Creativity

General Advice

We often want to invent an example or counterexample, and this requires some ingenuity.

- The TMUA Content Specification has a section on proof and conjecture and counterexamples, reproduced below.
- Note that for a disproof by counterexample, a single counterexample is enough.
- The more experience you have with examples of things, the better.
- You can invent counterexamples by experimentation. Critically evaluate your creations.
- Try to push the boundary.

TMUA Specification (April 2025, Section 2 §"Mathematical Proof")

- Follow a proof of the following types, and in simple cases know how to construct such a proof:
  - Direct deductive proof ("Since A, therefore B, therefore C, ..., therefore Z, which is what we wanted to prove.");
  - Proof by cases (for example, by considering even and odd cases separately);
  - Proof by contradiction;
  - Disproof by counterexample.
- Deduce implications from given statements.
- Make conjectures based on small cases, and then justify these conjectures.
- Rearrange a sequence of statements into the correct order to give a proof for a statement.
- Problems requiring a sophisticated chain of reasoning to solve.

Warm-up

Is there a 3D shape with exactly 13 edges and exactly 8 vertices?

Hint: do you know any 3D shapes that have **either** 13 edges **or** 8 vertices?

Extension: for which integers E and V can you find a 3D shape with exactly E edges and exactly V vertices?

Questions

TMUA 2021 Paper 2 Question 4 (modified)

Consider this claim about positive whole numbers a and b and c:

> If a is a factor of bc, then a is a factor of b or a is a factor of c.

Find a counterexample where a and b and c are all even numbers.

Find a counterexample where a and b and c are all odd numbers.

Find a counterexample where a and b and c are all square numbers.

Find a counterexample with $a < b < c$ and a counterexample with $a > b > c$.

TMUA 2020 Paper 2 Question 4 (modified)

Consider this claim about the positive integer N:

> If N is greater than 6 then N can be written as the sum of two non-prime integers that are each greater than 1.

Find a counterexample to the claim.

Extension: find all counterexamples. Hence find the smallest integer that could replace "6" in the claim to make it true.

TMUA 2022 Paper 2 Question 3 (modified)

Consider the following claim about the positive integer n

> if n is prime, then $n^2 + 2$ is prime

For each integer n from 1 to 10 inclusive, decide whether it is a counterexample to the claim.

For each integer n from 1 to 10 inclusive, decide whether it is a counterexample to the converse of the claim.

TMUA 2020 Paper 2 Question 20 (modified)
For each of the following claims, find a function f such that the claim is true but the converse of the claim is not true.

- $x \geq 0$ **only if** $\mathrm{f}(x) < 0$
- $x < 0$ **if** $\mathrm{f}(x) \geq 0$
- $x \geq 0$ **only if** $\mathrm{f}(x) \geq 0$
- $\mathrm{f}(x) < 0$ **if** $x < 0$
- $\mathrm{f}(x) \geq 0$ **only if** $x \geq 0$

TMUA 2020 Paper 2 Question 10
The real numbers a, b, c and d satisfy both

$$0 < a + b < c + d$$

and

$$0 < a + c < b + d$$

Which of the following inequalities **must** be true?

I $\quad a < d$
II $\quad b < c$
III $\quad a + b + c + d > 0$

A none of them
B I only
C II only
D III only
E I and II only
F I and III only
G II and III only
H I, II and III

TMUA 2020 Paper 2 Question 18

In this question, $\mathrm{f}(x) = ax^3 + bx^2 + cx + d$ and $\mathrm{g}(x) = px^3 + qx^2 + rx + s$ are cubic polynomials. If $\mathrm{f}(x) - \mathrm{g}(x) > 0$ for every real x, which of the following is/are **necessarily** true?

I $a > p$

II **if** $b = q$ **then** $c = r$

III $d > s$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

Part of an Interview

Adapted from an interview question used by James Munro for Maths interviews at Oxford. Reproduced here with permission.

We're going to look for functions $y(x)$ that satisfy

$$x^3 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x \frac{\mathrm{d}y}{\mathrm{d}x} - y = 0.$$

This is called a second-order differential equation, because it's an equation involving a second derivative. If that first term weren't there, this would be a first-order differential equation, because it would still involve a first derivative.

Let's start by looking for simple functions $y(x)$ that work. Can you think of anything?

In general for differential equations of this form, if you have a solution $y_1(x)$ then you can try $y(x) = v(x) \times y_1(x)$, and you will get a first-order differential equation for $\frac{\mathrm{d}v}{\mathrm{d}x}$. Give this a go!

(You should substitute $y(x) = v(x) \times y_1(x)$ into the differential equation, where $y_1(x)$ is the solution you've already found, and $v(x)$ is an unknown function. There will be terms involving $\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$ and v but after simplifying, one of these disappears.)

(Depending on whether you've seen integrating factors or not, we might solve this first-order differential equation for $\frac{\mathrm{d}v}{\mathrm{d}x}$ and then integrate for v.)