the ranked list was identified and assigned a USM of 70, and the corresponding raw mark denoted R_1.
4. The candidate at the $(p_1 + p_2)$ percentile from the top of the list was assigned a USM of 60 and the corresponding raw mark denoted R_2.
5. The line segment between $(R_1, 70)$ and $(R_2, 60)$ was extended linearly to USMs of 72 and 57 respectively, and the corresponding raw marks

denoted C_1 and C_2 respectively.

6. A line segment through $(C_2, 57)$ was extended towards the vertical axis, as if it were to join the axis at $(0, 10)$, but the line segment was terminated at a USM of 37 and the raw mark at the termination point was denoted C_3.

With these data a piecewise linear map was constructed with vertices at $\{(0,0),\ (C_3, 37),\ (C_2, 57),\ (C_1, 72),\ (100, 100)\}$.

Reports from the Assessors describing the apparent relative difficulty and the general standard of solutions for each question were then considered, and the Board decided that the values of $p_1 = 31\%$ and $p_2 = 48\%$ were suitable for all papers.

In line with previous years, for the Computational Mathematics assessment the linear map with gradient 2.5 was used to map from raw marks to USMs.

The vertices of the final maps used in each assessment are listed in Table 2.

Table 2: Vertices of final piecewise linear model

Paper	Vertices				
I	(0.0,0)	(21,40)	(34.6,57)	(61.6,72)	(100.0,100)
II	(0.0,0)	(16,40)	(36.7,57)	(68.2,72)	(100.0,100)
III	(0.0,0)	(31.7,37)	(55.2,57)	(94.2,72)	(120.0,100)
IV	(0.0,0)	(29.3,37)	(51,57)	(81,72)	(100.0,100)
V	(0.0,0)	(19,40)	(32,57)	(62,72)	(80.0,100)
CM	(0.0,0)	(40,100)			(0.0,0)

With the USMs, a provisional outcome class for each candidate was produced as follows: Write MI, MII, $MIII$, MIV and MV for the USMs on Papers I–V respectively, and CM for the USM on the Computational Mathematics assessment. Write Av_1 and Av_2 for the quantities

$$\frac{MI + MII + \frac{6}{5}MIII + MIV + \frac{4}{5}MV + \frac{1}{3}CM}{5\frac{1}{3}}$$

and

$$\frac{MI + MII + \frac{6}{5}MIII + MIV + \frac{4}{5}MV}{5}$$

respectively, symmetrically rounded. With these auxiliary statistics the provisional outcome class was determined by the definitions:

Distinction: both $Av_1 \geq 70$ and $Av_2 \geq 70$ and a USM of at least 40 on each paper and for the Computational Mathematics assessment;

Pass: not a Distinction and a USM of at least 40 on each paper and for the Computational Mathematics assessment;

Partial Pass: not a Pass or Distinction and a USM of at least 40 on three or more of Papers I–V;

Fail: not a Partial Pass, Pass, or Distinction, and a USM of less than 40 on three or more of Papers I–V.

The scripts of those candidates at the boundaries between outcome classes were scrutinised carefully to determine which attained the relevant qualitative descriptors and changes were made to move those into the correct class.

Mitigating Circumstances were then considered using the banding produced by the Mitigating Circumstances Panel, and appropriate actions were taken and recorded.

Table 3 gives the rank list ordered by the average of Av_1 and Av_2 (as defined above), showing the number and percentage of candidates with USM greater than or equal to each value.

Table 3: Rank list of average USM scores

		Candidates with USM $\geq x$	
USM (x)	Rank	Number	Percentage
91.88	1	1	0.56
91.12	2	2	1.12
89.28	3	3	1.69
87.16	4	4	2.25
85.96	5	5	2.81
83.88	6	6	3.37
83.36	7	7	3.93
83.08	8	8	4.49
81.96	9	9	5.06
81.06	10	10	5.62
80.56	11	11	6.18
78.76	12	12	6.74
78.62	13	13	7.3
78.04	14	14	7.87
77.89	15	15	8.43
77.6	16	16	8.99
77.56	17	17	9.55
76.64	18	18	10.11
76.56	19	19	10.67
76.24	20	20	11.24

Table 3: Rank list of average USM scores (continued)

		Candidates with average USM $\geq x$	
USM (x)	Rank	Number	Percentage
76.2	21	21	11.8
75.35	22	22	12.36
74.8	23	23	12.92
74.16	24	24	13.48
74.08	25	25	14.04
73.92	26	26	14.61
73.72	27	27	15.17
73.31	28	28	15.73
73.24	29	29	16.29
72.56	30	30	16.85
72.12	31	31	17.42
71.88	32	32	17.98
71.72	33	33	18.54
71.68	34	34	19.1
71.16	35	35	19.66
71.11	36	36	20.22
70.8	37	38	21.35
70.8	37	38	21.35
70.76	39	39	21.91
70.68	40	41	23.03
70.68	40	41	23.03
70.44	42	44	24.72
70.44	42	44	24.72
70.44	42	44	24.72
70.28	45	46	25.84
70.28	45	46	25.84
70.16	47	47	26.4
70.04	48	48	26.97
69.89	49	49	27.53
69.8	50	50	28.09
69.68	51	51	28.65
69.52	52	52	29.21
69.48	53	53	29.78
69.32	54	54	30.34
69.16	55	55	30.9
69.08	56	56	31.46
69	57	57	32.02
68.95	58	58	32.58
68.92	59	60	33.71
68.92	59	60	33.71

Table 3: Rank list of average USM scores (continued)

		Candidates with average USM $\geq x$	
USM (x)	Rank	Number	Percentage
68.84	61	61	34.27
68.6	62	62	34.83
68.56	63	63	35.39
68.52	64	64	35.96
68.48	65	67	37.64
68.48	65	67	37.64
68.48	65	67	37.64
68.44	68	69	38.76
68.44	68	69	38.76
68.4	70	70	39.33
68.12	71	71	39.89
68.08	72	72	40.45
67.84	73	73	41.01
67.36	74	75	42.13
67.36	74	75	42.13
67.28	76	76	42.7
67.24	77	77	43.26
67.16	78	78	43.82
67.08	79	79	44.38
66.96	80	81	45.51
66.96	80	81	45.51
66.95	82	82	46.07
66.84	83	83	46.63
66.68	84	86	48.31
66.68	84	86	48.31
66.68	84	86	48.31
66.56	87	88	49.44
66.56	87	88	49.44
66.52	89	89	50
66.4	90	91	51.12
66.4	90	91	51.12
66.32	92	92	51.69
66.08	93	93	52.25
66.04	94	94	52.81
65.92	95	95	53.37
65.76	96	96	53.93
65.56	97	97	54.49
65.52	98	98	55.06
65.46	99	99	55.62
65.4	100	100	56.18

Table 3: Rank list of average USM scores (continued)

USM (x)	Rank	Candidates with average USM $\geq x$ Number	Percentage
65.12	101	101	56.74
65	102	102	57.3
64.84	103	103	57.87
64.82	104	104	58.43
64.68	105	106	59.55
64.68	105	106	59.55
64.48	107	107	60.11
64.32	108	108	60.67
63.88	109	109	61.24
63.76	110	110	61.8
63.48	111	111	62.36
63.44	112	112	62.92
63.28	113	113	63.48
63	114	114	64.04
62.96	115	115	64.61
62.92	116	116	65.17
62.6	117	117	65.73
62.49	118	118	66.29
62.46	119	119	66.85
62.44	120	120	67.42
62.16	121	121	67.98
62.12	122	122	68.54
61.84	123	123	69.1
61.52	124	124	69.66
61.45	125	125	70.22
61.42	126	126	70.79
61.35	127	127	71.35
61.09	128	128	71.91
61.08	129	129	72.47
61.04	130	130	73.03
61	131	131	73.6
60.91	132	132	74.16
60.84	133	133	74.72
60.68	134	134	75.28
60.16	135	135	75.84
59.88	136	136	76.4
59.84	137	137	76.97
59.31	138	138	77.53
59.2	139	139	78.09
59.12	140	140	78.65

Table 3: Rank list of average USM scores (continued)

		Candidates with average USM $\geq x$	
USM (x)	Rank	Number	Percentage
59.04	141	142	79.78
59.04	141	142	79.78
58.84	143	143	80.34
58.52	144	144	80.9
58.32	145	145	81.46
58.21	146	146	82.02
58	147	147	82.58
57.08	148	148	83.15
56.92	149	149	83.71
56.52	150	150	84.27
56.48	151	151	84.83
56.4	152	152	85.39
56.24	153	153	85.96
56.04	154	154	86.52
55.8	155	155	87.08
55.64	156	156	87.64
55.48	157	157	88.2
55.44	158	158	88.76
55.16	159	159	89.33
55.08	160	161	90.45
55.08	160	161	90.45
53.84	162	162	91.01
53.72	163	163	91.57
53.56	164	164	92.13
52	165	165	92.7
51.96	166	166	93.26
51.52	167	167	93.82
50	168	168	94.38
49.15	169	169	94.94
48.6	170	170	95.51
47	171	171	96.07
41.92	172	172	96.63
41.34	173	173	97.19
40.88	174	174	97.75
40.64	175	175	98.31
37.4	176	176	98.88
29.55	177	177	99.44
20	178	178	100

Recommendations for next year's Moderators and Teaching Committee

1. It is recommended that markers completing assessor reports be told that a detailed mapping between raw marks and USMs will be arrived at by the Board of Moderators later and so their report does not need to include this.

2. It is recommended that assessor reports be produced for the Computational Mathematics assessments.

3. The Board noted that there are definitions of gender that do not partition populations into those who are male and those who are female and asks Teaching Committee for guidance on whether the reporting in §B could usefully be different or expanded in future years to capture this or other equal opportunity issues.

B. EQUAL OPPORTUNITY ISSUES AND BREAKDOWN OF THE RESULTS BY GENDER

Table 4 shows the performances of candidates by gender. Here gender is the gender as recorded on eVision.

Table 4: Breakdown of results by gender

Outcome	Number								
	2023			2022			2021		
	Female	Male	Total	Female	Male	Total	Female	Male	Total
Distinction	9	43	52	8	45	53	7	53	60
Pass	38	75	113	43	73	116	50	74	124
Partial Pass	4	7	11	2	4	6	2	5	7
Incomplete	0	0	0	0	2	2	0	0	0
Fail	1	1	2	0	3	3	3	0	3
Total	52	126	178	53	125	178	62	134	196
Outcome	Percentage								
	2023			2022			2021		
	Female	Male	Total	Female	Male	Total	Female	Male	Total
Distinction	17.23	34.15	29.21	15.09	36.00	29.78	11.29	39.55	30.61
Pass	73.08	59.52	63.53	81.13	58.40	65.17	80.65	55.22	63.27
Partial Pass	7.69	5.56	6.17	3.77	3.20	3.37	3.23	3.73	3.57
Incomplete	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.02	0.74
Fail	1.92	0.79	1.12	0.00	2.40	1.69	4.84	0.00	1.53

C. STATISTICS ON CANDIDATES' PERFORMANCE IN EACH PART OF THE EXAMINATION

Table 5: Numbers taking each paper

Paper	Number of Candidates	Average raw mark	Std dev of raw marks	Average USM	Std dev of USMs	Number failing
I	177	49.79	14.05	64.82	10.55	4
II	175	55.02	17.17	65.85	11.48	3
III	177	77.88	19.69	66.33	11.75	4
IV	178	66.52	14.39	65.1	9.66	1
V	177	49.19	14.03	65.6	11.63	5
CM	177	30.56	7.39	76.37	18.94	8

Tables 6–11 give the performance statistics for each individual assessment, showing for each question the average mark, first over all attempts, and then over the attempts used; the standard deviation over all attempts; and finally the total number of attempts, first those that were used, and then those that were unused.

Table 6: Statistics for Paper I

Question	Average mark		Std	No. of Attempts	
Number	All	Used	dev	Used	Unused
Q1	14.29	14.29	3.18	177	0
Q2	5.95	6.00	3.81	123	3
Q3	8.82	8.82	4.11	162	0
Q4	8.88	8.92	4.24	64	2
Q5	11.16	11.16	3.25	160	0
Q6	10.39	10.39	4.42	70	0
Q7	8.67	8.67	3.37	119	0

Table 7: Statistics for Paper II

Question	Average mark		Std	No. of Attempts	
Number	All	Used	dev	Used	Unused
Q1	15.29	15.29	3.58	171	0
Q2	9.66	9.66	4.66	108	0
Q3	11.03	11.03	3.62	71	0
Q4	10.18	10.18	3.96	55	0
Q5	12.01	12.01	4.41	143	0
Q6	13.03	13.03	4.10	152	0
Q7	5.44	5.44	4.50	171	0

Table 8: Statistics for Paper III

Question Number	Average mark All	Used	Std dev	No. of Attempts Used	Unused
Q1	13.83	13.85	3.93	156	1
Q2	15.42	15.5	4.17	108	1
Q3	11.74	11.8	3.22	90	1
Q4	13.75	13.75	4.53	103	0
Q5	10.90	10.90	4.19	131	0
Q6	13.75	13.75	4.43	120	0
Q7	14.10	14.10	4.63	173	0
Q8	11.96	11.96	5.02	101	0
Q9	10.11	10.11	4.34	74	0

Table 9: Statistics for Paper IV

Question Number	Average mark All	Used	Std dev	No. of Attempts Used	Unused
Q1	16.45	16.45	2.38	152	0
Q2	13.33	13.33	4.01	49	0
Q3	15.21	15.21	3.72	155	0
Q4	11.01	11.01	3.80	177	0
Q5	9.34	9.34	5.15	169	0
Q6	11.58	16.63	8.27	8	4
Q7	15.01	15.01	3.52	178	0

Table 10: Statistics for Paper V

Question Number	Average mark All	Used	Std dev	No. of Attempts Used	Unused
Q1	14.89	15.12	5.18	130	2
Q2	14.02	14.14	4.89	118	1
Q3	10.63	10.70	4.81	106	1
Q4	16.14	16.14	4.09	174	0
Q5	5.62	5.67	4.22	98	3
Q6	7.01	7.09	3.38	81	3

Table 11: Statistics for Computational Mathematics

Question Number	Average mark All	Used	Std dev	No. of Attempts Used	Unused
Project A	15.41	15.39	3.74	158	1
Project B	14.40	14.35	4.80	46	1
Project C	15.85	15.87	2.97	146	1

D. COMMENTS ON PAPERS AND ON INDIVIDUAL QUESTIONS

Paper I

Question 1. Candidates generally correctly answered part (a): listed the elementary row operations in (i), wrote down their matrices and inverses in (ii), and found the RREF in (iii). A common mistake in (a)(i) was to forget that one can only rescale a row of a matrix by a *non-zero* scalar. Part (b) was found to be slightly more difficult. In (b)(i), some students used the heuristic argument *every time you multiply by the matrix A, the non-zero diagonal moves up by one*, without giving a formal proof. The correct approach was to proceed by induction, which many of the candidates attempted, but only some of them presented a fully correct proof. In part (b)(ii), many students had the correct guess for A^n, but not all of them proved that their guess was correct by induction. There were also many mistakes in the right top entry of A^n.

Question 2. This question seems to have been very difficult for the candidates. The seemingly simple part (a)(i) had few correct definitions of a vector space; instead, a common answer was to say that *it is closed under addition and scalar multiplication*, which is the definition of a subspace of a vector space. Almost every candidate correctly defined a linear transformation in (a)(ii). The rest of the questions centered around counting problems related to vector spaces over finite fields and it seems that the lack of familiarity with finite field prohibited the students from correctly attempting them. While parts (a)(iii) and (b)(i) received many partially correct attempts, parts (b)(ii) and b(iii) had only a few correct solutions. The hardest part was the *only if* implication in (b)(iii), where few students realized that $\text{rank}(T) \leq 1$ because the codomain is 1-dimensional.

Question 3.

Question 4. The bookwork on this question was tricky, as the proof pulls together many ideas from the course. However, many candidates either understood the key ideas or were able to reproduce the proof by rote (I felt, unfortunately, that the latter was more common). The bookwork proof was a red herring for (a)(ii), as many candidates then wrongly assumed a diagonalizable matrix was orthogonally diagonalizable. Parts (b) and (c) were generally stronger. I would have liked to see better proofs in (b)(iii), where candidates stated the dimension was 'too big' to avoid the symmetric matrices without precise justifcation (stating the formula $\dim(A + B) = \dim(A) + \dim(B) - \dim(A \cap B)$ would suffice). Although (c)(i) could be answered by giving an explicit list of $n - 1$ vectors in $\ker(A)$, this was not done very well (I think because many natural approaches run into dividing-by-zero problems unless you are careful with cases). Instead, rank-nullity provided a clearer path to full marks on this part of the problem. Parts (c)(ii) and (c)(iii) were generally well done by those that attempted them.

Question 5. Q5 was mostly well-answered. It was intended to build on the observation that a group G is commutative if and only if the map $G \to G; x \mapsto x^2$ is a homomorphism (or similarly for inversion).

In general part (a) was an introduction to H, the 3×3 Heisenberg group over the field $\mathbb{Z}_3$, which can be useful in answering part (b). For (a)(i) a small number did not know what addition and multiplication tables were. For (a)(ii), a number of attempts failed to say that associativity was inherited, or that they were applying the subgroup test depending on which approach was taken.

For (b)(ii) almost any non-commutative group will do, but crucially not the group H from (a). $G = S_3$ (or S_n for $n \geq 3$) was a good and popular choice, because $(123) = (12)(23)$ and 3-cycles are sent to the identity by T while transpositions are sent to themselves. $G = Q_8$ and $G = D_8$ also appeared; in these cases if T were a homomorphism then its kernel would be trivial (since neither group has any element of order 3), and so the image, which equals G, is commutative by (b)(iv), a contradiction. A common error was to suggest $G = \mathrm{M}_n(\mathbb{R})$ – but this is not a group – or $G = \mathrm{GL}_n(\mathbb{R})$ and then say that multiplication is 'generically' not commutative. $G = \mathrm{GL}_n(\mathbb{R})$ is a valid example but the deduction needs some justification that the 'generic' set of non-commuting pairs of matrices has non-empty intersection with the set of (pairs of) cubes of invertible matrices – this can be done by explicit example. Taking $G = H$ (the group from (a)) is *not* a valid example; in this case T *is* a homomorphism.

(b)(iii) was hard with the idea being to apply the hint twice to get $a^4b^4 = (ab)^4$ and deduce that $a^3b^3 = (ba)^3 = b^3a^3$ from the homomorphism property. The best answers also realised that they had to show that the image of T is a group.

For (b)(iv) many assumed G was finite, which was fine provided they explained where this was involved: in particular, in this case an injective map (from G to itself) is necessarily surjective. Finiteness is not necessary, and a number of nice answers noted that $T(ab) = T(a)T(b) = T(b)T(a) = T(ba)$ because the image of T is commutative, and hence if T is injective then $ab = ba$.

Finally for (b)(v) some realised that a non-commutative group in which every element has order 3 would work because then T is trivial, and the best answers realised that the group H from (a) has this property.

Question 6. Q6 was the least popular question, though it was the easiest to complete with a number of near-perfect solutions. It was intended as a proof that there are infinitely many primes presented in the language of group theory.

For (a)(i) it was natural to refer to the well-ordering principle, but it was also fine to explain why S_n is finite and then note that every finite set has a minimum. Some mentioned the completeness axiom for the reals, and then had to show that $\inf S_n \in S_n$.

For (b) most of the problems came with assuming that various possibly infinite sets had finite sizes. Nice answers to (b)(ii) noted that the full strength of the fact that G/H is a partition is not needed, and in fact covering along with (b)(i) is enough. (b)(iv) was perhaps the hardest part but there were some nice solutions either along the lines of showing the map $G/(H \cap K) \to G/H \times G/H; x(H \cap K) \mapsto (xH, xK)$ is a well-defined injection; or that if $G \subset x_1H \cup \cdots \cup x_nH$ and $H \subset y_1(H \cap K) \cup \cdots \cup y_m(H \cap K)$ (which it is for some $y_1, \ldots, y_m$ by (b)(iii)), then $G \subset \bigcup_{i,j} x_iy_j(H \cap K)$.

The motivation behind (b)(v) is from its application in (c). We think of $G = \mathbb{Z}$ and $H_i = p_i\mathbb{Z}$ where $p_1, \ldots, p_k$ is a hypothetical complete list of primes. Then $U := H_1 \cup \cdots \cup H_k$ is the set of integer multiples of at least one of the primes $p_1, \ldots, p_k$ – by (a)(ii) this is everything except 1 and -1. In particular $\mathbb{Z} \setminus U$ is non-empty, and in (b)(v) we see it is a union of cosets of $H_1 \cap \cdots \cap H_k = p_1 \cdots p_k\mathbb{Z}$, so it is infinite. This contradicts the fact that it is $\{-1, 1\}$. By way of comparison, Euclid notes that $p_1 \cdots p_k + 1$ is in the coset $1 + H_1 \cap \cdots \cap H_k \subset \mathbb{Z} \setminus U$ but is not equal to 1 or -1.

Question 7. Q7 was the hardest question, though the bookwork was straight-forward. It is a proof that the only automorphisms of S_n (for $n \neq 6$) are inner automorphisms, meaning ones that arise by conjugation; famously S_6 has an automorphism that is *not* an inner automorphism.

The bookwork in (a)(i) was done well with the most common error being to only show one direction of the 'if and only if'. (a)(ii) was also well-known, though some attempts misinterpreted the question as asking for the order of the *set of* permutations of a given cycle type.

For (b)(i) a number of attempts noted that $\phi(ab)$ must have order 2 and so be in T_k for some k. Better answers also noted that homomorphisms map conjugate elements to conjugate elements and the T_ks are conjugacy classes, and so $\phi(T_1) \subset T_k$ for some k. Finally it also follows from this that $\phi^{-1}(T_k)$ is a conjugacy class of order 2 elements, and so $T_1 \subset \phi^{-1}(T_k) \subset T_s$ for some s which must therefore be 1. For (b)(ii) the hint implies $\phi(T_1) = T_1$ and so there is a permutation σ such that $\phi(1a) = (\sigma(1)\sigma(a)) = \sigma(1a)\sigma^{-1}$. Unfortunately this σ may depend on a and a number of answers fell down by assuming it was universal.

Paper II

Question 1. Question 1, on sequences, specifically lim sup, was generally well done. It was good to see that most candidates got the idea that the sequence of suprema was monotonic decreasing, and hence (as it was also bounded) converged to a limit. The calculations of examples were also quite well done, although some candidates got confused and thought the question was asking about the limit (which does not exist) of the sequence b_n rather than the lim sup.

Question 2. on convergence tests for series, proved more difficult than expected. Even part (b) was not done particularly well, and the more challenging examples in part (c) proved difficult for many candidates. A common mistake was using the bound $|sinx| \leq 1$ rather than $|sinx| \leq |x|$ in (c) (ii). Many candidates didn't give enough detail to apply the alternating series test in (c) (iii). There were a few very good answers however

Question 3. Question 3 also proved rather more difficult than anticipated, though not to the extent of Question 2. Many candidates didn't give sufficient detail in part (a), with failure to consider the extreme cases $R = 0$ or $R = \infty$ being a very common omission. In part (b), question (ii) was probably the best done (in general candidates seemed more confident with the ratio test than other tests).

Question 4. This was the most difficult question in Analysis II, with the fewest number of attempts. In part (a)(i), many solutions missed the inequality $0 < |x-p|$ in the definition of a limit point of a set (i.e., an isolated point is not a limit point). There was often no or incorrect justification of why $\mathbb{Z}$ has no limit points ($\varepsilon = 1/2$ does always not work when $p \in \mathbb{R} \setminus \mathbb{Z}$). Part (a)(ii) was generally fine, but there were frequently issues with the logic of the argument. Part (a)(iii) proved to be more difficult. The inclusion of $\overline{E}$ in the intersection of the closed sets containing E was usually fine, but the reverse inclusion was either missing or the argument incorrect or incomplete. Many claimed that $\overline{E}$ was closed without proving this. It requires showing that limit points of $\overline{E}$ lie in $\overline{E}$. Alternatively, one might argue that, for $p \notin \overline{E}$, the set $\mathbb{R} \setminus (p-\varepsilon, p+\varepsilon)$ is a closed set containing E, for ε sufficiently small. In part (b)(i), many missed the assumption that $0 < |x-p|$ in the definition of the limit of a function. The definition of when a function f converges as $x \to p$ was sometimes omitted or missed the assumption that there exists an ℓ that such that the limit is ℓ. The second half of part (b)(ii) was again more difficult. Many examples included functions that were not defined on all of $\mathbb{R}$, such as $1/x$, which was one of the assumptions. A simple exampel is given by $f(x) = e^x$ and $E = \mathbb{R}$.

Question 5. This questions was generally well done and was popular. In part (a)(ii), the Cauchy criterion states that uniform convergence is *equiva-*

lent to being Cauchy. Part (a)(iii) was generally fine. Some solutions used that f_n converges uniformly to f and f_n is bounded, then f is bounded without proof. In part (b)(i), when showing that $f_n g_n$ converges to fg uniformly, some solutions used that f_n and g_n were uniformly bounded without referencing (a)(iii), or that f and g were bounded without explaining why. Part (b)(ii) was harder, with relatively fewer solutions, and some incorrect examples where f_n or g_n was not uniformly convergent.

Question 6. This question was also popular and comparable in difficulty to Question 5. In Part (a)(ii), many solutions did not state the Constancy Theorem over a general interval, but instead chose a closed interval. Yet they used the result in (a)(iii) in full generality. A continuous function might not always extend from an open interval to a closed interval. In part (b)(i), many solutions introduced the function $f(x) - x$ and applied Rolle's theorem. While this is correct, it is not necessary, and one could directly invoke the Mean Value Theorem. There were very few completely correct solutions to (b)(iii), though many people obtained either the existence of a fixed point, often using the Intermediate Value Theorem, or that if there is a fixed point, then it is the limit of x_n via the Mean Value Theorem.

Question 7. 7a was done successfully by a lot of candidates, some forgot to show that f is integrable. In 7b, a lot of candidates missed the part that f_n was asked to be continuous in (i) and (ii). In (iii), many noticed that it was enough to find a family of fonctions not integrable (provided this family converges uniformly). 7c was more difficult and less candidates managed to treat it successfully.

Paper III

Question 1. Part (a) was answered well by most candidates, although a significant minority didn't find y explicitly in terms of x, which was needed to obtain the solution that satisfied the given initial condition. In Part (b) some candidates suggested a variety of unsuccessful substitutions. Again it was expected that the final solution explicitly satisfied the initial condition. There were some really excellent answers to part (c) but too many candidates seemed daunted by the integrations needed, and their solutions weren't simplified sufficiently for them to obtain elegant solutions.

Question 2. This question tested a variety of techniques and overall it was answered very well. Some answers to part (a) and part (b)(ii) were correct but appeared to come out of nowhere; without justification these did not score full marks. In part (d) several solutions had an incorrect range for θ, or, rather, did not justify their non-standard range.

Question 3. Part (a) was answered very well although several candidates did not get the mixed second-order derivative contribution correct. In part

(b) the second critical point was sometimes missing or incorrect. Whilst some answers to part (c)(i) were very good, too many students used a Lagrange multiplier without justification, and most solutions did not give the appropriate assumptions requested. Part (c)(ii) was done very well.

Question 4. This question was generally done well. Most people did (a)(i) and (a)(ii) correctly. Common mistakes in (a)(iii) concerned the fact that we need a partition of the sample space Ω rather than the set of values $\mathcal{S}$ taken by the random variable X, and/or restricting to only finite partitions, which loses generality. (b)(i), (b)(ii) and (b)(iii) were again generally done correctly, with the most common errors being calculation slips. Quite a few candidates lost marks for asserting that (iv) was immediate from (iii) by simply forgetting about the conditioning; only those who gave a properly justified argument using the partition theorem for expectations (or equivalent) got full marks here. Most people found (v) straightforward.

Question 5. This seems to have been found harder than the other two probability questions. Many candidates lost the mark in (a)(i) for being too vague: answers which didn't at least specify the probability of a successful trial got 0 marks. (a)(ii) and (a)(iii) were done well, with marks mainly lost for calculation errors. Many students seem to have misunderstood (b). In particular, despite the fact that the question clearly states that N is a random variable, many just assumed that it was always equal to n (which made (b)(iii) particularly confused). Candidates lost marks in (b)(i) for not giving a reasonable justification for their answer. A common error was to say that the parameter of the Bernoulli distribution for X_i was p rather than r. Candidates who did this were not penalised again in the following parts, since it does not render them any easier to set $r = p$. Despite an explicit instruction in the hint that a proof of the random sum formula was not necessary, surprisingly large numbers of candidates opted to waste time proving it regardless (and not always correctly!). However, relatively few candidates gave the clear statement asked for, and many lost marks for not at least mentioning that $X_1, X_2, \ldots$ need to be i.i.d. and also independent of N. (b)(iii) was done correctly by only a small minority, and only an even smaller minority gave a complete justification involving the uniqueness of the p.g.f.; the others lost one mark. Substantial partial credit was awarded in part (c) for spotting how it maps onto the set-up in (b), but only a small number of candidates gave a full and correct justification. Others performed direct calculations to obtain the distribution, which got full marks if correct, but that was the case only for a relatively small number of people.

Question 6. This question was done well. Most people did (a)(i) and (a)(ii) correctly. In (a)(iii), full credit was only given for answers which mentioned that we may use countable additivity because we have a union of a countable number of disjoint events (or equivalent); many people simply ignored the instruction to justify their argument carefully. (b)(i) and (b)(ii) were done correctly by most people, with marks most commonly lost for sign errors in the integration, which might have been caught by sanity-checking: a density function should not be negative, and the expectation of a positive random variable cannot be negative! (b)(iii) was again done well in many cases, with the most common error being in the manipulation of the floor function. Only a tiny minority of students thought to use $\mathbb{E}[\lfloor X \rfloor] = \sum_{k\geq 1} \mathbb{P}(\lfloor X \rfloor \geq k)$ in (b)(iv); the more complicated calculation involving the probability mass function was done correctly by a substantial fraction but far from all candidates; partial credit was given for sensible assertions involving the zeta function but an incomplete calculation. In (b)(iv), many people correctly got some elements of the answer, and received partial credit; a smaller number saw their way through to a complete argument, and full marks were only given if, for example, there was some sensible justification of the statement that $\mathbb{E}[\lceil X \rceil] = \mathbb{E}[\lfloor X \rfloor] + 1$.

Question 7. This question was generally done well. Some people struggled to remember the definitions of bias and MSE in (a)(ii), and this fed through into (b)(ii). Most people calculated the MLE correctly in (a)(iii), and about half checked that their estimator maximised the likelihood. Either graphical or calculus justifications were accepted here, but if they were absent a mark was lost. Many people spent a long time deriving the expectation in (b)(i) by induction, rather than simply using the gamma density given in the question. Unfortunately the inductive method was much slower and had more scope for calculation errors. In (b)(ii), marks were commonly lost for unjustified calculation steps. In (b)(iii), some people struggled to get the confidence interval the right way round, and a substantial number lost marks for not saying they were using the CLT.

Question 8. This question seems to have been perceived as hard. Many seem to have found it difficult to get started, perhaps because of the long preamble. Part (a) was generally done well, though. The main ways in which marks were lost here were either forgetting the factor of $1/\sqrt{n}$ in endpoints of the confidence interval, getting confused about which quantile of $N(0, 1)$ to use, or incorrect manipulations of inequalities. In part (b), very few people used the hints given in the preamble to the question. In (b)(i), people who correctly remembered the definition of covariance usually managed to get the independence, but very few people then specified the marginal distributions as required in the question. Most people who reached (b)(ii), knew what they had to show, but got tangled up in the calculations of means and variances. Few people made serious attempts at (b)(iii). Those who were able to do the first step where one extracts $X_1 - \bar{X}$ from the sum found the rest fell quickly into place. Others wasted time expanding the squares and sums, and getting confused in the calculations. Sometimes the final deduction was missing.

Question 9. This seems to have been found hard. In parts (a)(i) and (a)(ii), many people clearly hadn't memorised the estimators, so were deriving them blind; this led to quite a lot of errors which might have been avoided. Many people failed to notice that $\sum_{i=1}^{n} x_{i1} = 0$ is given in the statement of the question, which considerably simplifies the calculations. Quite a few people just set up the simultaneous equations and then said "solve for this" rather than actually solving. This was perhaps a consequence of time pressure, but obviously couldn't be awarded the marks! In (a)(iii), many people correctly stated that the variables could be correlated, but didn't then say why this meant that the interpretation might be problematic; those lost one mark. In (b)(i), most people were unable to give a convincing account of the purpose of PCA, and there was a common confusion between PCA and clustering. Parts (b)(ii) and (iii) were done well. Full marks were only awarded in (b)(ii) for solutions which mentioned the role played by the variance. The reparametrisation in (c)(i) was generally fine, but there were very few convincing answers to (c)(ii): for the advantages, not many people made the connection to (a)(iii) and, for the disadvantages, very few mentioned interpretability.

Paper IV

Question 1. Q1: This question was attempted by a vast majority of the candidates, and has overall been extremely well done. The bookwork in (a) was generally well done, though some candidates struggled to relate the volume of the parallelepiped with the proposed product. Parts (b) and (c) were generally well done by the candidates, but some students did not attempt (c), maybe due to a lack of time. In part (d), the majority of the students did very well until subpart (iv) but failed, or did not attempt the last part (v).

Question 2. This question was not popular, and a small proportion of the students attempted it. It was also the most difficult question of the set. For part (a), several students struggled to identify the paraboloid and the different cases of the third surface. In part (b), there was a mixed response, with some students doing very well, using different techniques, including eigenvectors, to arrive at the right answer, but several students were stuck at the start of the question. In part (c), for those who attempted, many had good intuition, but the solutions often lacked sufficiently formal arguments.

Question 3. This question was also popular with candidates, and was also very well done. The different parts of part (a) were well done, even if several students failed to properly identify the value of θ^2 where the arc length increases fastest. The response to part (b) showed more variability, and maybe students did not attempt it, maybe due to lack of time. Subpart (i) was overall fine, but several students made errors in subpart (ii), often from the beginning of the question.

Question 4. This question was attempted by all but one candidate. The bookwork in (a) was generally well done, though many candidates derived the expression for $\ddot{\mathbf{r}}$ from first principles, rather than using the hint. Part (b) was found to be very challenging with a common error being to take the RHS of the inhomogeneous equation as the particular solution, when in fact it required a small modification (because the prefactor of u in the ODE was not unity). Similarly many candidates assumed that an arbitrary rotation could be applied to the axes without considering the effect of such a rotation on the initial conditions. Finally, very few candidates correctly described the shape of the orbit in part (b)(iii); many candidates (correctly) stated that the period was $2\pi/n$ but did not realise that this meant the shape could not be an ellipse.

Question 5. This question was very popular, but was also found to be difficult. The bookwork of part (a) was generally well done, as was that in the first part of (b). However, determining when the motion was bounded proved more difficult than expected, with very few correctly explaining why this required equality to be attained in the inequality of part (b). In part (c) only a very few candidates were able to show that, for the particular choice of f, this condition corresponds to a quadratic in r^2 and hence determine when two real solutions might exist.

Question 6. This question was not popular with candidates, but those who attempted it generally scored high marks. Parts (a) and (b)(i) were particularly well done, with candidates generally confident in the calculation of moment's of inertia from first principles (including for the shifted axis in (b)(i)). Some candidates made algebraic slips in part (b) leading to an incorrect result for the limit $\alpha \ll 1$, $\mu \gg 1$ discussed in (b)(iv); such errors could have been detected by realizing that this limit corresponds physically to the classic simple pendulum and hence $\omega = (g/d)^{1/2}$.

Question 7. (a)(i): This was very routine and done identically and correctly by almost all candidates. (a)(ii): This was again done similarly by all candidates, with several minor mistakes which can be made along the way: finding the particular solution to the normalised equation with RHS 1 but forgetting to multiply by 5 for the final answer, accidentally multiplying *also* the homogeneous solution by 5, or multiplying by the hcf (7) rather than by 5. One can also include the detail that the equation $37x + 71y = 0$ has only the solutions $(71, -37)\mathbb{Z}$ specifically because 71 and 37 are coprime. (b), 1$^{\text{st}}$ part: It was clear that $\frac{1}{p}$ was a fixed point of the RHS $g(x)$, or equivalently a root of $g(x) - x$ after taking the limit on both sides of the equation; the latter was done with varying levels of rigour, with some candidates making claims such as $x_{k+1} \to x_k$ as $k \to \infty$. Many tried to find an interval around $\frac{1}{p}$ to which to apply the contraction mapping theorem, although this was not necessary. Note that Horner's method is not relevant. (b): 2nd part: Many candidates were not confident with calculating the order, so just stated a claim as to what it was, intuiting that it is likely cubic due to the cubic polynomial. A good solution was to Taylor expand $g(x)$ about $\frac{1}{p}$ (in fact to 3rd order this will be exact, since it is already a cubic polynomial) and observe that $g'(\frac{1}{p}) = g''(\frac{1}{p}) = 0$. An even neater solution was to notice that subtracting $\frac{1}{p}$ from $g(x)$ factorises nicely to give

$$\frac{\left|x_{k+1} - \frac{1}{p}\right|}{\left|x_k - \frac{1}{p}\right|^3} = p^2.$$

(c)(i), 1st part: This was mostly done well, apart from a few candidates who defined the gradient and Hessian with entries using the full derivative d rather than the partial derivative ∂. Some candidates gave the transpose of the Hessian rather than the Hessian, or (almost universally) forgot that it is not *a priori* symmetric. (c)(i), 2nd part: This was done well as it just required stating the formula, but care should be taken to specify that both the Hessian and gradient are evaluated at the current iterate $\mathbf{x}_k$ (even though in (ii), the Hessian turns out to be constant). Several other mistakes were possible when stating this formula, such as 'dividing' by the Hessian matrix rather than applying its inverse, dividing the gradient term by F, and using $\mathbf{x}_k$ in place of $\nabla F(\mathbf{x}_k)$ (which may be an easy mistake to make since they are both vectors). (c)(ii): Candidates should state that the first iterate found, $\mathbf{x}_1 = (0, 1)$, is either a critical point, or a fixed point of the iteration, so that no further Newton steps are needed. Many silly mistakes where made which caused candidates to do several Newton iterations rather than just one, such as using a plus sign in the Newton formula in place of a minus, miscalculating the Hessian due to several ± 4 entries in ∇F (which made candidates calculate the off-diagonal entries of the Hessian as 0), and many, many candidates computed the inverse of the Hessian incorrectly by computing its determinant to be $24 + 16$ rather than $24 - 16$. No candidate explicitly made the observation that since F is quadratic, its criticality condition is linear, so that Newton is guaranteed to converge in one step.

Paper V

Question 1. For 1(a)(ii), a common mistake was to differentiate $x = r$ cos(theta) and $y = r$ sin(theta) as if r was a constant on theta. It turns out that these two mistakes compensate, those who did them did not receive full credit, even though they got the right result.

The simplest way of solving 1(b)(i) was to parametrize the curve C and apply the fundamental theorem of calculus. Some candidates attempted to treat this question using the Kelvin-Stokes formula, but failed to give a fully rigorous proof. In particular, the existence of a surface S bounded by C should have been addressed : if C is knotted, S might not be an embedded disk (one could have taken a non-embedded disk, but that would have required some explanation as well).

Most candidates succeeded in 1(b)(ii) and used the unit circle for the curve C.

Question 2.

Question 2 was in general well received, with several students answering all questions correctly. Part a) i) was simple and well answered. Part a) ii) caused most problems, with a large proportion of students not even attempting it. Those who did attempted it often were able to set up the proof, introduce the two parametrisations, but were not able to correctly (and rigorously) show that the integral is independent of the parametrisation. Part b) was easy, with most students getting full marks. Part c) caused minor problems, with a small number of students incorrectly parametrising the circle or getting the wrong answer due to algebraic mistakes when evaluating the integrals.

Question 3. Question 3 caused problems to several students. Part a) was on average good, with several students getting full marks, although many students failed to get the correct answer because they failed to correctly parametrise the surface and evaluating the normal necessary to calculate the surface integral or due to calculation errors. Part b) was the trickiest, with only a few students getting full marks. Most students recognised that the key to the proof was that the statement is true for all closed curves. However, most candidates were unable to produce a rigorous proof and were thus awarded half marks. Part c) caused minor problems, with several students realising the need to use Stokes theorem, but still failing to prove the identity.

Question 4. Candidates generally did not specify the function at which it converges and did not take into account the boundary values $x = 0,\ \pi$. But generally this part was done correctly. (b) (i) The main problem was in the algebraic computations (integration, signs, constant) and to specify the actual series after computing the coefficients. (ii) This part was relatively fine, some candidates forgot to add the series truncated at large number of terms. (iii) Most candidates gave a reasonable justification in terms of disco

Question 5. 5) a) This was generally well done, with most candidates recalling the derivation of the heat equation from first principles. b) i) A common mistake here was misidentifying the correct boundary condition at $x = 0$. Many candidates applied a Dirichlet boundary condition here, and among those who correctly applied a Neumann condition it was common to omit the constant k and hence reach an incorrect solution. ii) Most candidates struggled with this question. In particular, a common early difficulty was in finding the form of the separable solution to the heat equation with the correct boundary conditions, while those who did this subsequently struggled to find the coefficients of the Fourier series. iii) Without a solution to (ii), most candidates were unable to consider its behaviour at late times. Those who found the solution to (ii) generally had no issues with (iii). c) More candidates struggled with this question than expected. Identifying the boundary conditions again caused difficulty, and while many candidates recognised that continuity was required at $x = l$, fewer identified the condition required here on the derivatives of T. Those who did generally reached the correct solution.

Question 6. Deriving the wave equation was generally well done, and most candidates did well on this question. The most common mistake here was in incorrectly identifying the boundary conditions. In particular, the boundary condition at $x = 0$ was frequently misidentified. b) Most candidates struggled with this question, particularly at the early stages. In particular, many candidates ignored the prompting of the question to consider different separable solutions for positive and negative x and as such immediately ran into problems finding solutions to the equation. Those who did often used incorrect boundary conditions to identify the Fourier coefficients. Few candidates identified the correct equations for the allowed frequencies of the wave, or explained why there must be infinitely many solutions. c) Given the difficulties with (b), most candidates were unable to attempt (c).

$$\frac{\dot{H}}{H} = \frac{F_{xx}}{F} - 1 = \omega.$$

Not relabelling $\omega + 1 = \tilde{\omega}$ led to a significant number of sign errors in the solution for H. Another common issue was using a generic Fourier series coefficient formula integrated over $-\pi$ to π; the question needed this to be derived which only a handful of candidate did. Overall, many candidates did present good solutions to the question.

E. COMMENTS ON PERFORMANCE OF IDENTIFIABLE INDIVIDUALS

F. MODERATORS AND ASSESSORS

Moderators: Prof. Andrew Dancer (Chair), Prof. Christina Goldschmidt, Prof. Dominic Vella, Prof. Tom Sanders, Prof. Renaud Lambiotte, Dr. Cath Wilkins, Prof. Andras Juhasz.

Assessors: Dr. Aleksander Horawa, Dr. Antonio Girao, Dr. Ric Wade, Dr. Maria Christodoulou, Dr. Guillem Cazassus, Dr. Francis Aznaran, Dr.

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