Sequences and series

TMUA Specification (April 2025, Section 1 §MM2)

- Sequences, including those given by a formula for the n^{th} term and those generated by a simple recurrence relation of the form $x_{n+1} = f(x_n)$.
- Arithmetic series, including the formula for the sum of the first n natural numbers.
- The sum of a finite geometric series.
- The sum to infinity of a convergent geometric series, including the use of $|r| < 1$.
- Binomial expansion of $(1 + x)^n$ for positive integer n, and for expressions of the form $(a + f(x))^n$ for positive integer n and simple $f(x)$.
- The notations $n!$ and $\binom{n}{r}$.

Revision

- A sequence a_n might be defined by a formula for the n^{th} term like $a_n = n^2 - n$.
- A sequence a_n might be defined with a "recurrence relation" like $a_{n+1} = f(a_n)$ for $n \geqslant 0$, if we're given the function $f(x)$ and also given a value for the first term a_0. (I'm counting from zero, so my "first term" is a_0, but it's also common to start with a_1, in which case the recurrence relation could be given as $a_{n+1} = f(a_n)$ for $n \geqslant 1$).
- The sum of the first n terms of a sequence a_k can be written with the notation $\displaystyle\sum_{k=0}^{n-1} a_k$ (if the first term is a_0) or $\displaystyle\sum_{k=1}^{n} a_k$ (if the first term is a_1).
- An arithmetic sequence (also known as an arithmetic progression) is a sequence where the difference between terms is constant. The terms can be written as a, $a + d$, $a + 2d$, $a + 3d$, $\ldots$, where a is the first term and d is the common difference.
- The sum of the first n terms of an arithmetic sequence with first term a and common difference d is $\dfrac{n}{2}(2a + (n-1)d)$, which you can remember as "first term plus last term, times the number of terms, divided by two" or $\dfrac{n}{2}(a + l)$, writing l for the last term.
- In particular, the natural numbers $1, 2, 3, \ldots, n$ are an arithmetic sequence, and the sum of the first n natural numbers is $\dfrac{n(n+1)}{2}$.
- A geometric sequence (also known as a geometric progression) is a sequence where the ratio between consecutive terms is constant. The terms can be written as a, ar, ar^2, ar^3, $\ldots$ where a is the first term and r is the common ratio.

- The sum of the first n terms of a geometric sequence with first term a and common ratio r is $\dfrac{a(1-r^n)}{1-r}$. One way to remember this is to remember what happens if we multiply the sum of the first n terms of a geometric series by $(1-r)$,

$$(1-r)(a+ar+\cdots+ar^{n-1})=(a-ar)+(ar-ar^2)+\cdots+(ar^{n-1}-ar^n)=a-ar^n.$$

- For a geometric sequence a_n, the sum to infinity is written as $\displaystyle\sum_{k=0}^{\infty}a_k$. If the common ratio r satisfies $|r|<1$ then this is equal to $\dfrac{a}{1-r}$. If $|r|\geqslant 1$ then this sum to infinity does not converge (it does not approach any particular real number).

- (Binomial Theorem) If n is a positive whole number then

$$(x+y)^n=\binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+\cdots+\binom{n}{r}x^ry^{n-r}+\cdots+\binom{n}{n-1}xy^{n-1}+\binom{n}{n}y^n$$

where $\dbinom{n}{r}=\dfrac{n!}{r!(n-r)!}$ and $n!$ means $n\times(n-1)\times(n-2)\times\cdots\times 2\times 1$ for a whole number n.

- In the Binomial Theorem, x and y could be real numbers or could even be functions, so you could use the Binomial Theorem on expressions like $(1+x)^n$ or even $(a+f(x))^n$.

- A periodic sequence has terms that repeat in a cycle; there is some L such that for all $n\geqslant 0$, we have $a_{n+L}=a_n$. The order of a periodic sequence is the minimum such L.

Revision Questions

1. A sequence is defined by $a_n = n^2 - n$. What is a_3? What is a_{10}? Find $a_{n+1} - a_n$ in terms of n. Find $a_{n+1} - 2a_n + a_{n-1}$ in terms of n.
2. A sequence is defined by $a_0 = 1$ and $a_n = a_{n-1} + 3$ for $n \geqslant 1$. Find $a_0 + a_1 + \cdots + a_{10}$. Find a_{1000}.
3. A sequence is defined by $a_0 = 1$ and $a_n = \frac{a_{n-1}}{3}$ for $n \geqslant 1$. Find $a_0 + a_1 + \cdots + a_{10}$. Find a_{1000}. Does the sum of all the terms of this sequence converge? If it does, what is the sum to infinity?
4. A sequence is defined by $a_0 = 1$ and $a_n = 3a_{n-1} + 1$ for $n \geqslant 1$. A sequence b_n is defined by $b_n = A \times 3^n + B$ where A and B are real numbers. Find values for A and B such that $a_n = b_n$ for all $n \geqslant 0$.
5. A sequence is defined by $a_n = An^2 + Bn + C$ where A, B, and C are real numbers. Find A, B, and C in terms of a_0, a_1, and a_2.
6. When does the sum $1 + x^3 + x^6 + x^9 + x^{12} + ...$ converge? Simplify it in the case that it converges.
7. When does the sum $2 - x + \dfrac{x^2}{2} - \dfrac{x^3}{4} + \ldots$ converge? Simplify it in the case that it converges.
8. If the first term of an arithmetic progression is 5 and the common difference is 3, what is the 15$^{\text{th}}$ term?
9. The sum of the first k terms of an arithmetic progression is equal to the sum of the next k terms. What can you deduce?
10. If the sum of the first n terms of an arithmetic progression is $3n^2 + 5n$, what is the n^{th} term?
11. What is the sum of the first 100 positive even integers (starting at 2)?
12. The first term of a geometric progression is 3 and the third term is 27. Find two possibilities for the sum of the first 5 terms.
13. A sequence is defined by $a_0 = 3$ and then for $n \geqslant 1$ a_n is the sum of all previous terms. Find a_n in terms of n for $n \geqslant 1$.
14. A sequence is defined by $C_0 = 1$ and then for $n \geqslant 0$,
$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}.$$
Find C_1 and C_2 and C_3 and C_4.
15. Expand $(2x + 3y)^3$
16. What is the coefficient of the x^2 term in the polynomial $w(x) = (3x - 1)^4$?
17. What is the sum of the coefficients of the polynomial $(x + 2)^3$? What about $(x + 2)^{300}$?

TMUA Questions

TMUA 2020 Paper 1 Question 4

The 1st, 2nd and 3rd terms of a geometric progression are also the 1st, 4th and 6th terms, respectively, of an arithmetic progression.
The sum to infinity of the geometric progression is 12.
Find the 1st term of the geometric progression.

A 1
B 2
C 3
D 4
E 5
F 6

[See the next page for hints]

TMUA 2020 Paper 2 Question 14

An arithmetic sequence T has first term a and common difference d, where a and d are non-zero integers.
Property P is:

> **For some** positive integer m, the sum of the first m terms of the sequence is equal to the sum of the first $2m$ terms of the sequence.

For example, when $a = 11$ and $d = -2$, the sequence T has property P, because

$$11 + 9 + 7 + 5 = 11 + 9 + 7 + 5 + 3 + 1 + (-1) + (-3)$$

i.e. the sum of the first 4 terms equals the sum of the first 8 terms.
Which of the following statements is/are **true**?

I For T to have property P, it is **sufficient** that $ad < 0$.
II For T to have property P, it is **necessary** that d is even.

A neither of them
B I only
C II only
D I and II

[See the next page for hints]

Hints

TMUA 2020 Paper 1 Question 4

- It's usually a good idea to give names to unknown quantities. For this question, you might write a and r and d for the unknown first term of the geometric progression, the common ratio of the geometric progression, and the common difference of the arithmetic progression.
- Take care with the arithmetic progression, where we're asked to skip terms. Is the sixth term going to be $a + 5d$ or $a + 6d$ or $a + 7d$?
- When I'm given multiple pieces of information that each lead to an equation, I like to re-write those equations all in one place, each on its own row, with the equals signs aligned. It doesn't affect the mathematics to line them up, of course, but I think it does help me to "see" what I might do next.
- When you're faced with some complicated simultaneous equations, focus on what you want to know, and try to eliminate the rest. In this question, we want to know the first term of the geometric progression.
- There are multiple routes through the algebra for this question.
- If your answer doesn't match any of the options, look back through your rough work instead of starting again.

TMUA 2020 Paper 2 Question 14

- Another way to say "For T to have property P, it is sufficient that $ad < 0$" that's more emotive is "no matter what the values of a and d are, if their product is negative, then I absolutely promise you that property P holds". Is that promise true?
- Another way to say "For T to have property P, it is necessary that d is even" that's more emotive is "I've checked every single case where property P holds, and in every case the number d is always even". Do you believe that case-check has been done correctly?
- Note that if both statements are true, then whenever $ad < 0$, property P must hold, and therefore d must be even. That's clearly nonsense. Negative numbers can be odd! So there's something else going on here.
- Now that we've thought about the statements, we should engage with the arithmetic sequence part of the question. What is the sum of the first m terms of the sequence?

TMUA 2020 Paper 2 Question 15
Which one of the following is a **necessary and sufficient** condition for

$$\sum_{k=1}^{n} \sin\left(\frac{k\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

to be true?

A $n = 1$
B n is a multiple of 3
C n is a multiple of 6
D n is 1 more than a multiple of 3
E n is 1 more than a multiple of 6
F n is 1 more than a multiple of 6 or n is 2 more than a multiple of 6

[See the next page for hints]

TMUA 2021 Paper 1 Question 13
The function f is such that, for every integer n

$$\int_{n}^{n+1} \mathrm{f}(x)\,\mathrm{d}x = n + 1$$

Evaluate

$$\sum_{r=1}^{8} \left(\int_{0}^{r} \mathrm{f}(x)\,\mathrm{d}x\right)$$

A 36
B 84
C 120
D 165
E 204
F 288

[See the next page for hints]

Hints

TMUA 2020 Paper 2 Question 15

- This question is in radians. You can tell because there's no degree sign $^\circ$.
- Most of the options refer to multiples of 3 or 6. Those values would make the fraction inside the brackets π or 2π, and I know that $\sin x$ is periodic with period 2π. So maybe this sequence is periodic too.
- We could just evaluate this for small values of n and see what happens. Because it's a sum over more and more terms, I think I would keep track of the values of $\sin\left(\frac{k\pi}{3}\right)$ and a running total, working from $k = 1$ up to $k = 6$.

TMUA 2021 Paper 1 Question 13

- Do not try to "solve" for the function f. Do not try to guess a function that works. That's not what's going on with this question.
- What's the relationship between $\displaystyle\int_n^{n+1} \mathrm{f}(x)\,\mathrm{d}x$ and $\displaystyle\int_0^r \mathrm{f}(x)\,\mathrm{d}x$?
- Check that relationship makes sense when $r = 0$.
- The number 8 is not that large. Just do it? (Sometimes I don't have a fancy summation formula... it's time to roll up my sleeves and actually add some numbers together!)
- Like the previous question, I would use a table with a running total here.

TMUA 2022 Paper 1 Question 8

A geometric sequence has first term a and common ratio r, where a and r are positive integers and r is greater than 1.

The sum of the first n terms of this sequence is denoted by S_n.

It is given that the terms of the sequence satisfy

$$S_{30} - S_{20} = kS_{10}$$

for some positive integer k.

What is the smallest possible value of k?

A 2^{10}

B 2^{20}

C 2^{30}

D $\dfrac{2^{10}}{2^{10} - 1}$

E $2^{10}(2^{10} - 1)$

[See the next page for hints]

TMUA 2022 Paper 2 Question 8

A selection, S, of n terms is taken from the arithmetic sequence $1, 4, 7, 10, \ldots, 70$.

Consider the following statement:

($*$) There are two distinct terms in S whose sum is 74.

What is the smallest value of n for which ($*$) is **necessarily** true?

A 12

B 13

C 14

D 21

E 22

F 23

[See the next page for hints]

Hints

TMUA 2022 Paper 1 Question 8

- Note that we're not asked to solve for a, the first term of the geometric sequence. Can you see why?
- Use your knowledge of geometric sequences to convert the equation $S_{30} - S_{20} = kS_{10}$ into an equation involving a and r.
- Look for cancellation, look for simplification. You could make a substitution in the equation, if you see the same expression turn up lots of times, if you like.
- My equation boils down to something very simple involving k and r. At that point I look back to the end of the question to check what I'm being asked. It says "smallest possible value of k", which has got nothing to do with my equation. Have I missed something? Then I look back up at the start of the question to see if there are any of what I call terms and conditions for the constants k and r. There are lots of terms and conditions at the top of the question! I'd forgotten about those while I was doing the algebra. Good thing I checked.

TMUA 2022 Paper 2 Question 8

- There is an important word in $(*)$ that you might miss the first time you read it.
- The selection S could be 4 and 70. Then there are two distinct terms in S whose sum is 74. That's just 2 terms! Why isn't the answer 2?
- Note that we're not told how this selection was done; when we consider whether $(*)$ is necessarily true, we're supposed to consider all possible selections.