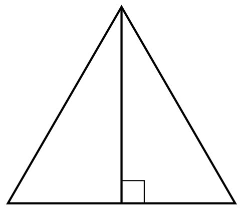
Trigonometry – Solutions

Revision Questions

1. Split the equilateral triangle into two right-angled triangles.



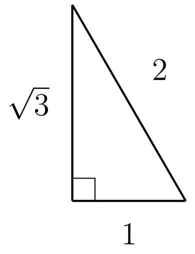
Each has base $\dfrac{a}{2}$ and height $\dfrac{\sqrt{3}a}{2}$, so the total area is $\dfrac{\sqrt{3}a^2}{4}$.

2. From the graph of $\sin x$, we expect two solutions in that range. If I draw a little triangle with opposite side 1 and hypotenuse 2 then I recognise this as half an equilateral triangle (so one solution is $30°$, then using $\sin(180° - x) = \sin x$ another is $150°$).

3. From the graph of $\tan x$, we expect solutions to be $45° + 180°n$ for any whole number n. In the given range, this is $x = 45°$ or $x = 225°$.

4. If we write $u = 45x$ then we've got $\tan(u) = 1$ so $u = 45° + 180°n$ just like in the previous part. But careful, because if $45x = 45° + 180°n$ then $x = 1° + 4°n$, so the solutions are the angles $1°, 5°, 9°, \dots, 357°$, not just $1°$ and $5°$ (there are 90 solutions).

5. Let's draw a picture



The three area formulas give the area as

$$\frac{1}{2} \times 1 \times \sqrt{3} \times \sin\left(\frac{\pi}{2}\right) \quad \text{or} \quad \frac{1}{2} \times \sqrt{3} \times 2 \times \sin\left(\frac{\pi}{6}\right) \quad \text{or} \quad \frac{1}{2} \times 1 \times 2 \times \sin\left(\frac{\pi}{3}\right)$$

all of which are equal to $\dfrac{\sqrt{3}}{2}$.

6. First I can expand the square to get $\cos^2 x + 2\cos x \sin x + \sin^2 x$.

Now I can use $\cos^2 x + \sin^2 x = 1$ and I can write $2\cos x \sin x$ as $2u$.

So the expression is $2u + 1$.

7. This is a geometric series with first term $a = 1$ and common ratio $r = -\sin^2 x$, so the sum to infinity is $(1 + \sin^2 x)^{-1}$ which is $(2 - \cos^2 x)^{-1}$ in terms of $\cos x$.

   The sum only converges if the common ratio r satisfies $|r| < 1$. This is not the case if $x = \dfrac{\pi}{2}$.

8. I'll use $\cos^2 x = 1 - \sin^2 x$ and substitute this into the expression.

   $\cos^4 x + \cos^2 x = (1 - \sin^2 x)^2 + (1 - \sin^2 x) = 2 - 3\sin^2 x + \sin^4 x$.

9. $\cos(450° - x) = \cos(90° - x)$ using the fact that $\cos x$ is periodic with period $360°$.

   Then $\cos(90° - x) = \sin x$.

10. We can use $\cos(90° - x) = \sin x$ and $\sin(90° - x) = \cos x$, and also $\sin(180° - x) = \sin x$ and $\cos(180° - x) = -\cos x$.

    We have $(\sin x)(\sin x) - (\cos x)(-\cos x) = \sin^2 x + \cos^2 x = 1$.

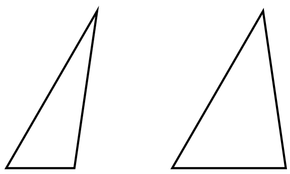
11. Use the cosine rule: $|AC|^2 = 8^2 + 7^2 - 2 \times 8 \times 7 \times \cos\left(\frac{\pi}{3}\right)$, so $|AC| = \sqrt{57}$.

    Use the $\frac{1}{2}ab\sin\gamma$ formula for the area: $\frac{1}{2} \times 8 \times 7 \times \sin\left(\frac{\pi}{3}\right) = 14\sqrt{3}$.

    Use the sine rule: $\dfrac{\sin\left(\frac{\pi}{3}\right)}{\sqrt{57}} = \dfrac{\sin\angle BCA}{8}$ so $\sin\angle BCA = \dfrac{4}{\sqrt{19}}$

12. Use the cosine rule: $7^2 = |BC|^2 + 8^2 - 2 \times 8 \times |BC| \times \cos\left(\frac{\pi}{3}\right)$.

    This is a quadratic for $|BC|$ that rearranges to $(|BC| - 3)(|BC| - 5) = 0$. So either $|BC| = 3$ or $|BC| = 5$. There are two distinct cases.

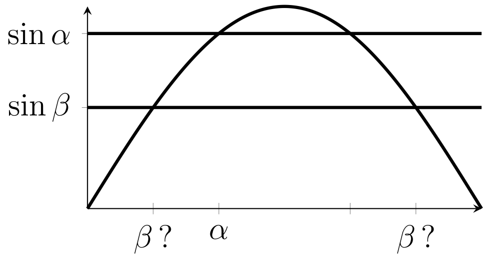


13. The sine rule gives $\sin\beta = \dfrac{b\sin\alpha}{a}$. If the denominator is less than the numerator, then this fraction is greater than 1. But the maximum value of $\sin\beta$ is 1, so there are no solutions if $a < b\sin\alpha$.

    Then if $a = b\sin\alpha$ the equation is $\sin\beta = 1$ which has unique solution $\beta = \frac{\pi}{2}$ in the range $0 < \beta < \pi$.

    If $a > b\sin\alpha$ then there are two solutions for β, and one of them is obtuse. If α is also obtuse then this isn't a real triangle (triangles can't have two obtuse angles, or the sum of the angles would exceed $180°$).

    Let's consider the case where $a \geqslant b$; we would have $\sin\beta = \frac{b\sin\alpha}{a} \leqslant \sin\alpha$. The solutions are an acute angle with $\beta \leqslant \alpha$, and an obtuse angle with $\beta \geqslant \pi - \alpha$. That obtuse angle is too large, once you realise that $\alpha + \beta$ would already exceed π.



14. The cosine rule is $a^2 = b^2 + c^2 - 2bc\cos\alpha$.

    Since $-1 \leqslant \cos\alpha \leqslant 1$, we have $b^2 - 2bc + c^2 \leqslant b^2 + c^2 - 2bc\cos\alpha \leqslant b^2 + 2bc + c^2$.

    So $(b-c)^2 \leqslant a^2 \leqslant (b+c)^2$. Since $a > 0$ and $(b+c) > 0$, it follows that $a \leqslant b + c$.

    In fact, we can't actually have $a = b + c$ because that would be a triangle with a 180° angle. So $a < b + c$.

15. If we write down the cosine rule for each of the angles, we get

$$\begin{aligned} 8^2 &= 13^2 + 15^2 - 2\times 13\times 15\times\cos\alpha \\ 13^2 &= 15^2 + 8^2 - 2\times 15\times 8\times\cos\beta \\ 15^2 &= 8^2 + 13^2 - 2\times 8\times 13\times\cos\gamma \end{aligned}$$

    and these rearrange to

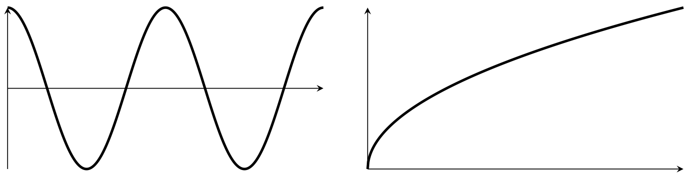
$$\cos\alpha = \frac{11}{13}, \quad \cos\beta = \frac{1}{2}, \quad \cos\gamma = \frac{1}{26}$$

    from which we see that $\beta = 60°$.

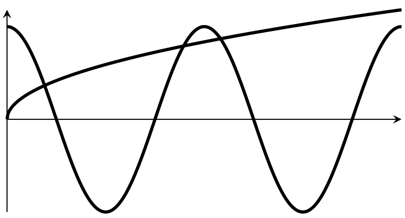
TMUA Questions

TMUA 2020 Paper 1 Question 12

- I wanted to start by drawing a sketch, but of course that's the whole question. If I could draw an accurate enough sketch, then I could just count the solutions!
- Instead, I sketched $3\cos x$ and $\sqrt{x}$ separately.



- I think that, eventually, $\sqrt{x}$ will be so large that we won't get any more solutions after that point. I can safely say that there will be no solutions once $\sqrt{x} > 3$, which is when $x > 9$. That's about 3π, I reckon, looking back at my sketch for $3\cos x$ to compare, so quite early on.
- Looking back at the $3\cos x$ graph reminded me that $\cos x$ is sometimes negative. We will only get a solution if $\cos x$ is positive. So perhaps we will have a solution between 0 and $\frac{\pi}{2}$, and then another one between $\frac{3\pi}{2}$ and $\frac{5\pi}{2}$?
- Time to draw the combined sketch. I'm happy with the first part of my sketch, where $\cos x$ decreases to zero while $\sqrt{x}$ increases. They must cross somewhere in between.
- For the part near $x = 2\pi$, I realised that I'll probably get (at least?) two solutions in the range $\frac{3\pi}{2} < x < \frac{5\pi}{2}$, not just one. That's because I checked the values of each function at $x = \frac{3\pi}{2}$ and $x = 2\pi$ and $x = \frac{5\pi}{2}$.
- Each of these changes of lead gives at least one solution where the curves cross. I suppose there could be some funny business, like extra crossing points or a point of tangency. That doesn't seem very likely to me, so I think we'll just have two solutions near $x = 2\pi$. Then by the time we get to 3π, no more solutions.



- So I have three solutions overall.
- The answer is D.

Extension

- Use a calculator or Desmos to investigate the case with $\frac{5}{2}\cos x$ instead of $3\cos x$.
- If I were using this as the start of an interview question, then next I would ask you about $3\cos x = x^{-1}$. This has infinitely many solutions, but perhaps you can tell me the approximate values for some of them?

TMUA 2021 Paper 1 Question 6

- I don't like the mix of $\sin x$ and $\cos x$ in this question, so I'm going to immediately rewrite the $\sin^2 x$ as $1 - \cos^2 x$.
- I'm also going to write $u = \cos x$. The fraction is now

$$\frac{u+3}{6+5u+u^2}.$$

- I have no idea how to find the maximum or minimum of this expression, but just in case it's helpful, I could factorise the denominator.
- The denominator factorises to $(u+3)(u+2)$.
- So the fraction simplifies to $\frac{1}{u+2}$. That's much better!
- Now I have to remember that $u = \cos x$, and I have to be a bit careful. The range of $\cos x$ is from -1 to 1. But where will the maximum and minimum of $\frac{1}{u+2}$ be?.
- If it helps, I could sketch a graph of $\frac{1}{u+2}$ against u. It's a translation of the standard $y = \frac{1}{x}$ graph two units to the left, and it has an asymptote at $u = -2$.
- So $\frac{1}{u+2}$ is a decreasing function as u varies from -1 to 1.
- So the maximum of $\frac{1}{u+2}$ for $-1 \leqslant u \leqslant 1$ comes at $u = -1$ and has value 1, and the minimum of $\frac{1}{u+2}$ comes at $u = 1$ and has value $\frac{1}{3}$.
- The positive difference between those numbers is $\frac{2}{3}$.
- The answer is D.

Extension

- If I were using this as the start of an interview question, I would ask you to sketch the function.

  I'm perhaps working towards asking you about a more difficult function like

$$f(x) = \frac{2\cos x + 5}{7 + 5\cos x - \sin^2 x}.$$

  Can you find the maximum and minimum values of that function?

  [Hint: what have I added?]

TMUA 2021 Paper 1 Question 19

- This question is essentially just a very complicated composite function, and my plan is to work backwards to undo this function, step-by-step.
- Given $\sin^2\left(4^{\cos\theta} \times 60^\circ\right) = \dfrac{3}{4}$ we can write

$$\sin\left(4^{\cos\theta} \times 60^\circ\right) = \frac{\sqrt{3}}{2} \quad \textbf{or} \quad \sin\left(4^{\cos\theta} \times 60^\circ\right) = -\frac{\sqrt{3}}{2}.$$

- I know when sin takes the values $\pm\dfrac{\sqrt{3}}{2}$. Lots of times, in fact! Do I have to list negative solutions? I know that $4^{\cos\theta}$ will be positive, so I don't think that I need to. So my two cases are now many cases;

$$4^{\cos\theta} \times 60^\circ = 60^\circ \text{ or } 120^\circ \text{ or...} \qquad \textbf{or} \qquad 4^{\cos\theta} \times 60^\circ = 240^\circ \text{ or } 300^\circ \text{ or...}$$

- On the left, I can see that these correspond to $4^{\cos\theta} = 1$ or 2 or... . I reckon that the next one would be 7, which is out of range because $4^{\cos\theta}$ is at most 4.
- On the right, I can see that these correspond to $4^{\cos\theta} = 4$ or 6 or... . I'm only really interested in the first of those.
- So now I must solve $4^{\cos\theta} = 1$ or 2 or 4. That happens when $\cos\theta$ is 0 or $\frac{1}{2}$ or 1.
- A sketch of $\cos\theta$ helps me to remember that this happens when $\theta = 0$ or $\theta = 30^\circ$ or $\theta = 90^\circ$ or $\theta = 270^\circ$ or a couple of values after that, before we get back round to 360° and the patten repeats.
- I've stopped there though, because I've secretly been counting! The question wants a range for x such that we have three solutions, notably including the one at $\theta = 0$, in the range $0^\circ \leqslant \theta \leqslant x$.
- This happens if x is between 90° and 270°.
- The answer is B.

Extension

- Check your understanding with this similar question. Show that there are exactly four values of x in the range $0 \leqslant x < 2\pi$ such that

$$\frac{6 \times \arctan\left(3^{\cos x}\right)}{\pi}$$

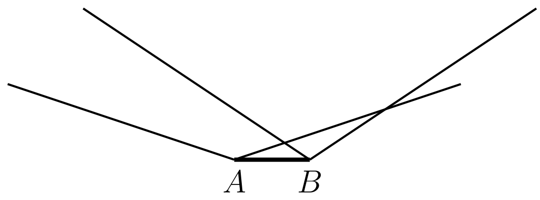
  is a whole number.

  In this question, $\arctan y$ is the value of x in the range $-\frac{\pi}{2} < x < \frac{\pi}{2}$ that has $\tan x = y$.

  Note that I'm using radians here, just to mix things up.

TMUA 2021 Paper 2 Question 18

- This looks like the ASA case. Why would this be ambiguous? What else is going on?
- Something I noticed is that we're given x and y, not the angles A and B. That matters, because for $0 < x < 1$ there are exactly two angles that have $\sin A = x$. One is obtuse and one is acute, and they sum to π.
- I drew some pictures to try to imagine different possibilities. I've taken the side AB to be horizontal, and I'm drawing lines where we might find C. I've got a pair of lines from each point, for the two possibilities for the angle, and I've tried to make the lines from A shallower, to reflect the fact that $\sin A < \sin B$.



- It took me a surprisingly long time to remember that triangles can't have two obtuse angles. So that's a choice that I'm not allowed to make for angles A and B. This is good, because I briefly believed that the answer would be "you always get four triangles" (that would be option A, I suppose).
- If angles A and B are both acute, then I managed to convince myself that there's a unique triangle; once you've fixed the angles, you're in the ASA case, and it's pretty clear that there will be somewhere to put C.
- With one obtuse angle, it's not clear that the lines will actually meet; they might diverge.
- This reminds me of the ambiguous case in the revision notes, specifically the part where you think your angles correspond to a real triangle, but then you realise that they already sum to more than π.
- This happens if I pick the obtuse angle for A and the acute angle for B. That's because $x < y$ so $\sin A < \sin B$, which makes $\alpha > \pi - \beta$, and then the angles are too large. On my diagram, the lines going left from A and B don't cross.
- On the other hand, if I pick the obtuse angle for B, then this will give a genuine triangle. In my diagram, that's the crossing point on the right where the steep line from B manages to "catch up" with the shallow line from A.
- So I've convinced myself that there will be two different triangles that the student could draw, for any choices of x and y that satisfy $0 < x < y < 1$.
- The answer is C.

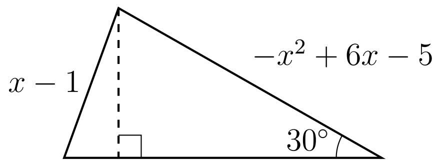
Extension

- Check your understanding; what if we're told that $\tan A = x$ and $\tan B = y$ instead of $\sin A = x$ and $\sin B = y$?

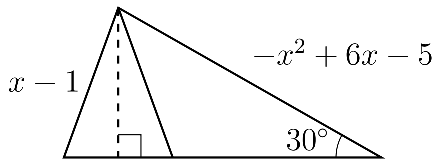
  With that change, what can you say about the angle at C?

TMUA 2022 Paper 1 Question 17

- This looks like the SSA case, where (given an acute angle like 30°), there might be two non-congruent triangles.
- Looking at the diagram, if this is a real triangle, then I should be able to drop a perpendicular from the top corner down to the base.



- My idea here is that I can work out the length of the dashed line; it's $\frac{1}{2}(-x^2 + 6x - 5)$.
- I can also say that $(x - 1)$ must be greater than this length, because it's the hypotenuse of a little triangle on the left.
- In fact, I can see that I could get to the other non-congruent triangle (with the same two sides and not-included angle) by reflecting that little right-angled triangle over to the other side of the dashed line.



- This could go wrong though, if $(x - 1)$ is too big, because then the little triangle won't be so little, and my alternative position for the lower-left corner will end up beyond the lower-right corner of the triangle. That happens if $x - 1 > -x^2 + 6x - 5$, thinking about the critical value where the little triangle reflects perfectly onto the side.
- So putting that together, I want

$$x - 1 > \frac{1}{2}\left(-x^2 + 6x - 5\right) \quad \text{and} \quad x - 1 < -x^2 + 6x - 5$$

- The quadratic factorises as $(x - 1)(5 - x)$ and then I can cancel the factor of $(x - 1)$, assuming that $x > 1$ as the side length $(x - 1)$ is positive, and I get $1 > \frac{1}{2}(5 - x)$ and $1 < 5 - x$.
- That's $3 < x < 4$.
- The answer is D.

Extension

- Alternatively, we could write down the inequalities in the revision notes. I didn't have this option, because I hadn't written that bit of the revision notes when I did this question! Check that the inequalities in the notes give the same answer.

TMUA 2022 Paper 2 Question 20

- I made a table to track the maximum and minimum values of each function. I've got to be careful because, in general when you have some range like $a < x < b$ and you apply a function f, the maximum output might occur at a, or at b, or at some in-between point. It depends on whether the function f is increasing or decreasing or a bit of both.

- My table starts like this, describing the minimum and maximum values of $\cos(x)$.

	min	max
f_1	-1	1

- Then I'm asked to apply sine, because $f_2(x) = \sin(f_1(x))$. What happens? The input to the sine could be as large as one radian. Is that a lot? In particular, is that more or less than $\frac{\pi}{2}$? I'm interested because I want to know if $\sin(f_1(x))$ could be as large as 1.

- We know that $\frac{\pi}{2} > 1$. So the inputs for sine here are pretty small, and f_2 never gets as large as 1. Instead, the maximum is $\sin 1$, whatever that is. This rules out option A. Similarly, a negative value of f_1 gives the minimum of f_2 as $-\sin 1$.

	min	max
f_2	$-\sin 1$	$\sin 1$

- Now we apply cosine to those values. There's a bit of a trap here, because I might try to continue the pattern so far, and just write down $\cos(\sin 1)$ for the maximum. That would forget that cosine achieves the value of 1 when the input is 0. As the range of f_2 includes 0, we will see that value.

- So the maximum of f_3 is 1. This rules out option B. What's the minimum? Well, $\sin(1) < 1 < \frac{\pi}{2}$. So when we apply cosine, the minimum value is $\cos(\sin(1))$ (not the maximum!).

	min	max
f_3	$\cos(\sin 1)$	1

- Now it's time to apply sine. The maximum $m_4 = \sin(1) = m_2$ which rules out options D and F, and the minimum is $\sin(\cos(\sin 1))$. Note that, unlike the previous case, I'm relatively relaxed here because sine has no turning points in the relevant range.

	min	max
f_4	$\sin(\cos(\sin 1))$	$\sin 1$

- Finally we apply cosine. Looking at the remaining options, it looks like the question is just whether $m_5 = 1$ or not. We don't have zero in the range of f_4, so I can't see any way for f_5 to give value 1. I think that $m_5 < 1$, which eliminates option C.

- The answer is E.

Extension

- What happens if the question is in degrees instead?

- What happens if I replace every mention of $\sin u$ in the question with $\sin 2u$, and every mention of $\cos u$ with $\cos 2u$?