Sequences and series – Solutions

Revision Questions

1. $a_3 = 3^2 - 3 = 6$. $a_{10} = 90$.

   $a_{n+1} - a_n = ((n+1)^2 - (n+1)) - (n^2 - n) = 2n$.

   $a_{n+1} - 2a_n + a_{n-1} = (a_{n+1} - a_n) - (a_n - a_{n-1}) = 2n - 2(n-1)$ using the previous part, and this is 2.

2. This is the sum of the first 11 terms of an arithmetic sequence with first term $a = 1$ and common difference 3. So the sum is $\frac{11}{2}(2 + 10 \times 3) = 176$. Also, $a_{1000} = 3001$.

3. This is the sum of the first 11 terms of a geometric sequence with first term $a = 1$ and common ratio $\frac{1}{3}$. So the sum is $\frac{(1-3^{-11})}{1-3^{-1}} = \frac{3}{2}(1 - 3^{-11})$. Also, $a_{1000} = 3^{-1000}$.

   The common ratio is between -1 and 1 so the sum to infinity does converge. The sum to infinity is $\frac{1}{1-3^{-1}} = \frac{3}{2}$.

4. In particular, we would need $a_0 = b_0$ so $1 = A + B$. Also, we would need $a_1 = b_1$ so $4 = 3A + B$. So $A = \frac{3}{2}$ and $B = -\frac{1}{2}$.

   We should check that this works for all n, so let's check that if $a_{n-1} = b_{n-1}$ then $a_n = b_n$. We know that $a_n = 3a_{n-1} + 1$ and we can work out $3b_{n-1} + 1 = 3 \times \frac{3}{2}3^{n-1} - \frac{3}{2} + 1 = \frac{3}{2}3^n - \frac{1}{2}$ which is exactly b_n. So if $a_{n-1} = b_{n-1}$ then $a_n = b_n$. The sequences match for all n.

5. At first sight, this doesn't look like enough information; we haven't been told the values of any of the terms in the sequence! The key is that we're asked to give our answer in terms of the first three terms of the sequence without solving for what those are.

   For example, if we substitute $n = 0$ then we find $a_0 = C$. So we've got an expression for C in terms of a_0.

   Now substitute $n = 1$ and $n = 2$ to get $a_1 = A + B + C$ and $a_2 = 4A + 2B + C$. We want A and B in terms of the variables a_0, a_1, a_2, and we can use the fact that $C = a_0$ to eliminate C. These are simultaneous equations for A and B with solution

   $$A = \frac{1}{2}(a_2 - 2a_1 + a_0), \qquad B = \frac{1}{2}(-a_2 + 4a_1 - 3a_0)$$

6. This is a geometric sequence with common ratio x^3, and the sum to infinity converges if $|x^3| < 1$, which is precisely $|x| < 1$.

   In that case, it converges to $(1 - x^3)^{-1}$.

7. This is a geometric sequence with common ratio $-\frac{x}{2}$, and the sum to infinity converges if $\left|-\frac{x}{2}\right| < 1$, which is precisely $|x| < 2$.

   In that case, it converges to $\dfrac{2}{1 + \frac{x}{2}} = \dfrac{4}{2 + x}$.

8. The 15^{th} term will be equal to the first plus 14 times the common difference, so $5 + 14 \times 3 = 47$.

9. This is strange because the next k terms should each be kd more than the corresponding term k places before it in the sequence. For the sum to be the same, we would need $d = 0$ (a constant sequence) or $k = 0$ (no numbers in the statement!).

10. The n^{th} term is the same thing as the sum of the first n terms minus the sum of the first $(n-1)$ terms, so we want $(3n^2 + 5n) - (3(n-1)^2 + 5(n-1)) = 6n + 2$.

    Alternatively, just set $n = 1$ and $n = 2$ to find the first term is 8 and the sum of the first two terms is 22 so the second term is 14 and the common difference is therefore 6.

11. $2 + 4 + 6 + 8 + \cdots + 200$ is the sum of an arithmetic progression with first term $a = 2$, common difference $d = 2$ and number of terms $n = 100$, so the sum is

$$\frac{n}{2}(2a + (n-1)d) = 50 \times 202 = 10100.$$

12. What's the common ratio? We have $a = 3$ and $ar^2 = 27$ so either $r = 3$ or $r = -3$. The sum of the first five terms is

$$\frac{a(1-r^5)}{1-r} = \frac{3(1-(\pm 3)^5)}{1-(\pm 3)} = \quad \text{either 363 or 183.}$$

13. First note that $a_1 = 3$ (the only previous term is 3) and then $a_2 = 3 + 3 = 6$. After that, $a_3 = 6 + 3 + 3 = 12$ and it seems like the terms double each time.

    That's true because $a_n = a_{n-1} + a_{n-2} + \cdots + a_0$ and $a_{n-1} = a_{n-2} + \cdots + a_0$ so $a_n = a_{n-1} + a_{n-1} = 2a_{n-1}$.

    So after a_0 this is a geometric sequence with $a_n = 3 \times 2^{n-1}$ for $n \geqslant 1$.

14. For C_1 I'll put $n = 0$ into the equation, so that the left-hand side reads C_1. The right-hand side is then

$$\sum_{i=0}^{0} C_i C_{-i}.$$

    That looks pretty weird written out like that, but the sum from $i = 0$ to 0 is just the term with $i = 0$, which is C_0^2. So $C_1 = C_0^2 = 1$.

    Now for C_2, I'll put $n = 1$ into the equation. The right-hand side is more interesting now; I get $\sum_{i=0}^{1} C_i C_{1-i}$ which is $C_0C_1 + C_1C_0$. So $C_2 = C_0C_1 + C_1C_0 = 2$.

    Then $C_3 = C_2C_0 + C_1C_1 + C_0C_2 = 5$ and $C_4 = C_3C_0 + C_2C_1 + C_1C_2 + C_0C_3 = 14$.

15. We can use the binomial theorem here to get $(2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3$ which is $8x^3 + 36x^2y + 54xy^2 + 27y^3$.

16. The x^2 term is $\binom{4}{2}(3x)^2(-1)^2$, so the coefficient is 54.

17. $(x+2)^3 = x^3 + 6x^2 + 12x + 8$, and so the sum of the coefficients is $1 + 6 + 12 + 8 = 27$. Hopefully you spotted that this is 3^3, and that the sum of the coefficients of any polynomial is just the value of that polynomial at $x = 1$. Then the sum of the coefficients of $(x+2)^{300}$ is 3^{300}.

TMUA Questions

TMUA 2020 Paper 1 Question 4

- I decided to call the first term of the geometric progression a.
- That's also the first term of the arithmetic progression, and it's the number that I'm trying to find.
- I usually call the common ratio of a geometric progression r, and I usually call the common difference of an arithmetic progression d.
- With that notation, the 1st, 2nd, 3rd terms of the geometric progression are a, ar, ar^2.
- The 1st, 4th and 6th terms of the arithmetic progression are a, $a+3d$, and $a+5d$. That one almost caught me out! I could check that I'm not out by one by writing out the first six terms explicitly as a, $a+d$, $a+2d$, $a+3d$, $a+4d$, $a+5d$, but in this case I think I'm confident enough that I don't need to do that.
- So we have (writing the equations with the equals signs aligned)
$$\begin{aligned} ar &= a + 3d \\ ar^2 &= a + 5d. \end{aligned}$$
- What else? I notice that I haven't done anything with the sum to infinity yet. That's $\dfrac{a}{1-r}$ and we're told that this is 12.
- I've written "Want a. Eliminate d" on my rough work. I can see from the first two equations above that it will be easy to eliminate d (because d appears in one place in each equation, only multiplied by constants not other unknown numbers).
- Multiply the first equation by 5, multiply the second by 3, and take the difference to get $5ar - 3ar^2 = 2a$.
- Both sides involve a, so unless that's zero I can divide both sides by a (I'm slightly upset to do this, because I'm looking for a, but I really like simplifying equations).
- I get the quadratic $3r^2 - 5r + 2 = 0$, which I can solve for $r = 1$ or $r = 2/3$.
- The first is deeply suspicious to me, because it's supposed to be the ratio of a geometric progression that we can sum to infinity. I think that calling it r, the letter I always use, is helpful here, because it helps me recognise associated facts like "we're supposed to have $|r| < 1$".
- So $r = 2/3$. Then I can go back to one of the previous equations to find a, preferably an equation that only involves a and r. Ah, that's $\dfrac{a}{1-r} = 12$. So $a = 4$.
- The answer is D.

Extension

- The question said "the 1st, 4th and 6th terms" and that seems a bit unusual.

  Why doesn't the question say "the 1st, 2nd and 3rd terms" for both progressions?

TMUA 2020 Paper 2 Question 14

- I wanted to start by understanding the property P that an arithmetic sequence might or might not have. It sounds a bit unlikely to me (adding more terms should give a bigger answer, am I right?).
- The sum of the first m terms of an arithmetic sequence is $m \times \dfrac{(2a + (m-1)d)}{2}$ and the sum of the first $2m$ terms is $2m \times \dfrac{(2a + (2m-1)d)}{2}$.
- Investigating, we can have
$$m \times \frac{(2a + (m-1)d)}{2} = 2m \times \frac{(2a + (2m-1)d)}{2}$$
if
$$2a + (3m-1)d = 0.$$
- Alternatively, we could recognise that the $(m+1)^{\text{th}}$ to the $(2m)^{\text{th}}$ terms form an arithmetic progression, with sum $m \times \dfrac{a + md + a + (2m-1)d}{2}$ and we would like this to be zero.
- Either way, the property P is "for some positive integer m, we have $2a + (3m-1)d = 0$".
- To check against the example in the question, set $a = 11$ and $d = -2$, and then there is such an integer ($m = 4$) because $22 + 11 \times (-2) = 0$. I was not right at the top of this question; adding more terms might not give a bigger answer, if those terms are negative!
- Time to look at the statements. The second one looks more approachable to me, because I can see factors of 2 in this equation; I can see that $2a$ is even. Does d have to be even? Maybe not? Maybe $3m-1$ could be even. That's not a proof yet, but it's given me a direction to explore.
- To make up an interesting example, I set $m = 3$ because that gives even $3m-1$. Working backwards, I set $d = -1$, an odd number, because I am trying to prove a point. Then to satisfy "$2a + (3m-1)d = 0$", I have $a = 4$ (I made d negative so that a would work out to be positive). That gives me an example of an arithmetic sequence. Statement II is not true.
- Time to look at the first statement. Is it enough just to take $ad < 0$? Looking at my equation, it looks a bit unlikely that if I picked random numbers for a and d that $(3m-1)$ would be a whole number, let alone m. I checked that with $a = 1$ and $d = -10$ there's no integer m that works. Statement I is also not true. These statements are both rubbish!
- The answer is A.

Extension

- If I were using this as the start of an interview question, then next I might explore that idea about choosing a sequence "at random". Perhaps we start with the case where $d = -5$ and a is an integer chosen uniformly at random from 1 to 1000. What's the probability that such a sequence has property P?

TMUA 2020 Paper 2 Question 15

- The question uses the language "necessary and sufficient", but that's just encouraging us to solve this *properly* (for example, we can't shortcut this by finding a single value of n that works, or by saying "I reckon some positive numbers work").
- I made a table of values for $\sin\left(\frac{k\pi}{3}\right)$. At this point I'm just looking for patterns, I'm not committed to actually working out the sum of the first n terms.

	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π	repeat					
sin	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	0	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	0

- I've written "repeat" because I know that sin is periodic with period 2π, so the next batch of values will be the same. This is perhaps a bit sloppy, but I'm not going to pretend that I wrote out $\frac{7\pi}{3}$ and so on!
- Now that I've seen the values, I'm willing to calculate the sum for various values of n (this is a sort of running total; I'll add the values from left to right, one at a time, keeping track of the total "so far". That's always how I think of sums where the lower limit is fixed and the upper limit is a variable like n).

	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π	repeat					
sin	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	0	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	0
Σ	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\sqrt{3}$	$\frac{\sqrt{3}}{2}$	0	0	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	...			
	✓			✓			✓					

- The table shows the point at which I got bored! We want numbers like 1 and 4 and 7, the ones that are 1 more than a multiple of 3.
- The answer is D.

Extension

- There's a factor of $\frac{\pi}{3}$ in the question, and we saw that the sum could be zero. What happens if we change that factor; can this sum ever be zero? If we set the factor to $\frac{\pi}{2}$ or $\frac{\pi}{5}$ then we get a similar-looking question.
- (Hard) If we change the factor to 1 then we have the sum of $\sin(k)$ for integers k. Can that ever be zero? [Hint: multiply the sum by $\sin\left(\frac{1}{2}\right)$ and use double-angle identities.]

TMUA 2021 Paper 1 Question 13

- This is the sum of eight integrals, and I don't know the function that I'm integrating!
- To make progress, I need to find a link between the integral I'm given $\int_n^{n+1} \mathrm{f}(x)\,\mathrm{d}x$ and the integral I want $\int_0^r \mathrm{f}(x)\,\mathrm{d}x$.
- That's just a sum, isn't it? To find $\int_0^3 \mathrm{f}(x)\,\mathrm{d}x$, I would add the values of $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x$ and $\int_1^2 \mathrm{f}(x)\,\mathrm{d}x$ and $\int_2^3 \mathrm{f}(x)\,\mathrm{d}x$. So this question wants a sum of a sum. Let's do this one sum at a time.
- At this point I made a table to work out $\int_0^k \mathrm{f}(x)\,\mathrm{d}x$ for various values of k.
- It's important to start strong, otherwise I'll probably be out by one for the whole table. The first thing in my table will be $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x$ which is just the given integral with $n = 0$, so it has value $0 + 1 = 1$.
- The second term in my table will include $\int_1^2 \mathrm{f}(x)\,\mathrm{d}x = 1 + 1 = 2$, and the sum of those two integrals will be $1 + 2 = 3$. The rule is just "add the next number to the total so far". I can do that if I keep track of the number I'm adding in my head.

k	1	2	3	4	5	6	7	8
$\int_0^k \mathrm{f}(x)\,\mathrm{d}x$	1	3	6	10	15	21	28	36

- The question wants me to add these together. I paused here for a moment to think about whether I was missing a trick. Perhaps there's some nice formula for the sum of triangular numbers. If there is, then I don't know it (and it's not on the TMUA Content Specification). Ah well, here we go!
- $1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 = 120$.
- The answer is C.

Extension

- Numbers of the form $\frac{1}{2}N(N+1)$ for $N = 1, 2, \ldots,$ are called triangular numbers. The sum of the first N triangular numbers has an expression of the form $aN^3 + bN^2 + cN$ for some real constants a, b, c. Using that information, and some small values for the sum that you can calculate, find a and b and c.
- Construct a function f that has the property in the question. You might like to do this as a piecewise function, defining the values of f between n and $n + 1$ for each number n.

TMUA 2022 Paper 1 Question 8

- This question is all about sums of terms of a geometric sequence, so as soon as possible I would like to write down

$$S_{10} = \frac{a(1-r^{10})}{1-r} \quad \text{and} \quad S_{20} = \frac{a(1-r^{20})}{1-r} \quad \text{and} \quad S_{30} = \frac{a(1-r^{30})}{1-r}.$$

- The equation in the question is then

$$\frac{a(1-r^{30})}{1-r} - \frac{a(1-r^{20})}{1-r} = k\frac{a(1-r^{10})}{1-r}.$$

- I'm not going to worry about precisely what the question wants to know about k yet. I'm going to try to simplify this expression. I'll multiply through by $1-r$ and divide by a. This gives

$$\left(1-r^{30}\right) - \left(1-r^{20}\right) = k - kr^{10}$$

- The left-hand side is $r^{20} - r^{30}$ and I can pull out a factor of r^{20}. Brilliantly, the other factor on the left is then $(1-r^{10})$, so we have

$$r^{20}\left(1-r^{10}\right) = k - kr^{10}$$

- There's a shared factor on each side. So (unless $r = \pm 1$... it's not), we have $r^{20} = k$. I can't believe how much that's simplified!

- We're asked for the smallest positive value of k, and we're told that r is a positive integer greater than 1. There's a really nice relationship between r and k that makes it clear now that the smallest value of k corresponds to the smallest value of r.

- So we should take $r = 2$, the smallest value we're allowed, and then $k = 2^{20}$.

- The answer is B.

Extension

- If I were using this as the start of an interview question, then next I would ask about other relationships between sums of terms of a geometric series that would simplify in the same way; that moment where we could divide both sides by a factor like $(1-r^{10})$ is the thing I'm interested in.

- We could have noticed that the difference between S_{20} and S_{10} refers to a sum of terms that themselves form a geometric sequence (with a different "first" value, but the same common ratio, as the original geometric sequence). Redo the algebra using that observation. You should get the same equation!

- What happens if we allow $r = 1$? (notice that my very first step, writing down the sums, is no longer true.)

- What happens if we allow $r = -1$?

TMUA 2022 Paper 2 Question 8

- I copied out 1, 4, 7, . . . , 70 and worked out the common difference (3) and the number of terms (24). The terms have the form $1 + 3n$ for some integer value of n from 0 to 23.
- At first sight, I thought that the statement was simply true because there are two distinct terms, 4 and 70, that sum to 74. I missed the fact that we're talking about a selection S, with n terms, maybe not all 24.
- I can salvage something from that misunderstanding; I know that if the selection contains both 4 and 70 then there will be two distinct terms in S whose sum is 74. Is there some reason why the selection would have to include one of those? Not that I can see. What else?
- Going further, we can keep pairing terms of the sequence that sum to 74, so 7 pairs with the number before 70 in the sequence, and 10 pairs with 64, and so on.
- Let's be careful though. The statement $(*)$ says "distinct", which means that we're interested in sums that involve two different numbers. Does it say that for a reason? 74 is even... is 37 in this sequence? Yes it is, because $37 = 1 + 3 \times 12$. Sneaky!
- So I have eleven pairs of numbers that I'm interested in, with each of the numbers 4 up to 34 paired with a number from 70 down to 40 respectively. What can I say about these pairs? If S contains both of any pair, then statement $(*)$ will hold, because I can use the pair that's both there to make 74. These are the only ways to make 74 with numbers from the arithmetic sequence in the question.
- The numbers 1 and 37 don't seem to be part of this argument so far.
- The question asks for the smallest value of n for which $(*)$ is necessarily true. So I should imagine my nemesis selecting values for S, purposefully making sure that $(*)$ doesn't hold, until at some point they are forced to... do what? Include a pair that sums to 74, I suppose, and I know all of those.
- My nemesis could select one number from each of my eleven pairs, and the number 1, and the number 37 (a selection of 13 terms) and I would be powerless to find a pair that sums to 74. But then after that I will win, because they will be forced to take two from the same pair. That happens when they get greedy and select 14 numbers.
- The answer is C.

Extension

- Check your understanding; is the following statement true or false "if $n > 13$ then a selection of n terms will include the number 1 and the number 37 and some pair of distinct terms whose sum is 74".
- Check your understanding; replace the sequence with 1, 4, 7, . . . , 100, and replace the number 74 in $(*)$ with the number 140. What is the smallest value of n for which $(*)$ is necessarily true?