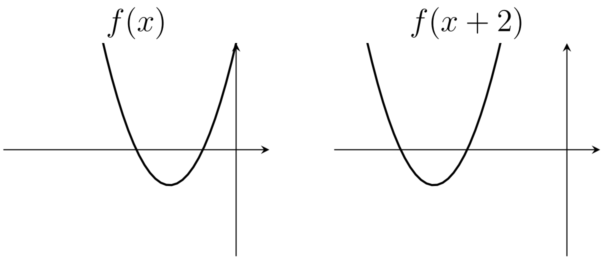
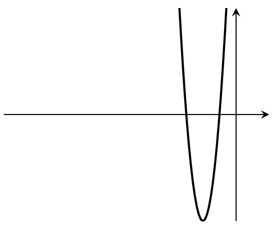
Graphs of functions – Solutions

Revision Questions

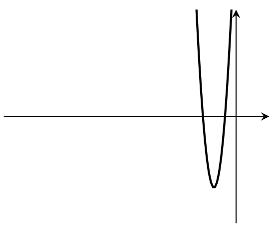
1. Note that $x^2+4x+3=(x+3)(x+1)$. The graph of $y=f(x+2)$ is the graph of $y=f(x)$ after it has been translated two units to the left.



For $y=3f(2x)$, the graph is "squashed" by a factor of 2 parallel to the x-axis, then "stretched" by a factor of 3 parallel to the y-axis.



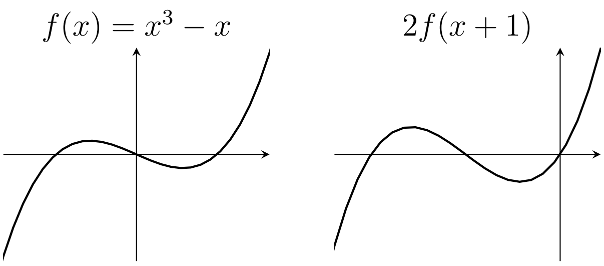
For $y=2f(3x)$, the graph is squashed by a factor of 3 parallel to the x-axis, then stretched by a factor of 2 parallel to the y-axis



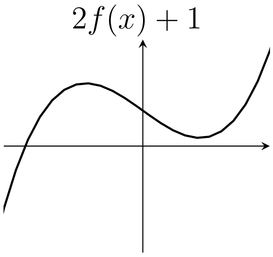
It's not the same as the previous graph. For example, the roots aren't in the same places.

$g(x)=x$ works for the last part; then $y=20x$ in both cases.

2. The graph of $y=f(x)$ and the graph of $y=2f(x+1)$;



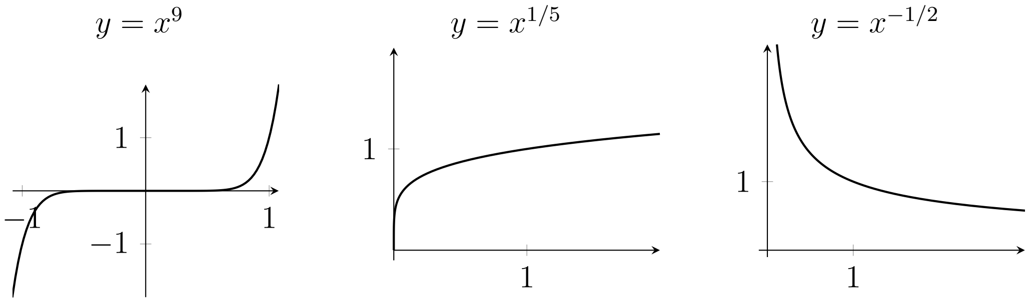
The graph of $y = 2f(x) + 1$;



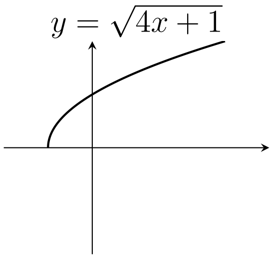
It's not the same as the previous graph. We could compare, for example, the values of $2f(x+1)$ and $2f(x)+1$ when $x = 0$.

$g(x) = x/3 + c$ for any constant c works for the last part.

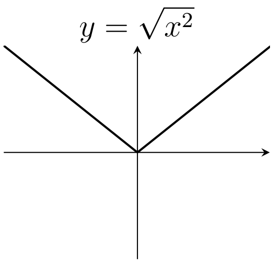
3. For large n, $y = x^n$ is close to zero between -1 and 1, except for values near $x = \pm 1$. For small positive $n < 1$, $y = x^n$ is close to 1 between 0 and 1, except for values near $x = 0$. For negative n, the graph increases without bound near $x = 0$.



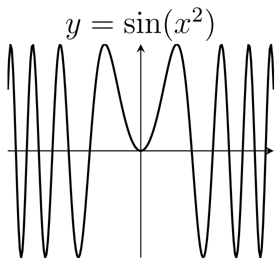
4. Note that $\sqrt{4x+1} = 2\sqrt{x + \frac{1}{4}}$ so this is a translation of the graph of $y = \sqrt{x}$ by $\frac{1}{4}$ units in the negative x-direction followed by a stretch parallel to the y-axis with scale factor 2.



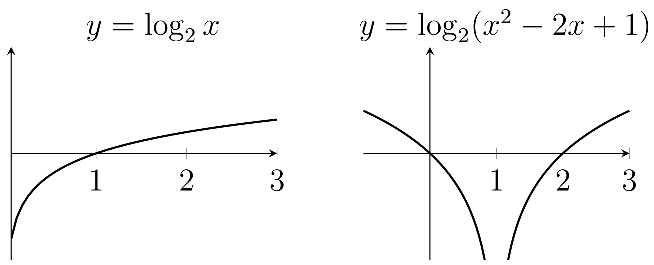
5. If $x \geqslant 0$ then $\sqrt{x^2} = x$ but if $x < 0$ then $\sqrt{x^2} = -x$, because $\sqrt{u}$ is always the non-negative root.



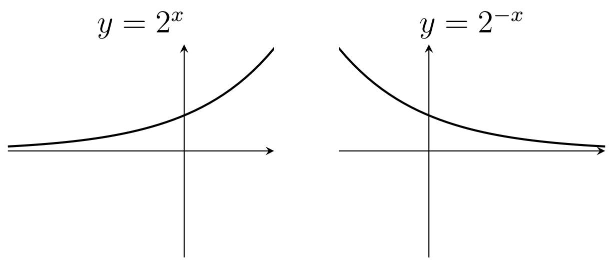
6. The function $\sin(x^2) = 0$ when $x^2 = 180°n$ for n an integer, so the graph crosses the x-axis more and more frequently as x increases. The graph has reflectional symmetry in the y-axis.



7. Note that $\log_2(x^2 - 2x + 1) = \log_2((x-1)^2)$. For $x > 1$ this is just $2\log_2(x-1)$. For $x < 1$, it is $2\log_2(-(x-1))$, because for negative x, $\sqrt{x^2} = -x$.

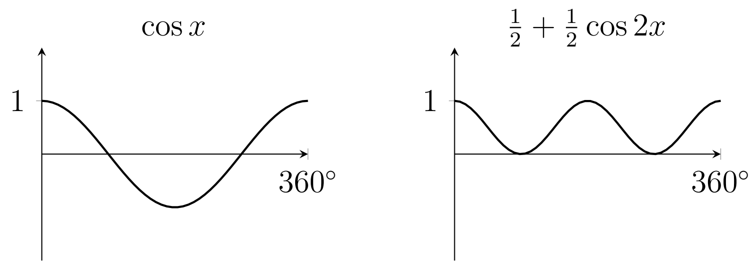


8. The graph of $y = 2^x$, and the graph of $y = 2^{-x}$.



Each graph is the reflection in the y-axis of the other graph.

9. The graph of $y = \frac{1}{2} + \frac{1}{2}\cos 2x$ is related to the graph of $y = \cos x$ by a stretch parallel to the x-axis with scale factor $\frac{1}{2}$, a stretch parallel to the y-axis with scale factor $\frac{1}{2}$, and then a translation parallel to the y-axis by $\frac{1}{2}$ a unit. After those transformations, the local minima of $\cos x$ will be transformed to be on the x-axis.

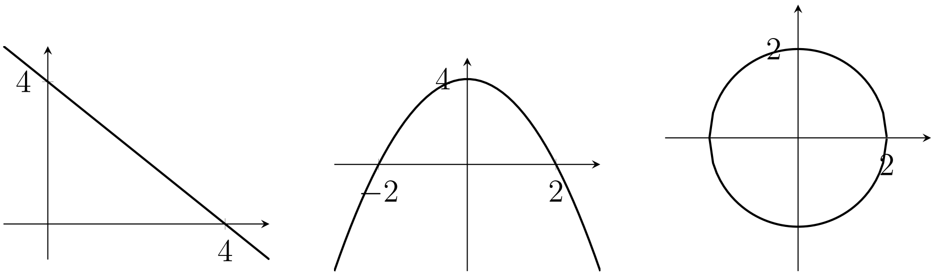


10. The equation $y = 4 - x$ is the equation of a straight line.

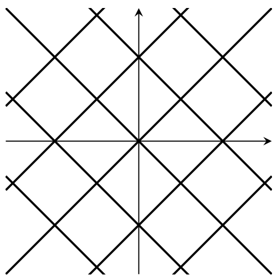
    The equation $y = 4 - x^2$ is the equation of a parabola.

    The equation $y^2 = 4 - x^2$ is the equation of a circle with radius 2 centred on the origin.

$y = 4 - x$ $\quad$ $y = 4 - x^2$ $\quad$ $y^2 = 4 - x^2$



11. If $\cos x = \cos y$ then either $x = y + 360°n$ for some integer n, or $x = -y + 360°n$ for some integer n. These are the equations of straight lines.



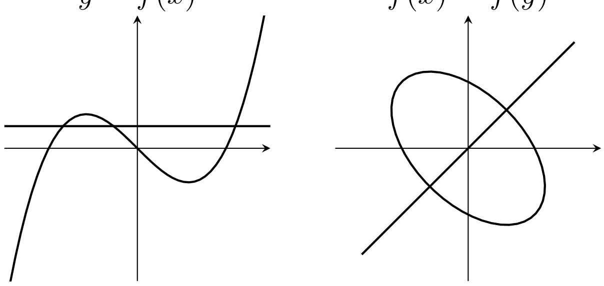
12. It's helpful to consider different values of the function separately. Let's suppose that $f(x) = c$ and $f(y) = c$ for some particular value of c.

    If c is large enough then there is just one input to f that gives the output c, so we must have $x = y$. In fact, it's always possible that $x = y$, so that line is part of our graph.

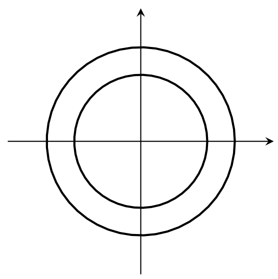
    For some values of c though, there are three possible inputs to f that give that output. As we change c, these alternative solutions appear at the turning point of $y = f(x)$. In terms of x and y, these solutions trace out a neat round graph. That part of the graph is actually an ellipse, but you are not expected to spot that!

    Algebraically, if $x^3 - x = y^3 - y$ then we can rearrange for $x^3 - y^3 - x + y = 0$ and factorise for $(x - y)(x^2 + xy + y^2 - 1)$. You are not expected to know that $x^2 + xy + y^2 = 1$ is the equation of an ellipse.

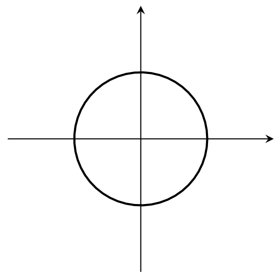
$y = f(x)$ $\quad$ $f(x) = f(y)$



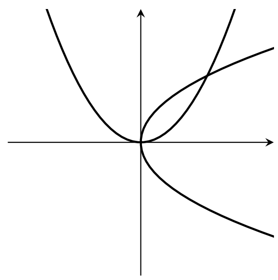
13. The equation $x^4+2x^2y^2+y^4-3x^2-3y^2+2=0$ simplifies to $(x^2+y^2)^2-3(x^2+y^2)+2=0$, which is a quadratic for x^2+y^2, with roots $x^2+y^2=1$ or $x^2+y^2=2$. This is a pair of circles.



14. The equation $x^6+3x^4y^2+3x^2y^4+y^6=1$ simplifies to $(x^2+y^2)^3=1$ so we have $x^2+y^2=1$ and this is a circle.



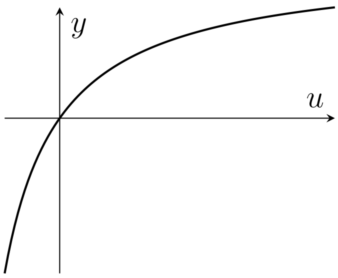
15. The equation rearranges to $x^3-xy=x^2y^2-y^3$. Take out a factor of x on the left and y^2 on the right to factorise this as $x(x^2-y)=y^2(x^2-y)$. So either $x^2=y$ or $x=y^2$. This is a pair of parabolas.



TMUA Questions

TMUA 2020 Paper 2 Question 5

- The function involves 2^x twice, so I'd like to think about what happens to 2^x for different values of x.
- If x is very large and positive, 2^x is very large indeed. What does that tell me about the graph? Informally, I'll have $y = \dfrac{\text{large}}{1+\text{large}}$. That's about 1.
- For very negative x, 2^x is about 0. What does that tell me about the graph? I'll have $y \approx 0$.
- When $x = 0$, I get $\dfrac{1}{1+1} = \dfrac{1}{2}$. Halfway in between.
- I also sketched $y = \dfrac{u}{1+u}$, just to check my understanding. This is an increasing function of u for $u > -1$.



- Asymptotes are not mentioned on the TMUA Content Specification, but the idea is that, if $f(x)$ approaches a value for large $|x|$ (either very positive or very negative), then we can draw a horizontal dotted line with that value, if we want to. You've maybe met this idea for graphs like $y = \dfrac{1}{x}$ which we say has a horizontal asymptote at $y = 0$, thinking about the behaviour of y for very large x.
- The graph in this question has a horizontal asymptote at $y = 1$, thinking about the values when x is very large, and it also has a horizontal asymptote at $y = 0$, thinking about the values when x is very negative.
- The answer is A.

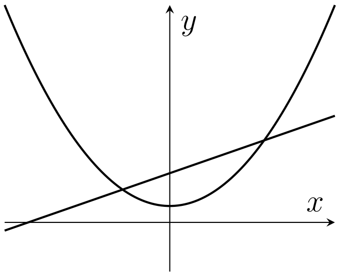
Extension

- Let $\mathrm{f}(x) = \dfrac{2^x}{1+2^x}$. Prove that $\mathrm{f}(-x) = 1 - \mathrm{f}(x)$.

  Interpret this as a symmetry of the graph of $y = \mathrm{f}(x)$.
- True or false? "If $\mathrm{f}(x)$ is increasing and $\mathrm{g}(x)$ is increasing, then $\mathrm{f}(\mathrm{g}(x))$ is increasing."
- True or false? "If $\mathrm{f}(x)$ is decreasing and $\mathrm{g}(x)$ is decreasing, then $\mathrm{f}(\mathrm{g}(x))$ is decreasing."

TMUA 2021 Paper 2 Question 13

- I sketched $y - x = 3$ and $y - x^2 = 1$ on the same axes.



- I had to think hard about the inequalities. Which region do I want? The region that's "below" both curves, I think, because I can re-write each of them as $y <$ [something].

- It's strange to me that y can go very negative; when I drew the curves I was expecting R to be the finite region bounded between the curves (possibly because I'm used to integration questions that ask me to find the area of such a region).

- The first inequality asks us whether the values of x are limited for every point in R. No, x can be very negative. At this point I went back to look at the inequalities very carefully, because this is precisely the sort of inequality that I'd get if R were the finite region between the curves. It's not though.

- II looks like a trap to me; I can see why you might think that this is true, by multiplying the inequalities together. But we know that this can go wrong, if both brackets are negative. Can I actually do that for the region R? Yes if I move away from the curves far enough, I think. Looking for a counter-example, I'll set $x = 0$ and then I'm looking for y such that $y < 3$ and $y < 1$ (to obey the given inequalities) and also $y^2 \geqslant 3$ (to contradict II). I'll set $y = -2$. One counter-example $(0, -2)$ is enough to disprove II.

- III asks whether the values of y are limited. No, y can be large. Again, this is the sort of inequality that I might get if R were the finite region between the curves. It's still not though.

- None of them are true.

- The answer is A.

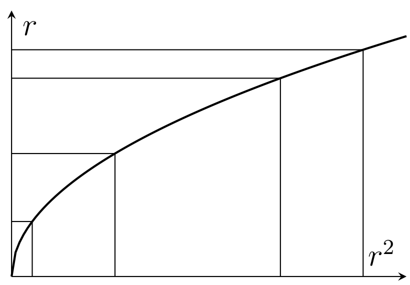
Extension

- Change $y - x^2 < 1$ to $y - x^2 > 1$. Which of I, II, III is/are true for every point in R?

- If I were using this as the start of an interview question, then I might build towards a sketch of graph of $(y - x)(y - x^2) = 3$. We might start by exploring how many points are on that graph for each value of x. For large x, what are the approximate values of y (in terms of x) for those points?

TMUA 2021 Paper 1 Question 17

- I'll want x^2+y^2 to be equal to $\dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$, or one of those plus 2π or 4π, or so on.
- Those are concentric circles (the centre is the origin) with different radii.
- Since $x^2+y^2=r^2$ for a circle, the radius is the square root of each of the numbers I listed above.
- These solutions for r^2 come in pairs, with a gap of $\dfrac{4\pi}{6}$ between the two solutions in the pair.
- Thinking about the radius r instead of r^2, I would like to take the square root of those numbers. What will happen to the gap between a pair of solutions?
- Because the graph of $y=\sqrt{x}$ grows at a slower rate for larger values of x, if we look at a larger pair of solutions for r^2, then the gap between the values of r will be smaller than it would have been for a smaller pair.



- So I'm expecting pairs of circles, with a gap between the pair that decreases the further from the origin I look.
- Options B and E have equally-spaced pairs.
- Option C doesn't seem to have pairs, or any change in the gaps.
- I have no idea what's going on with option D, but I don't like it.
- Option A seems to fit the description above.
- The answer is A.

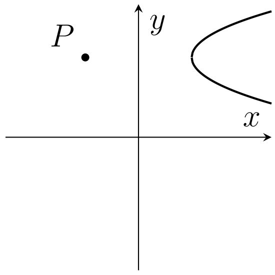
Extension

- Check your understanding; sketch the solutions to $\cos\left(x^2+y^2\right)=\frac{1}{2}$.
- If I were using this as the start of an interview question, then next I might switch to a different function entirely, and ask you about the graph

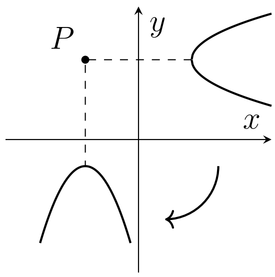
$$\left(x^2+y^2\right)^{1/2}\sin\left(x^2+y^2\right)=1.$$

TMUA 2021 Paper 1 Question 18

- I drew a graph. This is a "sideways" parabola where the roles of x and y have been swapped compared with the usual parabolas I've seen before. For some values of x there are no points on the graph, for some values there are two solutions for y from the quadratic formula, and for one value of x there's a single value of y on the graph.



- I can complete the square to find $x = (y-3)^2 + 2$, so the turning point (if we can call it that, the parabola is sideways!) is at $(2, 3)$.
- Nice, that turning point is on the same horizontal line as the point P.
- So after rotation the turning point will be directly below P, a distance of 4 away, the same distance as before the rotation.
- I drew a sketch of this. I've drawn a parabola for the new curve (partly because all the options are parabolas), and I've drawn it with a negative leading coefficient (pointing downwards).



- The leading term of the quadratic will probably be $-x^2$, looking at my sketch and looking at the options.
- Then, for the turning point to be at $(-2, -1)$, it will need to be $y = -(x+2)^2 - 1$.
- Expand that out for $y = -x^2 - 4x - 5$.
- The answer is B.

Extension

- Check your understanding; suppose that the original curve was $x = (y-2)^2 + 2$ instead.
- You don't have to know this, but in general the way to rotate a graph by 90 degrees clockwise **about the origin** is to replace x with $(-y)$ and, simultaneously, replace y with x. How can you use this fact to do this question, where the rotation is not about the origin?

TMUA 2022 Paper 1 Question 10

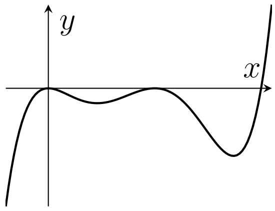
- A sequence of translations is just one big translation... or perhaps I should say one big translation parallel to the x-axis and one big translation parallel to the y-axis.
- So after all of that, the graph $y = x^3$ will be transformed to $y = (x - a)^3 + b$.
- That's everything that I can make with translations!
- I'd like to multiply that out and compare it to the graphs in the question. If they're not in that form, then I can't make them with my translations (if they are then I probably can, but I will be careful and make sure that I can solve for a and b)
- Multiplying out gives $y = x^3 - 3ax^2 + 3a^2x - a^3 + b$.
- So III is out straight away, because the coefficient of x^3 doesn't match.
- For I, to get the coefficients to match, we would need $-3 = -3a$ and $9 = 3a^2$ and also something involving -27 and b. But there's no value of a that satisfies both of those equations, so no translation will give this graph.
- For II, to get the coefficients to match, we would need $-9 = -3a$ and $27 = 3a^2$... so we're off to a good start because $a = 3$ works for both of those... and also $-3 = -a^3 + b$. I could solve for $b = 24$ if I wanted to.
- So I can get the graph of II (translate 3 units parallel to the x-axis and then 24 units parallel to the y-axis).
- The answer is C.

Extension

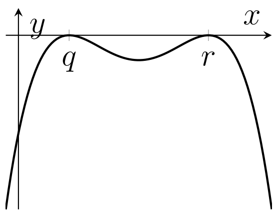
- An alternative method that I considered when I read this question was "look to see if there is a point with both $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$". Would this work? If yes, retry the question with this method. If not, refine this method and then retry the question.
- Find a necessary and sufficient condition involving A and/or B and/or C such that there is a sequence of translations that takes the graph $y = x^3$ to the graph
$$y = x^3 + Ax^2 + Bx + C.$$
- Find a sequence of translations and/or stretches parallel to the axes that takes the graph $y = x^3$ to graph III.
- Explain why there is no sequence of translations and/or stretches parallel to the axes that takes the graph $y = x^3$ to graph I.
- Find necessary and sufficient conditions involving A and/or B and/or C and/or D such that there is a sequence of translations and/or stretches parallel to the axes that takes the graph $y = x^3$ to the graph
$$y = Dx^3 + Ax^2 + Bx + C.$$

TMUA 2022 Paper 1 Question 18

- I started by sketching $y = \mathrm{f}(x)$. This polynomial has degree 5, and it has repeated roots at $x = 0$ and at $x = 1$. There is a root at $x = 2$. The values are positive for $x > 2$ only.



- I don't want to work out where the turning points are precisely, so I'll just draw something with the roots in the right place, and I'll try to remember that I haven't checked the size of the values of $\mathrm{f}(x)$ away from the x-axis.
- Next I sketched $y = \mathrm{g}(x)$. I can see that changing p will just stretch the graph parallel to the y-axis, so I'll ignore that for now. This polynomial has degree 4 and has repeated roots at $x = q$ and at $x = r$.



- I'm interested in places where the graphs cross. Thinking about $x < 0$, I'd like to know whether I'll get a crossing there. I know that $\mathrm{f}(0) = 0$ and $\mathrm{g}(0) < 0$, so the question is whether $\mathrm{f}(x)$ "catches up" and becomes more negative than $\mathrm{g}(x)$ as x gets very negative. This will happen, because the degree of f is larger.
- What is the "greatest number of distinct real solutions"? I suppose that the equation $\mathrm{f}(x) = \mathrm{g}(x)$ can be rearranged to a polynomial equation $\mathrm{h}(x) = 0$ where $\mathrm{h}(x) = \mathrm{f}(x) - \mathrm{g}(x)$. That has degree 5, so the equation has at most 5 real solutions.
- Can I choose q and r to guarantee four more distinct real solutions? Yes, if I make the graphs cross a lot!
- If I put the turning point of $\mathrm{g}(x)$ at q between $x = 0$ and $x = 1$, then I'll get two solutions, one between 0 and q, and one between q and 1. Similarly, if I put the other turning point of $\mathrm{g}(x)$ between $x = 1$ and $x = 2$, then I'll get two more solutions over there.
- This works for all p because, for any value of p, the graphs really will cross in those intervals. In the cases given for the other options, changing p might move the graph of $\mathrm{g}(x)$ away from the graph of $\mathrm{f}(x)$ in that region.
- The answer is B.

Extension

- For option C, find the minimum number of distinct real solutions as p changes.