Differentiation

TMUA Specification (April 2025, Section 1 §MM6)

- The derivative of $f(x)$ as the gradient of the tangent to the graph $y = f(x)$ at a point.
  - Interpretation of a derivative as a rate of change.
  - Second-order derivatives.
  - Knowledge of notation: $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, $f'(x)$, and $f''(x)$.
- Differentiation from first principles is excluded.
- Differentiation of x^n for rational n, and related sums and differences. This might require some simplification before differentiating.
- Applications of differentiation to gradients, tangents, normals, stationary points (maxima and minima only), strictly increasing functions [if $f'(x) > 0$] and strictly decreasing functions [if $f'(x) < 0$]. Points of inflexion will not be examined, although a qualitative understanding of points of inflexion in the curves of simple polynomial functions is expected.

Revision

- The tangent to a graph at a particular point is a line which has the same value and gradient as the graph at that point. The gradient of the graph means something like the instantaneous rate of change of the graph at that specific point (differentiation from first principles is excluded, so you're not expected to know how it's possible to work out the gradient at a single point on a graph!)
- The derivative of a function is a new function that tells you the gradient. If your function is $f(x)$ then it might be written as $f'(x)$. It's also common to see a graph or curve in the (x, y)-plane, and in that case the derivative might be written as $\frac{\mathrm{d}y}{\mathrm{d}x}$.
- For powers of x, we have some rules that let us work out the derivative.
- The derivative of x^a is ax^{a-1}, including for fractional exponents like $a = \frac{1}{2}$.
- If c is a constant then the derivative of $c \times f(x)$ is $c \times f'(x)$.
- The derivative of $y_1 + y_2$ is (the derivative of y_1)+(the derivative of y_2). Perhaps this looks too obvious to need stating, but remember that, for example, the square of $y_1 + y_2$ is not equal to (the square of y_1)+(the square of y_2).
- Example: if we want the tangent to the graph $y = x^3 + 2x$ at $x = 1$, then we need the value of y (which is 3), and the value of the derivative, which is the value of $3x^2 + 2$ when $x = 1$, which is 5. To find that derivative, I've used the rule for adding together powers of x, the rule for a function that's multiplied by 2, and I've remembered that $x = x^1$. The derivative of a line is its gradient, so we can write $y = 5x + c$ and solve for c using the value at $x = 1$ to get $y = 5x - 2$.

- The normal to a graph at a point is a line through that point which is perpendicular to the tangent. Two lines are perpendicular if their gradients multiply to -1.
- If the derivative is positive, that means that the function is increasing. If it's negative, that's a decreasing function. In general a function might increase in some regions and decrease in other regions.
- If the derivative changes sign at a point, that's a turning point. You'll have zero derivative at the turning point, but that's not actually sufficient for the derivative to *change* sign (e.g. x^3 has zero derivative at $x = 0$, but that's not a turning point because the derivative is positive on both sides). A point with zero derivative is called a stationary point.
- The derivative of the derivative is called the second derivative. You can work out the derivatives one at a time. So the second derivative of x^a would be the derivative of ax^{a-1}, which is $a(a-1)x^{a-2}$. The second derivative is the rate of change of the derivative. For the second derivative of a function $f(x)$, we might use the notation $f''(x)$. In cases where we wrote the first derivative as $\frac{\mathrm{d}y}{\mathrm{d}x}$, then we might use the notation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ for the second derivative.
- A turning point is a local maximum if the second derivative is negative at that point. It's a local minimum if the second derivative is positive. "Maxima" is the plural of "maximum". "Minima" is the plural of "minimum". Overall, the function might have several local maxima, or none, and it might increase without bound (like $y = x$ for example).
- If you're looking for the overall maximum value of a function, then you might need to look beyond the turning points. For example; if $\mathrm{f}(x)$ is defined as $2x^4 - x^2$ when $-1 \leqslant x \leqslant 1$, then the maximum value comes at the endpoints $x = \pm 1$, not at the local maximum at $x = 0$.
- You're not required to know about points of inflexion, but you should be aware that the second derivative of the curve can change sign; for example, $y = x^3 - x$ has such a point when $x = 0$ because the second derivative is $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x$. You should also be aware that the second derivative might be zero at a stationary point. In such a case you might need to sketch the graph to work out if it's a turning point (and if so, whether it's a maximum or a minimum).
- Remember that you know how to find the turning point of a quadratic without differentiating. You could complete the square instead.
- Remember that in the context of circle geometry, you know facts about tangents that do not require you to differentiate anything.

Revision Questions

1. Differentiate $x^{17} - x^{-17}$ with respect to x.
2. Differentiate $2\sqrt{x} + 3\sqrt[3]{x}$ with respect to x.
3. Differentiate $\dfrac{x+3}{x^2}$ with respect to x.
4. Differentiate $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!}$. As a reminder, $n!$ means $n \times (n-1) \times \cdots \times 2 \times 1$.
5. Find the tangent to the curve $y = x^3 - x$ at $x = 1$.
6. Find the tangent to the curve $x^2 + y^2 + 2x - 6y - 15 = 0$ at the point $(3, 0)$.
7. Find the normal to the parabola $y = x^2$ at $x = 3$.
8. Find the normal to the curve $y = \sqrt{1 - x^2}$ at $x = p$ in terms of p, given $0 < p < 1$.
9. Find the minimum value of the function $\mathrm{f}(x) = x^4 - 4x^2 + 3$.
10. Find the turning points of the curve $y = x^4 - 2x^3 + x^2$. Identify whether the turning points are maxima or minima. For which values of x is $\mathrm{f}(x) = x^4 - 2x^3 + x^2$ an increasing function? For which values of x is it decreasing? Sketch the curve.
11. Consider the function $\mathrm{f}(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$. Check that there is a stationary point at $x = 1$, and decide whether this is a local minimum or a local maximum.

TMUA Questions

TMUA 2021 Paper 1 Question 11

The function f is given by

$$\mathrm{f}(x) = x^{\frac{1}{7}}(x^2 - x + 1)$$

Find the fraction of the interval $0 < x < 1$ for which $\mathrm{f}(x)$ is decreasing.

A $\frac{2}{15}$ **B** $\frac{1}{5}$ **C** $\frac{1}{3}$ **D** $\frac{1}{2}$ **E** $\frac{2}{3}$ **F** $\frac{4}{5}$ **G** $\frac{13}{15}$

[See the next page for hints]

TMUA 2021 Paper 1 Question 12

The minimum value of the function $x^4 - p^2x^2$ is -9
p is a real number.
Find the minimum value of the function $x^2 - px + 6$

A -3 **B** $6 - \frac{3\sqrt{2}}{2}$ **C** $\frac{3}{2}$ **D** 3 **E** $\frac{9}{2}$ **F** $6 + \frac{3\sqrt{2}}{2}$

[See the next page for hints]

Hints

TMUA 2021 Paper 1 Question 11

- The question asks for the fraction of the interval where $\mathrm{f}(x)$ is decreasing. We should try to find all values of x where $\mathrm{f}(x)$ is decreasing, and then work out the fraction.
- We could multiply out $\mathrm{f}(x)$. It's not going to be fun, but it will give us something we can work with.
- Don't bother sketching this function.
- "Decreasing" means something about the derivative $\mathrm{f}'(x)$.
- Once you've worked out $\mathrm{f}'(x)$, it will be messy. Try to simplify or factorise a bit, before you move to inequalities or turning points.
- Taking out a factor of $x^{1/7}$ from $\mathrm{f}'(x)$ (the opposite of the advice in the first bullet point!) might help you work with the derivative.

TMUA 2021 Paper 1 Question 12

- You could differentiate both of these functions.
- Find expressions for both minimum values in terms of p.
- It's time for an equation; the given information tells you something about p, and you want to know the value of the other minimum. Hopefully this is a matter of solving one equation and substituting the value into your other expression.
- Don't panic if your equation for p does not have a unique solution. Check both, or convince yourself that it doesn't matter which solution you choose.

TMUA 2021 Paper 2 Question 6

Consider the following two statements about the polynomial $\mathrm{f}(x)$:

P: $\mathrm{f}(x) = 0$ for exactly three real values of x

Q: $\mathrm{f}'(x) = 0$ for exactly two real values of x

Which one of the following is correct?

A P is **necessary** but **not sufficient** for Q.

B P is **sufficient** but **not necessary** for Q.

C P is **necessary and sufficient** for Q.

D P is **not necessary** and **not sufficient** for Q.

[See the next page for hints]

TMUA 2021 Paper 2 Question 8

Consider the following statement about the polynomial $\mathrm{p}(x)$, where a and b are real numbers with $a < b$:

$(*)$ There exists a number c with $a < c < b$ such that $\mathrm{p}'(c) = 0$.

Which one of the following is true?

A The condition $\mathrm{p}(a) = \mathrm{p}(b)$ is **necessary and sufficient** for $(*)$

B The condition $\mathrm{p}(a) = \mathrm{p}(b)$ is **necessary** but **not sufficient** for $(*)$

C The condition $\mathrm{p}(a) = \mathrm{p}(b)$ is **sufficient** but **not necessary** for $(*)$

D The condition $\mathrm{p}(a) = \mathrm{p}(b)$ is **not necessary** and **not sufficient** for $(*)$

[See the next page for hints]

Hints

TMUA 2021 Paper 2 Question 6

- Sketch a few polynomials with $f(x) = 0$ for exactly three real values of x. Do all of your sketches have exactly two points where $f'(x) = 0$? If yes, then perhaps P is sufficient for Q. Keep sketching!
- Sketch a few polynomials with exactly two points where $f'(x) = 0$. Do all of your sketches have exactly three real values of x for which $f(x) = 0$? If yes, then perhaps P is necessary for Q. Keep sketching!
- Returning to the first set of sketches, note that the points where $f'(x) = 0$ do not have to be "between" the roots, but there must be exactly two of them in total.
- This question might sound like it's about cubic polynomials. It is not about cubic polynomials.

TMUA 2021 Paper 2 Question 8

- This question looks quite similar to the previous one!
- Remember that "P is sufficient for Q" means "if P then Q".
- Remember that "P is necessary for Q" means "if Q then P".
- Sketch a few polynomials with $p(a) = p(b)$. Do all of your sketches have a point in between with zero derivative?
- Sketch a few polynomials with a point where the derivative is zero. Do all of your sketches have $p(a) = p(b)$? Really? I haven't even told you what a and b are yet!

TMUA 2022 Paper 1 Question 15

A rectangle is drawn in the region enclosed by the curves p and q, where

$$p(x) = 8 - 2x^2, \qquad q(x) = x^2 - 2$$

such that the sides of the rectangle are parallel to the x- and y-axes.
What is the maximum possible area of the rectangle?

A $\frac{26}{9}$ **B** $\frac{52}{9}$ **C** $\frac{4\sqrt{6}}{3}$ **D** $\frac{8\sqrt{6}}{3}$ **E** $4\sqrt{2}$ **F** $8\sqrt{2}$ **G** $\frac{20\sqrt{10}}{9}$ **H** $\frac{40\sqrt{10}}{9}$

[See the next page for hints]

TMUA 2022 Paper 2 Question 17

A student answered the following question:
"a and b are non-zero real numbers. Prove that the equation $x^3 + ax^2 + b = 0$ has three distinct real roots if $27b\left(b + \frac{4a^3}{27}\right) < 0$."
Here is the student's solution:

I We differentiate $y = x^3 + ax^2 + b$ to get $\frac{dy}{dx} = 3x^2 + 2ax = x(3x + 2a)$.
Solving $\frac{dy}{dx} = 0$ shows that the stationary points are at $(0, b)$ and $\left(-\frac{2a}{3}, b + \frac{4a^3}{27}\right)$

II If $27b\left(b + \frac{4a^3}{27}\right) < 0$, then b and $b + \frac{4a^3}{27}$ must have opposite signs, and so one of the stationary points is above the x-axis and one is below.

III If the cubic has three distinct real roots, then one of the stationary points is above the x-axis and one is below.

IV Hence if $27b\left(b + \frac{4a^3}{27}\right) < 0$, then the equation has three distinct real roots.

Which one of the following options best describes the student's solution?

A It is a completely correct solution.
B The student has instead proved the converse of the statement in the question.
C The solution is wrong, because the student should have stated step II after step III.
D The solution is wrong, because the student should have shown the converse of the result in step II.
E The solution is wrong, because the student should have shown the converse of the result in step III.

[See the next page for hints]

Hints

TMUA 2022 Paper 1 Question 15

- Draw a diagram!
- Work out what the question means by "the region enclosed by the curves p and q". Does your diagram have a single finite region bounded by the curves?
- Sketch a little rectangle in the region with the sides parallel to the axes. You can make this rectangle larger by making it wider or taller. Whichever you choose, how far can you push this?
- Time for algebra; write down the height of the rectangle in terms of its width. Write down the area.

TMUA 2022 Paper 2 Question 17

- Read the student's solution line by line, but regularly remind yourself of what they're supposed to be proving.
- It's unlikely that the student has made any small algebraic mistakes (e.g. finding the stationary points incorrectly) because that wouldn't be a very interesting question! Indeed, none of the answers refer to the differentiation in step I.
- For this question, you need to know that the converse of "if [this] then [that]" is "if [that] then [this]". For example, the converse of "if it rains then I wear a coat" is "if I wear a coat then it rains". The converse of a true statement might be false (often, but not always).
- The statement that they're trying to prove is phrased as "[this] if [that]", which is a little bit awkward to read. We could re-write it as "Prove that if $27b\left(b + \frac{4a^3}{27}\right) < 0$ then the equation $x^3 + ax^2 + b = 0$ has three distinct real roots". That looks like their last line, which is a good sign for them I guess!
- I'm always looking out for lines where the student has previously shown [this], and now the student is proving "if [that] then [this]". That sets off an alarm bell for me, because the student is probably going to conclude "therefore [that]", which doesn't follow logically from the work they've done.
- Example; imagine that a student writes "2 is even. All multiples of 4 are even. Therefore 2 is a multiple of 4." The sentence in the middle isn't false, but it's not going in the right direction to match up nicely with the previous statement.