Exponentials and logarithms

TMUA Specification (April 2025, Section 1 §MM5)

- $y = a^x$ and its graph, for simple positive values of a.
- Laws of logarithms:
  - $a^b = c \Leftrightarrow b = \log_a c$
  - $\log_a x + \log_a y = \log_a(xy)$
  - $\log_a x - \log_a y = \log_a \left(\dfrac{x}{y}\right)$
  - $k \log_a x = \log_a(x^k)$ including the special cases:
  - $\log_a \left(\dfrac{1}{x}\right) = -\log_a x$
  - $\log_a a = 1$.
- Questions requiring knowledge of the change of base formula will not be set.
- The solution of equations of the form $a^x = b$, and equations which can be reduced to this form, including those that need prior algebraic manipulation.

Revision

- The solution x to the equation $a^x = b$ where a and b are positive numbers (with $a \neq 1$) is called $\log_a(b)$. In this expression, the number a is called the base of the logarithm.
- $\log_a(x)$ is a function of x which is defined when $x > 0$. Like with $\sin x$, sometimes the brackets are omitted if it's clear what the function is being applied to, so we might write $\log_a x$.
- $\log_a x$ is not defined for $x = 0$. If x is positive and close to zero, then $|\log_a x|$ is large.
- $\log_a x$ is a one-to-one function on $x > 0$; if $\log_a x = \log_a y$ then $x = y$. If $a > 1$ then $\log_a x$ is an increasing function.
- $\log_a(a^k) = k$ for any value of k. To see why, remember that the value of $\log_a(a^k)$ is defined to be the solution x to the equation $a^x = a^k$. That solution is just k. This includes the special cases $\log_a 1 = 0$ and $\log_a a = 1$.
- With fixed $a \neq 1$, you can think of $\log_a x$ as the inverse function for $f(x) = a^x$.
- It's also true that $a^{\log_a x} = x$. To see why, let $y = \log_a x$. That would mean that $a^y = x$. Now replace the y in that equation with the expression $\log_a x$.
- (Product rule) If $a > 0$ and $a \neq 1$ and $x > 0$ and $y > 0$, then $\log_a(xy) = \log_a(x) + \log_a(y)$.
- (Power rule) If $a > 0$ and $a \neq 1$ and $x > 0$ and k is real, then $\log_a(x^k) = k \log_a x$.
- You are not required to know that $\log_a b = \dfrac{\log_c b}{\log_c a}$ for positive a, b, c with $a \neq 1$ and $c \neq 1$.

Revision Questions

1. Simplify $(2^3)^4$ and $(2^4)^3$ and $2^4 2^3$ and $2^3 2^4$.
2. Solve $x^{-2} + 4x^{-1} + 3 = 0$.
3. Simplify $\log_{10} 3 + \log_{10} 4$ into a single term.
4. Write $\log_3(x^2 + 3x + 2)$ as the sum of two terms, each involving a logarithm.
5. Solve $\log_x(x^2) = x^3$.
6. Solve $\log_x(2x) = 3$ for $x > 0$.
7. Solve $\log_{x+5}(6x + 22) = 2$.
8. Let $a = \log_3 2$ and $b = \log_3 5$. Write the following in terms of a and b.
$$\log_3 1024, \quad \log_3 40, \quad \log_3 \sqrt{\frac{2}{5}}, \quad \log_3 \frac{1}{10}, \quad \log_3 1.024.$$
9. Expand $(2^x + 2^{-x})(2^y - 2^{-y}) + (2^x - 2^{-x})(2^y + 2^{-y})$.
   Expand $(2^x + 2^{-x})(2^y + 2^{-y}) + (2^x - 2^{-x})(2^y - 2^{-y})$.
10. Use logarithms to solve $2^x = 3$, then $0.5^x = 3$, then $4^x = 3$.
11. Suppose that a and b are positive numbers and $a \neq 1$. Using the definition of $\log_a b$ as the number x that satisfies $a^x = b$, explain why $\log_a b = 0$ if and only if $b = 1$.
12. Given $\log_{10}(\log_{10} x) = 6$, how many zeros are there at the end of the number x?
13. Use the product rule and the power rule to prove the "quotient rule", which is; if $a > 0$ and $a \neq 1$ and $x > 0$ and $y > 0$, then $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$.
14. Solve $2^x + 2^{-x} = 4$.
    How many solutions are there to $2^x + 2^{-x} = c$? Identify different cases in terms of c.
15. Prove that $\log_a(N + \sqrt{N^2 - 1}) = -\log_a(N - \sqrt{N^2 - 1})$ for any number $N \geqslant 1$ and any base $a > 1$.
16. Consider the equation $x^y = y^x$ with $x, y > 0$. Use logarithms to turn this into an equation of the form $f(x) = f(y)$. [Harder] Sketch $f(x)$.
17. Simplify $a^{k \log_a b}$ for positive numbers a, b, k with $a \neq 1$ and $b \neq 1$.
    Consider the number $x = (\log_a b)(\log_b c)$. By simplifying a^x, show that $x = \log_a c$.
    Deduce that $\log_b c = \dfrac{\log_a c}{\log_a b}$ for positive numbers a, b, c with $a \neq 1$ and $b \neq 1$.
18. Let $a = \log_3 2$ and $b = \log_3 5$. Use the result in the previous question to write $\log_6(45)$ in terms of a and b.
19. The revision notes say "if $a > 1$ then $\log_a x$ is an increasing function". What can you say if $a < 1$?

TMUA Questions

TMUA 2020 Paper 1 Question 7

Given that

$$2^{3x} = 8^{(y+3)}$$

and

$$4^{(x+1)} = \frac{16^{(y+1)}}{8^{(y+3)}}$$

what is the value of $x + y$?

A -23 **B** -22 **C** -15 **D** -14 **E** -11 **F** -10

[See the next page for hints]

TMUA 2021 Paper 1 Question 10

Use the trapezium rule with 3 strips to estimate

$$\int_{\frac{1}{2}}^{2} 2\log_{10} x \, \mathrm{d}x$$

A $\log_{10} \frac{\sqrt{6}}{2}$ **B** $\log_{10} \frac{3}{2}$ **C** $\log_{10} \frac{9}{4}$ **D** $\log_{10} 3$ **E** $\log_{10} \frac{81}{16}$ **F** $\log_{10} \frac{\sqrt{23}}{2}$

[See the next page for hints]

Hints

TMUA 2020 Paper 1 Question 7

- The question just asks for $x + y$, but we should probably try to solve for both x and y.
- The variable x only appears in exponents. The variable y only appears in exponents.
- How are the numbers 2, 4, 8, and 16 related?
- Try not to evaluate 8^3 as 512, unless you have to.

TMUA 2021 Paper 1 Question 10

- If you haven't met the trapezium rule yet, come back to this question after you have.
- Don't forget the factor of $\frac{1}{2}$, or the factor of 2, or the other factor of $\frac{1}{2}$.
- During your work, you can choose whether you prefer to write things like $4\log_{10} 3$ or $\log_{10} 81$. I personally prefer the first one.

TMUA 2022 Paper 1 Question 11

Evaluate

$$\sum_{n=1}^{100} \log_{10}\left(3^{1-n}\right)$$

- **A** $-4950 \log_{10} 3$
- **B** $4950 \log_{10} 3$
- **C** $-5050 \log_{10} 3$
- **D** $5050 \log_{10} 3$
- **E** $1 - 4950 \log_{10} 3$
- **F** $1 + 4950 \log_{10} 3$
- **G** $1 - 5050 \log_{10} 3$
- **H** $1 + 5050 \log_{10} 3$

[See the next page for hints]

TMUA 2021 Paper 2 Question 14

Consider the following simultaneous equations, where p is a real number:

$$p2^x + \log_2 y = 2$$
$$2^x + \log_2 y = 1$$

What is the complete range of p for which these simultaneous equations have a real solution (x, y)?

- **A** $p < 1$
- **B** $p \neq 1$
- **C** $p > 1$
- **D** $p < 1$ or $p > 2$
- **E** $p \neq 1$ and $p < 2$
- **F** $p > 1$ and $p < 2$
- **G** $p > 2$
- **H** All real values of p

[See the next page for hints]

Hints

TMUA 2022 Paper 1 Question 11

- Write out the first couple of terms of the sum, and the last term.
- Careful with the last term!
- Use laws of logarithms on each term.
- Remember that you can recognise certain sorts of sequences (in particular, arithmetic progressions and geometric progressions).

TMUA 2021 Paper 2 Question 14

- Try to solve these equations for x and y, and see what happens.
- The variable x only appears in terms like 2^x, and the variable y only appears in terms like $\log_2 y$. If you write $a = 2^x$ and $b = \log_2 y$, can you solve the equations for a and b?
- By making that substitution, we've obscured the fact that one of the variables appears in an exponential and one appears in a logarithm. Don't forget that! Given that we actually want to solve for x and y, not a and b, what might go wrong?

TMUA 2021 Paper 2 Question 17

Consider the following functions defined for $x > 1$:

$$\mathrm{f}(x) = \log_2\left(\log_2 \sqrt{x}\right)$$

$$\mathrm{g}(x) = \log_2\left(\sqrt{\log_2 x}\right)$$

Which one of the following is true for **all** values of $x > 1$?

A $0 \le \mathrm{f}(x) \le \mathrm{g}(x)$ **or** $\mathrm{g}(x) \le \mathrm{f}(x) \le 0$

B $0 \le \mathrm{g}(x) \le \mathrm{f}(x)$ **or** $\mathrm{f}(x) \le \mathrm{g}(x) \le 0$

C $\frac{1}{2} \le \mathrm{f}(x) \le \mathrm{g}(x)$ **or** $\mathrm{g}(x) \le \mathrm{f}(x) \le \frac{1}{2}$

D $\frac{1}{2} \le \mathrm{g}(x) \le \mathrm{f}(x)$ **or** $\mathrm{f}(x) \le \mathrm{g}(x) \le \frac{1}{2}$

E $1 \le \mathrm{f}(x) \le \mathrm{g}(x)$ **or** $\mathrm{g}(x) \le \mathrm{f}(x) \le 1$

F $1 \le \mathrm{g}(x) \le \mathrm{f}(x)$ **or** $\mathrm{f}(x) \le \mathrm{g}(x) \le 1$

[See the next page for hints]

TMUA 2022 Paper 2 Question 15

The real numbers x, y and z are all greater than 1, and satisfy the equations

$$\log_x y = z \quad \text{and} \quad \log_y z = x$$

Which one of the following equations for $\log_z x$ **must** be true?

A $\log_z x = y$

B $\log_z x = \dfrac{1}{y}$

C $\log_z x = xy$

D $\log_z x = \dfrac{1}{xy}$

E $\log_z x = xz$

F $\log_z x = \dfrac{1}{xz}$

G $\log_z x = yz$

H $\log_z x = \dfrac{1}{yz}$

[See the next page for hints]

Hints

TMUA 2021 Paper 2 Question 17

- Don't try to understand the options! Compare $f(x)$ and $g(x)$. For which values of x is $f(x) \leqslant g(x)$?
- You know that $\log_2 x$ is an increasing function of x, so you can just compare the arguments of that function if you want to know which will give the larger output.
- If it helps, write u for $\log_2 x$ to make things look simpler.
- The question wants bounds on $f(x)$ and $g(x)$ in various cases. If you work out values of x for which $f(x) \leqslant g(x)$, then that will give you a starting point to use when you try to find inequalities for the values of $f(x)$ and $g(x)$.

TMUA 2022 Paper 2 Question 15

- Write out what the given equations mean in terms of exponentials.
- Don't think too much about what it means for x to appear both as a base in the first equation, and then also as a value in the second statement. Aim for equations that you can work with!
- If you can eliminate y from your equations then you've shown that there is some relationship between x and z. That might let you rule out six of the eight options.
- You're aiming for a fact about $\log_z x$, so you're looking for an expression with a power of z that involves x.