Visualisation

General Advice

The questions on this worksheet are easier if you draw a graph or diagram or table to visualise the problem.

- If the question includes a function, you could draw a graph.
- If the question involves a positive whole number, you could consider small cases and track them in a table. Look at your table for patterns. If there are two positive whole numbers in the question, you could make a table where each row is labelled with a value of the first, and each column is labelled with a value of the second.
- There's a link between coordinate geometry and algebra, and you can use your intuition in both to get ideas about how to approach a question.

Warm-up

Alice climbs a mountain. At 09:00 she leaves base camp and begins a journey to the summit which might involve both uphill and downhill travel. At 17:00 she reaches the summit.

On the same day, Charlie is descending the same mountain. At 09:00 Charlie is at the summit. They take a different route from Alice. Charlie reaches base camp at 17:00.

Explain why, at some time during the day, Alice and Charlie were at the same altitude.

Questions

TMUA 2021 Paper 1 Question 9

Find the area enclosed by the graph of

$$|x| + |y| = 1$$

A $\frac{1}{2}$ **B** 1 **C** 2 **D** 4 **E** $\frac{\sqrt{2}}{2}$ **F** $\sqrt{2}$ **G** $2\sqrt{2}$

TMUA 2021 Paper 1 Question 14

This question uses radians.
Find the number of distinct values of x that satisfy the equation

$$(x+1)(3-x) = 2(1 - \cos(\pi x))$$

A 2 **B** 3 **C** 4 **D** 5 **E** 6 **F** 7

TMUA 2022 Paper 2 Question 12
Place the following integrals in order of size, starting with the smallest.

$$P = \int_0^1 2^{\sqrt{x}}\,\mathrm{d}x \qquad Q = \int_0^1 2^x\,\mathrm{d}x \qquad R = \int_0^1 \left(\sqrt{2}\right)^x\,\mathrm{d}x$$

A $P < Q < R$
B $P < R < Q$
C $Q < P < R$
D $Q < R < P$
E $R < P < Q$
F $R < Q < P$

MAT 2022 Q1J
The real numbers m and c are such that the equation

$$x^2 + (mx + c)^2 = 1$$

has a repeated root x, and also the equation

$$(x-3)^2 + (mx + c - 1)^2 = 1$$

has a repeated root x (which is not necessarily the same value of x as the root of the first equation). How many possibilities are there for the line $y = mx + c$?

(a) 0, (b) 1, (c) 2, (d) 3, (e) 4.

MAT 2023 Q1J
Let $\lfloor x \rfloor$ denote the largest whole number that is less than or equal to x.
For example, $\lfloor -\pi \rfloor = -4$.
A function $f(x)$ is defined as follows; if $0 < x < 2$ then

$$f(x) = \left(\frac{3}{4}\right)^{\lfloor \log_2(x) \rfloor}$$

and $f(x) = 0$ otherwise. Note that, for example, $f\left(\frac{1}{2}\right) = \frac{4}{3}$.

The value of $\int_0^2 f(x)\,\mathrm{d}x$ is

(a) 1, (b) 2, (c) 3, (d) 4, (e) 5.

Part of an Interview
Adapted from an interview question used by James Munro for Maths interviews at Oxford. Reproduced here with permission.

Let $f(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$ where a is a positive constant. Sketch $y = f(x)$ for $x > 0$.

Describe the behaviour of $f(x)$ when x is very small and when x is very large. For large x, if it's growing like a straight line with gradient m, what's the value of m? If there is a turning point on your graph, find the coordinates.

Next we're going to sketch $y = f(f(x))$ for $x > 0$, but without detailed calculation. For example, you should not substitute the expression for $f(x)$ into itself. Describe the behaviour of $y = f(f(x))$ when x is very small and when x is very large. Identify the turning point and explain how you know that it's a minimum without differentiating. Without further calculation, sketch $y = f(f(x))$.

Suppose I choose a large value of x and apply f repeatedly, by which I mean that I will calculate the sequence $f(x)$, $f(f(x))$, $f(f(f(x)))$, and so on. What do you think will happen to the values of that sequence? For instance, will they increase? Decrease? Get very close to zero? Get very close to some non-zero number? Get larger and larger without bound?

For $x > \sqrt{a}$, prove that $\left(f(x) - \sqrt{a}\right) < \frac{1}{2}\left(x - \sqrt{a}\right)$.

Discuss your result, thinking about the sequence above once more.