

## Algebra and functions

### TMUA Specification (April 2025, Section 1 §MM1)

- Laws of indices for all rational exponents.
- Use and manipulation of surds. Simplifying expressions that contain surds, including rationalising the denominator.
- Quadratic functions and their graphs; the discriminant of a quadratic function; completing the square; solution of quadratic equations.
- Simultaneous equations: analytical solution by substitution, e.g. of one linear and one quadratic equation.
- Solution of linear and quadratic inequalities
- Algebraic manipulation of polynomials, including:
  - expanding brackets and collecting like terms
  - factorisation and simple algebraic division (by a linear polynomial, including those of the form  $ax + b$ , and by quadratics, including those of the form  $ax^2 + bx + c$ )
  - use of the Factor Theorem and the Remainder Theorem
- Qualitative understanding that a function is a many-to-one (or sometimes just a one-to-one) mapping.
- Familiarity with the properties of common functions, including  $f(x) = \sqrt{x}$  (which always means the “positive square root”) and  $f(x) = |x|$ .

### Revision

- $a^m a^n = a^{m+n}$  for any positive real number  $a$  and any rational numbers  $m$  and  $n$ .
- $(a^m)^n = a^{mn}$  for any positive real number  $a$  and any rational numbers  $m$  and  $n$ .
- $a^{-n} = \frac{1}{a^n}$  for any positive real number  $a$  and any real number  $n$ .
- $a^0 = 1$  for any non-zero real number  $a$ .
- (Rationalising the denominator) It’s sometimes helpful to simplify an expression like  $\frac{a}{\sqrt{b}}$  to  $\frac{a\sqrt{b}}{b}$ , or an expression like  $\frac{a}{b + \sqrt{c}}$  like this

$$\frac{a}{b + \sqrt{c}} = \frac{a(b - \sqrt{c})}{(b + \sqrt{c})(b - \sqrt{c})} = \frac{a(b - \sqrt{c})}{b^2 - c}.$$

- (Completing the square) If  $a \neq 0$  then we can write  $ax^2 + bx + c$  in the form  $a(x + r)^2 + p$  because

$$ax^2 + bx + c = a \left( x + \frac{b}{2a} \right)^2 - \left( \frac{b^2 - 4ac}{4a} \right).$$

This is helpful if we're trying to prove that the quadratic is non-negative for all values of  $x$ , because anything squared is non-negative.

- Once we've written the quadratic in the form  $a(x + r)^2 + p$ , we know that there is a vertical line of symmetry at  $x = -r$ , because the values of the quadratic at  $-r \pm D$  will both be  $aD^2 + p$ , for any distance  $D$  away from the line of symmetry.
- The discriminant of the quadratic  $ax^2 + bx + c = 0$  is equal to  $b^2 - 4ac$ . If the discriminant is positive then the quadratic has two real solutions. If the discriminant is zero then there's one (repeated) real solution. If the discriminant is negative then there are no real solutions.
- (The quadratic formula) If  $b^2 - 4ac \geq 0$ , then the solution(s) of  $ax^2 + bx + c = 0$  are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- If  $b^2 - 4ac \geq 0$ , then  $ax^2 + bx + c$  can be written as  $a(x - \alpha)(x - \beta)$ , where  $\alpha$  and  $\beta$  are real numbers called the "roots", with values given by the quadratic formula above.
- (Difference of two squares) The expression  $x^2 - a^2$  factorises as  $(x - a)(x + a)$ .
- Given two linear equations for  $x$  and  $y$  like  $2x + 3y = 7$  and  $3x + 5y = 9$ , we can solve by rearranging the first for  $x$  and substituting that into the second equation, then rearranging the resulting equation for  $y$ , before back-substituting for  $x$ .
- We can add and subtract from each side of an inequality. For example, if  $x < 2$  then  $x + 3 < 2 + 3$ , and if  $y > -3$  then  $y - 5 > -3 - 5$ .
- We can multiply each side of an inequality by a fixed number, but if we multiply by a negative number then the direction of the inequality changes. For example,  $-6 < -3$ , but if we multiply both sides by  $-\frac{1}{3}$  then we need to flip the sign to get the true statement  $2 > 1$ .
- Squaring each side of an inequality is a bit like multiplying, and we need to be careful. If both sides of an inequality are positive, then squaring each side preserves the inequality, because  $x^2$  is an increasing function when  $x > 0$ .
- If we have a quadratic inequality like  $(x + 3)(x - 5) < 0$ , then we can consider the signs of the terms. In this case the inequality is satisfied for  $-3 < x < 5$ .
- A polynomial is an expression of the form  $a_0 + a_1x + a_2x^2 + \dots + a_kx^k$  for some whole number  $k \geq 0$ , and with real numbers  $a_0, a_1, \dots, a_k$ , known as the coefficients.
- The degree of a polynomial is the highest exponent of  $x$ , so the degree of any quadratic is 2, the degree of  $x^{17} - x^{12}$  is 17, and the degree of  $100x + \pi x^3 - x$  is 3, for example. The degree of a constant polynomial ( $y = a_0$ ) is zero.

- (Polynomial division) Given polynomials  $f$  and  $g$  where the degree of  $f$  is at least as large as the degree of  $g$ , we can divide  $f$  by  $g$ . By this I mean that we can find polynomials  $q$  (called the quotient) and  $r$  (called the remainder) such that  $f(x) = q(x)g(x) + r(x)$ , and such that the degree of  $r$  is strictly less than the degree of  $g$ .

If  $g$  has degree 1 then this remainder  $r$  will just be a number, but in general it's a polynomial.

Practically, we might find  $q$  and  $r$  by comparing coefficients on each side of the identity. I like to start with the highest exponent and work downwards.

Example: divide  $3x^3 - 2x^2 + x + 10$  by  $x^2 + 4x + 3$ .

I'm looking for  $q$  and  $r$  such that  $3x^3 - 2x^2 + x + 10 = q(x)(x^2 + 4x + 3) + r(x)$ , with the degree of  $r$  less than 2 (so just a linear polynomial). Looking at the highest exponent on the left, I can see that the degree of  $q$  must be 1. So perhaps I'll write  $q(x) = Ax + B$  and  $r(x) = Cx + D$  and solve for those coefficients. Rather than writing out everything in terms of  $A$ ,  $B$ ,  $C$ , and  $D$  all at once, I'm going to start with the highest exponent and work my way down.

The coefficient of  $x^3$  on the right will be  $A$ , and I want this to be 3 to match the coefficient of  $x^3$  on the left. Let's write that down; I have something like

$$3x^3 - 2x^2 + x + 10 = (3x + B)(x^2 + 4x + 3) + Cx + D$$

and it's time to think about the coefficient of  $x^2$ . On the left that's  $-2$  and on the right it's currently  $12$  from the  $3 \times 4$  that I've already got... plus  $B \times 1$ . I should take  $B = -14$ . I've run out of terms in the quotient  $q$ , so it's time to use the remainder  $r$  to mop up any leftover terms.

The coefficient of  $x$  on the left is  $1$ , but on the right so far I have  $3 \times 3 - 14 \times 4 = -47$ . So I should add  $48x$  via a term in the remainder. Similarly, the constant coefficient is currently  $-42$  on the right-hand side, but I want it to be  $10$  to match the left-hand side, so I'll add  $52$ .

$$3x^3 - 2x^2 + x + 10 = (3x - 14)(x^2 + 4x + 3) + 48x + 52.$$

With practice, it's possible to do this without ever writing the letters  $A$ ,  $B$ ,  $C$ , or  $D$ .

- (Factor Theorem) If  $p(a) = 0$  for a polynomial  $p(x)$ , then  $(x - a)$  is a factor of  $p(x)$ . This means that  $p(x) = (x - a)q(x)$  for some polynomial  $q(x)$ . Conversely, if  $(x - a)$  is a factor of  $p(x)$ , then  $p(a) = 0$ .
- If we know that  $p(a) = 0$  and we want to actually find  $q(x)$  such that  $p(x) = (x - a)q(x)$ , then we could use polynomial division, as demonstrated above.
- (Remainder Theorem) If a polynomial  $p(x)$  is divided by  $(x - a)$ , then the remainder is  $p(a)$ . This means that  $p(x) = (x - a)q(x) + p(a)$  for some polynomial  $q(x)$ .
- When sketching the graph of  $y = ax^2 + bx + c$ , we should consider whether  $a$  is positive or negative, whether the quadratic has any roots (and if so, where), and where it crosses the  $y$ -axis.

- Sometimes a function which is not a quadratic might secretly be a quadratic in a different variable. For example,  $y = e^{2x} + e^{x+3} - 1$  is not a quadratic, but if we write  $u = e^x$  then we have  $y = u^2 + e^3u - 1$ , which is a quadratic. This is sometimes called “changing variable” or “finding the hidden quadratic”.
- A “one-to-one” function has the property that no two distinct inputs give the same output. At university (but not on TMUA), we might use the adjective “injective” to describe such a function.  
Example:  $f(x) = x^3$  is a one-to-one function because, for any two inputs  $x_1$  and  $x_2$ , if  $(x_1)^3 = (x_2)^3$  then  $x_1 = x_2$ .  
Example:  $f(x) = x^2 + 1$  is not a one-to-one function because  $(-3)^2 = (3)^2$ . Note that I only have to give one counter-example.
- Sometimes people say “many-to-one” to describe a function that (might) have multiple inputs that give the same output.
- Functions cannot be “one-to-many”. For each input, there must be exactly one output.
- The function  $f(x) = \sqrt{x} = x^{1/2}$  is defined for  $x \geq 0$  as the non-negative number that squares to  $x$ . It’s called the square root of  $x$ .
- The function  $|x|$  just means  $x$  if  $x \geq 0$ , and it means  $-x$  if  $x < 0$ . It’s called the absolute value of  $x$ . It’s the distance between  $x$  and 0 on the number line.

**Revision Questions**

1. Given the polynomial  $p(x) = 2x^3 - 5x^2 + 7x - 3$ , evaluate  $p(2)$ .
2. Find a positive number  $x$  which satisfies  $x^2 = x + 1$ .
3. What is the discriminant of the quadratic  $2x^2 + 5x + 1$ ?
4. For which values of  $k$  does  $x^2 - x + k = 0$  have exactly two real solutions?
5. For which values of  $k$  does  $x^4 - x^2 + k = 0$  have exactly two real solutions?
6. How many real solutions does  $x^2 + bx + 1 = 0$  have? Find different cases in terms of  $b$ .
7. Write  $x^2 + 4x + 3$  in the form  $(x + a)^2 + b$ .
8. Find the maximum or minimum value of the function  $f(x) = -2x^2 + 8x + 5$  by completing the square. Is this value a maximum or a minimum?
9. Let  $p(x) = x^3 - 13x^2 - 65x - 51$ . Check that  $p(17) = 0$ . Factorise  $p(x)$ .
10. Divide  $2x^3 + 5x^2 - 5x - 19$  by  $x^2 + 4x + 3$ .
11. Divide  $p(x) = x^3 - 7x^2 + 15x + 1$  by  $x - 3$ . Check that the remainder matches the value of  $p(3)$ .
12. If  $(x - 2)$  is a factor of  $p(x)$ , what is the value of  $p(2)$ ?
13. Show that  $(x - 2)$  is a factor of  $f(x) = x^4 - 6x^3 + 13x^2 - 12x + 4$ , and factorise  $f(x)$ .
14. Find all the roots of the polynomial  $p(x) = x^3 - 6x^2 + 11x - 6$ .
15. Determine whether  $(x - 3)$  is a factor of  $p(x) = 2x^4 - 11x^3 + 19x^2 - 20x + 15$ .
16. In each of the following cases, choose a variable  $u$  in terms of  $x$  to make the function into a quadratic in  $u$ . There might be more than one sensible choice of  $u$  in each case.
  - $y = 2x^6 + x^3 + 1$ ,
  - $y = x + \sqrt{2x}$ ,
  - $y = 3e^{-3x} + 6e^{-6x}$ ,
  - $y = \frac{1 + x}{(1 - x)^2}$ .
17. Given that the polynomial  $q(x)$  has roots 2,  $-3$ , and 1, what could  $q(x)$  be? Write down at least two possible polynomials, with different degrees.
18. Given that the polynomial  $v(x) = x^3 + 2x^2 + ax + b$  has a double root at  $x = 1$ , find the values of  $a$  and  $b$ .
19. Solve the simultaneous equations  $x + 4y = 1$  and  $2x - y = 3$ .
20. Solve the simultaneous equations  $x^2 + 2x + xy + y^2 = 5$  and  $x + y = 2$ .

21. Solve the simultaneous equations  $x^2 + y = 2$  and  $x + y^2 = 2$ .
22. For which values of  $x$  is it true that  $x^2 + 4x + 3 > 0$ ?
23. Given  $-2 < a < 1$ , what can you say about  $a^2$ ?
24. Given  $a < b$  and  $c < d$ , what (if anything) can you say about the relationship between  $ac$  and  $bd$ ? What (if anything) can you say if you're also told that  $a > 0$  and  $c > 0$ ?

## TMUA Questions

### TMUA 2020 Paper 1 Question 3

Find the complete set of values of  $x$  for which

$$(x + 4)(x + 3)(1 - x) > 0 \quad \text{and} \quad (x + 2)(x - 2) < 0$$

- A  $1 < x < 2$
- B  $-2 < x < 1$
- C  $-2 < x < 2$
- D  $x < -2$  or  $x > 1$
- E  $x < -4$  or  $x > 2$
- F  $x < -4$  or  $-3 < x < 1$
- G  $-4 < x < -2$  or  $x > 1$

[\[See the next page for hints\]](#)

### TMUA 2020 Paper 1 Question 9

The quadratic expression  $x^2 - 14x + 9$  factorises as  $(x - \alpha)(x - \beta)$ , where  $\alpha$  and  $\beta$  are positive real numbers.

Which quadratic expression can be factorised as  $(x - \sqrt{\alpha})(x - \sqrt{\beta})$ ?

- A  $x^2 - \sqrt{10}x + 3$
- B  $x^2 - \sqrt{14}x + 3$
- C  $x^2 - \sqrt{20}x + 3$
- D  $x^2 - 178x + 81$
- E  $x^2 - 176x + 81$
- F  $x^2 + 196x + 81$

[\[See the next page for hints\]](#)

## Hints

### TMUA 2020 Paper 1 Question 3

- Start by thinking about the inequalities separately.
- Sketch the polynomial on the left-hand side of each inequality. What does the cubic look like? There are no repeated roots, so at each root, the sign of the polynomial ( $> 0$  or  $< 0$ ) changes.
- Once you've understood the values of  $x$  for which the two inequalities separately hold, it's time to think about the word "and" in the middle of the question. I drew a little table to keep track of the different overlapping ranges of  $x$ . I could imagine drawing and labelling a number line.

### TMUA 2020 Paper 1 Question 9

- You don't actually need to solve for  $\alpha$  and  $\beta$  to do this question (but you can if you like). I started to do it that way, but got bored at roughly the point where I needed to work out a square root of a complicated expression.
- Instead, you could multiply out these factorised expressions; if we can find the coefficients of this new quadratic expression then we're done.
- You might not immediately know the coefficients of this quadratic expression, but I think that with some work, you do know the squares of these coefficients.

**TMUA 2020 Paper 1 Question 20**

For how many values of  $a$  is the equation

$$(x - a)(x^2 - x + a) = 0$$

satisfied by exactly two distinct values of  $x$ ?

- A 0
- B 1
- C 2
- D 3
- E 4
- F more than 4

[\[See the next page for hints\]](#)

**TMUA 2021 Paper 1 Question 8**

The line  $y = 2x + 3$  meets the curve  $y = x^2 + bx + c$  at exactly one point.

The line  $y = 4x - 2$  also meets the curve  $y = x^2 + bx + c$  at exactly one point.

What is the value of  $b - c$ ?

- A -9
- B -5.5
- C -1
- D 5
- E 6
- F 14

[\[See the next page for hints\]](#)

## Hints

### TMUA 2020 Paper 1 Question 20

- The question doesn't use these words, but it's about repeated roots.
- The phrase "distinct values of  $x$ " in this question does not mean "distinct expressions in terms of  $a$  for  $x$ ". So if your roots turn out to be  $a + 6$  and  $a^2$  then those might be distinct values, but they might not be (because  $a$  might be 3).
- It might help to change the letter  $a$  to  $y$  and draw the graph for the points in the  $(x, y)$ -plane for which the equation holds.
- Don't ignore the  $(x - a)$  term.

### TMUA 2021 Paper 1 Question 8

- The question asks for  $b - c$ . At the start of the question, I can't guess whether I'll solve for  $b$  and  $c$  separately and then calculate  $b - c$ , or if I'll find  $b - c$  directly along the way, without needing to find  $b$  and  $c$  individually. I'll keep an eye out for  $b - c$  along the way.
- The phrase "exactly one" reminds me of repeated roots... but what's the equation?
- There are two lines so we will have two equations. This is good news, because there are two unknowns,  $b$  and  $c$ . Well, actually, there are lots more things that I don't know (like the positions of the points where the lines meet the curve), but I can see that if I know  $b$  and  $c$  then I'm done.

**TMUA 2021 Paper 2 Question 16**

$p$  and  $q$  are real numbers, and the equation

$$x|x| = px + q$$

has exactly  $k$  distinct real solutions for  $x$ .

Which one of the following is the complete list of possible values for  $k$ ?

- A 0, 1, 2
- B 0, 1, 2, 3
- C 0, 1, 2, 3, 4
- D 0, 2, 4
- E 1, 2, 3
- F 1, 2, 3, 4

[\[See the next page for hints\]](#)

**TMUA 2022 Paper 1 Question 13**

Given that

$$\left(a^3 + \frac{2}{b^3}\right)\left(\frac{2}{a^3} - b^3\right) = \sqrt{2}$$

where  $a$  and  $b$  are real numbers, what is the least value of  $ab$ ?

- A  $-\sqrt{2}$
- B  $\sqrt{2}$
- C  $-2\sqrt{2}$
- D  $2\sqrt{2}$
- E  $-\frac{\sqrt{2}}{2}$
- F  $\frac{\sqrt{2}}{2}$
- G  $-2^{\frac{1}{6}}$
- H  $2^{\frac{1}{6}}$

## Hints

### TMUA 2021 Paper 2 Question 16

- Sketch the function  $y = x|x|$ .
- The function  $y = px + q$  represents a straight line, and since we don't know the values of  $p$  and  $q$  it could be any straight line (technical note; not a vertical line because we're told that  $p$  is real).
- So we're supposed to think of ways (if possible) that a line could cross the function  $y = x|x|$  in exactly 0 or 1 or 2 or 3 or 4 places. Some of those cases are impossible, unless the answer is C.
- We could consider easy cases like " $p = 0$  and  $q = 1$ " or " $p = 1$  and  $q = 0$ " to see what happens.
- If we can imagine a case with particular behaviour, then that's probably good enough (I might not find precise values of  $p$  and  $q$  that produce that behaviour).

### TMUA 2022 Paper 1 Question 13

- Whenever I see something like  $a^3$  and  $a^{-3}$ , I'm hoping that they'll cancel each other out.
- Therefore, I'm willing to consider multiplying out the brackets in this question.
- The question asks for  $ab$ . It's not obvious if we'll solve for  $a$  and  $b$  separately, or just solve for  $ab$ . That said, we only have one equation, so I'm doubting my ability to find values of two unknowns.
- The question asks for the "least value" of  $ab$ . It's not obvious if we'll find all possible solutions for  $ab$  and select the least, or whether there will be some shortcut that means we only find the value we want. Probably I'm overthinking it and I should now get on with the question.
- $a$  doesn't appear in the equation except as  $a^3$ . So perhaps the cube  $(ab)^3$  will make an appearance before  $ab$  does.