Exponentials and logarithms – Solutions

Revision Questions

1. $(2^3)^4 = 2^{12}$. $(2^4)^3 = 2^{12}$. $2^4 2^3 = 2^7$. $2^3 2^4 = 2^7$.

2. This is a quadratic for $\frac{1}{x}$ with solutions $\frac{1}{x} = -1$ or $\frac{1}{x} = -3$. So $x = -1$ or $x = -\frac{1}{3}$. Alternatively, we could multiply both sides by x^2 and solve the quadratic that we get.

3. This is $\log_{10} 12$.

4. The quadratic inside the brackets factorises, and this is $\log_3(x+2) + \log_3(x+1)$. Other answers are possible, such as $\log_3(2(x+2)) + \log_3((x+1)/2)$.

5. The left-hand side is just 2 so we want $2 = x^3$. So $x = \sqrt[3]{2}$.

6. The left-hand side is $1 + \log_x 2$ so we want $\log_x 2 = 2$. So $x^2 = 2$ and $x = \sqrt{2}$ (not $-\sqrt{2}$ because we're told that $x > 0$).

7. Take $(x+5)$ to the power of each side to get $6x + 22 = (x+5)^2$. Expand the square and rearrange for $x^2 + 4x + 3 = 0$. The solutions are $x = -1$ or $x = -3$. Check these solutions; $\log_4(16) = 2$ and $\log_2(4) = 2$.

8.
   - $\log_3 1024 = \log_3 \left(2^{10}\right) = 10 \log_3 2 = 10a$.
   - $\log_3 40 = \log_3 8 + \log_3 5 = 3a + b$.
   - $\log_3 \sqrt{\frac{2}{5}} = \frac{1}{2} \log_3 \left(\frac{2}{5}\right) = \frac{1}{2}(a - b)$.
   - $\log_3 \left(\frac{1}{10}\right) = -\log_3 10 = -\log_3 2 - \log_3 5 = -a - b$.
   - $\log_3 1.024 = \log_3 1024 + \log_3 \left(\frac{1}{1000}\right) = 10a + 3(-a - b) = 7a - 3b$.

   There are other solutions, partly because $b = a \times \log_2 5$ (see the change of base formula below).

9. $2^{x+y} + 2^{y-x} - 2^{x-y} - 2^{-x-y} + 2^{x+y} - 2^{y-x} + 2^{x-y} - 2^{-x-y}$. That's $2^{x+y+1} - 2^{-x-y+1}$.
   $2^{x+y} + 2^{y-x} + 2^{x-y} + 2^{-x-y} + 2^{x+y} - 2^{y-x} - 2^{x-y} + 2^{-x-y}$. That's $2^{x+y+1} + 2^{-x-y+1}$.

10. $2^x = 3$ is what it means for x to be $\log_2 3$.

    If $0.5^x = 3$ then $2^{-x} = 3$ so $x = -\log_2 3$. Alternatively, just write down $x = \log_{0.5} 3$.

    If $4^x = 3$ then $2^{2x} = 3$ so $x = \frac{1}{2} \log_2 3$. Alternatively, just write down $x = \log_4 3$.

11. By definition, $\log_a b = 0$ if and only if $a^0 = b$. Since a is positive, we have $a^0 = 1$ and so $b = 1$.

12. We have $\log_{10} x = 10^6$, which is a million. So x is ten to the power of a million. That's got a million zeros at the end.

13. We have $\log_a \left(\dfrac{x}{y}\right) = \log_a \left(xy^{-1}\right) = \log_a x + \log_a y^{-1}$ using the product rule, and then $\log_a y^{-1} = -\log_a y$ using the power rule.

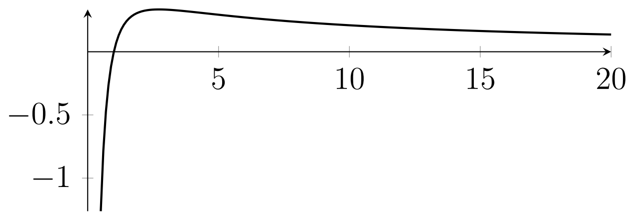
14. Multiply both sides by 2^x and rearrange to get $2^{2x} - 4 \times 2^x + 1 = 0$. This is a quadratic for 2^x. Solve it for $2^x = 2 \pm \sqrt{3}$. So $x = \log_2(2 \pm \sqrt{3})$.

    Following the previous working, we can see that we'll get two roots for 2^x if $c^2 - 4 > 0$. But we need these to be positive roots, so we need $c > 2$. If $c = 2$ there's a repeated root. If $c < 2$ there are no roots.

15. Move both terms onto the left-hand side and use the fact that $(N + \sqrt{N^2 - 1})(N - \sqrt{N^2 - 1}) = N^2 - (N^2 - 1) = 1$; that's the difference of two squares. Remember that $\log_a 1 = 0$.

    As a result, our solutions to the previous question, $\log_2(2 \pm \sqrt{3})$, are actually $\pm \log_2(2 + \sqrt{3})$, revealing a lovely symmetry. But you could have spotted that from the equation, of course!

16. I've taken logarithms base 3 on each side to get $\log_3(x^y) = \log_3(y^x)$. You probably picked a different base for your logarithms, it doesn't make much difference (see the last question below). I can simplify my equation to $y \log_3 x = x \log_3 y$ and rearrange to get $\frac{\log_3 x}{x} = \frac{\log_3 y}{y}$. You might choose to put this fraction the other way up, or to square both sides or something, so your $f(x)$ might not be the same as mine. Here's a sketch of $y = \frac{\log_3(x)}{x}$.



17. $a^{k \log_a b}$ is the same as $(a^{\log_a b})^k$ which is just b^k.

    We can use a similar bit of algebra with $k = \log_b c$.

    $$a^x = a^{\log_a b \log_b c} = \left(a^{\log_a b}\right)^{\log_b c} = (b)^{\log_b c} = c.$$

    So $a^x = c$ and therefore $x = \log_a c$. We have proved that $(\log_a b)(\log_b c) = \log_a c$.

    We can divide by $\log_a b$ to get

    $$\log_b c = \frac{\log_a c}{\log_a b}$$

18. We can use the result in the previous question to change the base to 3, and then we can rewrite the numerator and denominator separately;

    $$\log_6(45) = \frac{\log_3(45)}{\log_3 6} = \frac{\log_3 5 + \log_3 9}{\log_3 2 + \log_3 3} = \frac{b+2}{a+1}.$$

19. If $a < 1$ then $\log_a x$ is a decreasing function of x. You can deduce this from the corresponding equation $x = a^y$. Since a^y is a decreasing function of y for $a < 1$, large values of y correspond to small values of x, and vice versa.

    You might also note that $\log_a x = -\log_{(1/a)} x$ using the change of base formula.

TMUA Questions

TMUA 2020 Paper 1 Question 7

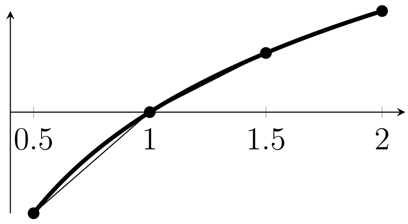
- The question includes powers of numbers like 2 and 4 and 8 and 16. These are all powers of 2, so we can re-write everything in terms of powers of 2.
- The equation $2^{3x} = 8^{(y+3)}$ can be re-written as $2^{3x} = 2^{3(y+3)}$.
- The equation
$$4^{(x+1)} = \frac{16^{(y+1)}}{8^{(y+3)}}$$
can be re-written as
$$2^{2(x+1)} = 2^{4(y+1)-3(y+3)}$$
- Since 2^x is an increasing function of x, we can just compare exponents.
- So we have the simultaneous equations
$$3x = 3(y+3)$$
$$2(x+1) = 4(y+1) - 3(y+3)$$
- The first equation simplifies to $x = y + 3$, and that's in a convenient form to substitute into the second equation. This gives $2y + 8 = y - 5$.
- The solution I get has $x = -10$ and $y = -13$, so $x + y = -23$.
- The answer is A.

Extension

- Is the function
$$\left(\frac{1}{2}\right)^{-x/2} \times 8^{3x} \times \left(2\sqrt{2}\right)^{3x/2}$$
an increasing function of x, or a decreasing function of x?
- Find the minimum value of $\mathrm{f}(x) = \dfrac{2^{\left(x^2\right)}}{\left(2^x\right)^2}$.

TMUA 2021 Paper 1 Question 10

- I drew a sketch of the function. The sketch made me think about the fact that the difference between 2 and $\frac{1}{2}$ is $\frac{3}{2}$, so if we use 3 strips then each will have width $\frac{1}{2}$. I thought about the sign of the function, but I'm not sure if that's going to be relevant.



- For the trapezium rule, we need the value of the function at various points. Writing $\mathrm{f}(x)$ for the function $2\log_{10} x$, the trapezium rule calls for

$$\frac{1}{2} \times \frac{1}{2} \times \left(\mathrm{f}\left(\frac{1}{2}\right) + 2 \cdot \mathrm{f}(1) + 2 \cdot \mathrm{f}\left(\frac{3}{2}\right) + \mathrm{f}(2) \right)$$

- Those values of f are, in order, $2\log_{10}\frac{1}{2}$, 0, $2\log_{10}\frac{3}{2}$, and $\log_{10} 4$.
- So we have

$$\frac{1}{4}\left(2\log_{10}\frac{1}{2} + 0 + 4\log_{10}\frac{3}{2} + \log_{10} 4\right),$$

  and the $2\log_{10}\frac{1}{2}$ cancels out the $\log_{10} 4$.
- So the answer is just $\log_{10}\frac{3}{2}$.
- The answer is B.

Extension

- Is this an overestimate or an underestimate?
- If I were using this as the start of an interview question, then next I might try to find out whether you know enough about integration and logarithms to work out the exact value of that integral. If not then I might use the interview to teach you that

$$\int \ln x \, \mathrm{d}x = x \ln x - x + c,$$

  where $\ln x$ means $\log_e x$ with e a particular irrational number that's about 2.7, and where c is the constant of integration. Or perhaps we might work that out together.
- Use the trapezium rule with 101 strips to estimate

$$\int_0^{101} \log_{10}\left(3^{1-x}\right) \, \mathrm{d}x.$$

TMUA 2022 Paper 1 Question 11

- I wrote out a few terms and the last term, like this

$$\log_{10}\left(3^0\right) + \log_{10}\left(3^1\right) + \log_{10}\left(3^2\right) + \cdots + \log_{10}\left(3^{-99}\right),$$

  where I had to think really hard about whether the last term would involve 3^{-99} or 3^{-100}.

- Each term looks like it can be simplified with a law of logarithms to bring down the exponent (from the options, I can see that we want everything to be in terms of $\log_{10} 3$).

- I now have

$$-(0 + 1 + 2 + \cdots + 99)\log_{10} 3$$

  where I've pulled the minus sign out the front, the $\log_{10} 3$ out the end, and I've kept the 0 at the front of the sum so that it's still clear that I've got 100 terms.

- In the brackets I can see the sum of the numbers from 1 to 99. That's $\frac{1}{2} \times 99 \times 100$, using the rule for the sum of the terms of an arithmetic progression, or the formula for triangle numbers $\binom{n}{2}$, whichever you prefer.

- That's $-4950 \log_{10} 3$.

- The answer is A.

Extension

- Check your understanding: suppose that a_n is a geometric progression with $a_1 = a$ and $a_{n+1} = ra_n$ for $n \geqslant 1$, where a and r are positive constants. Find an expression for

$$\sum_{n=1}^{100} \log_{10} a_n.$$

- This sum is related to the last bullet point in the Extension section of the previous question. Noting that the function $\log_{10}(3^{1-x})$ is linear, can you comment on whether the trapezium rule gives an overestimate or an underestimate?

- MAT 2016 Q1A asked for the product of the first 15 terms of the geometric progression with first term $a_1 = 1$ and subsequent terms defined by $a_{n+1} = la_n$ for $n \geqslant 1$, in terms of l. That question also involved careful manipulation of powers.

TMUA 2021 Paper 2 Question 14

- I would quite like to solve these equations, if I can.
- I see that x only appears as part of 2^x, and y only appears as part of $\log_2 y$. I'd like to solve for 2^x and $\log_2 y$, treating those as the variables, if I can.
- In my rough work I've underlined 2^x and written "problem!" because I can see that a solution for 2^x might not correspond to a solution for x.
- Taking the difference between the equations in the question will eliminate the bit that depends on y, whatever we're calling it. I'm left with $(p-1)2^x = 1$
- I've rearranged this for $p = 1 + 2^{-x}$. I know that 2^{-x} is positive, so I really want $p > 1$.
- Then $x = -\log_2(p-1)$, which would be fine.
- And y would be... something else. If I'm honest, I didn't write down what y would be, because it seems pretty clear that the equation $2^x + \log_2 y = 1$ could be rearranged for $\log_2 y$ in terms of this solution we've just found for x, and then y would be 2 to the power of whatever we get. That's all well-defined, and the question doesn't ask me to find y.
- If I'm being less lazy, then the solution is

$$x = -\log_2(p-1), \qquad y = 2^{\left(\frac{p-2}{p-1}\right)}.$$

- Anyway, the solution exists if and only if $p > 1$.
- The answer is C.

Extension

- Check your understanding: what is the complete range of p for which the simultaneous equations

$$\begin{aligned} p\cos x + \sin y &= 2 \\ \cos x + \sin y &= 1 \end{aligned}$$

  have at least one real solution (x, y)?
- What is the complete range of p for which the simultaneous equations

$$\begin{aligned} p \cdot \sqrt{x} + y^2 &= 2 \\ \sqrt{x} + y^2 &= 1 \end{aligned}$$

  have at least one real solution (x, y)?
- What is the complete range of p for which the simultaneous equations

$$\begin{aligned} p2^x + \log_2 y &= 2 \\ 2^x + \log_2 y &= 2 \end{aligned}$$

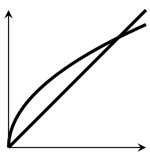
  have at least one real solution (x, y)?

TMUA 2021 Paper 2 Question 17

- The options are quite confusing to me. They each seem to involve one case where $f(x) < g(x)$ and one case where $g(x) < f(x)$. So perhaps I should start by working out which values of x give each of those behaviours.
- I rewrote $f(x) = \log_2\left(\frac{1}{2}\log_2 x\right)$ to make the two functions look more similar.
- Both functions end with an application of the function $\log_2 x$. I know that $\log_2 x$ is an increasing function of x, so

$$\log_2\left(\frac{1}{2}\log_2 x\right) < \log_2\left(\sqrt{\log_2 x}\right) \quad \text{if and only if} \quad \frac{1}{2}\log_2 x < \sqrt{\log_2 x}.$$

- To get to grips with that, I'm going to write $u = \log_2 x$. So now I'm looking at $\frac{1}{2}u < \sqrt{u}$.
- I know that $\frac{1}{2}u < \sqrt{u}$ if and only if $0 < u < 4$. I drew a little sketch to remind myself which way around that inequality goes. $\sqrt{u}$ starts fast but $\frac{1}{2}u$ eventually catches up.



- So $f(x) < g(x)$ if and only if $\log_2 x < 4$, and $f(x) > g(x)$ if and only if $\log_2 x > 4$. I'm not including the fact that $u > 0$ because this is guaranteed by the range of x in the question.
- For each of the possible answers, we need to know about the values in each case. It's not immediately clear whether I should look at $f(x)$ or $g(x)$, so I'll look at both.
- If $\log_2 x < 4$ then what can I say about $f(x)$? I could say $f(x) < \log_2\left(\frac{1}{2} \times 4\right) = 1$.

  If I had to say something about $g(x)$ it would be $g(x) < \log_2\left(\sqrt{4}\right) = 1$.
- In the other case, if $\log_2 x > 4$, then all the inequality signs flip, and so $g(x) > 1$ and $f(x) > 1$.
- I've missed the case where $\log_2 x = 4$ and both $f(x)$ and $g(x)$ are equal to 1.
- The answer is F.

Extension

- If I were using this as the start of an interview question, then next I might introduce a third function:

$$h(x) = \sqrt{\log_2(\log_2 x)}.$$

  Where is this function defined? For what value or values of x does $h(x) = f(x)$? For what value or values of x does $h(x) = g(x)$?

  [If your answers involve massive powers of 2, leave them as massive powers of 2.]
- These functions involve the three permutations of two "$\log_2$"s and one "$\sqrt{\ }$". You could invent your own extension, using the six possible permutations of two of each function!

TMUA 2022 Paper 2 Question 15

- I think that the following approach is unintended, but it's what I did, so I'll start by showing you how it goes.
- I know that the logarithms in the question, $\log_x y$ and $\log_y z$ and $\log_z x$, are related via the identity $(\log_x y)\left(\log_y z\right)(\log_z x) = 1$
- We're told that the first two are equal to z and x respectively.
- Therefore the third must be $\frac{1}{xz}$ to make the identity work. That's one of the options.
- I think that the intended route is more like this: start by writing out what the given information tells us in terms of exponentials.
- We have $y = x^z$ and $z = y^x$.
- So, substituting the first into the second to eliminate y, we have $z = x^{zx}$.
- I want to say something about $\log_z x$, so I want a statement where some power of z is equal to x.
- I can take each side to the power of $\frac{1}{xz}$ to get $z^{(1/xz)} = x$.
- Therefore $\log_z x = \frac{1}{xz}$.
- Either way, the answer is F.

Extension

- Use the change of base formula to prove that
$$\log_x y = \left(\log_y x\right)^{-1}$$
for all positive x and y, with neither equal to 1.
- Use the change of base formula to prove that
$$\log_{\frac{1}{b}} c = \log_b \frac{1}{c}$$
for all positive b and c, with b not equal to 1.
- Use the change of base formula to prove that
$$(\log_x y)\left(\log_y z\right)(\log_z x) = 1$$
for all positive x, y, and z, with none of them equal to 1.
- Use the change of base formula to prove that
$$(\log_a b)(\log_b c)(\log_c d)(\log_d a) = 1$$
for all positive a, b, c, and d, with none of them equal to 1.
- If I were using this as the start of an interview question, then next I might ask you to continue the pattern that I've set up with the previous two bullet points.