Fluency – Solutions

Warm-up

- Our first idea might be to multiply out the square, because we're really good at that sort of thing. If you've done that a few times though, you know that it is a lot of work when there are more than two or three terms inside the bracket. So, thinking ahead to the page of algebra that will produce, I'm going to look for a different plan instead.

- Can I take a square-root of both sides instead? Almost, but not quite, because I need to remember that there are two numbers that square to 1. But I prefer this idea to the one above.

- I have $x^8 + 4yx^6 + 6y^2x^4 + 4y^3x^2 + y^4 = \pm 1$. You might prefer to write that out as two different cases, if you like.

- I'd like to simplify this, perhaps by taking out a common factor, but nothing (not even x^2) is a factor of each term. It is interesting to me that x only appears with even exponents, or said differently, as powers of x^2.

- Is this something involving powers of x^2 and powers of y? Yes it is, with decreasing powers of x^2 from left to right, and increasing powers of y from left to right. I've seen that before in the binomial expansion.

- Let's make that precise; what would the exponent be? The first term is $(x^2)^4$ so I suppose it would have to be 4.

- Are the numbers 1, 4, 6, 4, 1 the binomial coefficients for $(x^2 + y)^4$? This is one of those things that you either know or that you might assume based on the evidence so far!

- I have $(x^2+y)^4 = \pm 1$. I now see that the -1 case was a red herring, because the expression on the left is a square and cannot be negative.

- By similar logic to the above, the solutions to $(x^2 + y)^4 = 1$ come in two potential cases; $x^2 + y = 1$ or $x^2 + y = -1$.

- These are the parabolas $y = 1 - x^2$ and $y = -1 - x^2$. A quick sketch matches only one of the options.

- This was MAT 2018 Q1I, if you're interested.

- The answer is (c).

Questions

TMUA 2021 Paper 2 Question 15

- This is a fluency test for the various ways to write the equation of a circle, and the ways we can switch between algebra and geometry.
- We could complete the square to rewrite the equation as

$$\left(x+\frac{a}{2}\right)^2+\left(y+\frac{b}{2}\right)^2=\frac{a^2}{4}+\frac{b^2}{4}-c$$

- So the centre of the circle is $\left(-\frac{a}{2},-\frac{b}{2}\right)$ and the radius is $\sqrt{\frac{a^2}{4}+\frac{b^2}{4}-c}$, provided that the expression inside the square-root is positive.
- Then, thinking geometrically, the circle is tangent to the y-axis if and only if the radius is equal to the distance from the y-axis to the centre, so that the circle just reaches the y-axis but doesn't cross it.
- The distance is $\left|\frac{a}{2}\right|$, and I have to include the absolute value signs because the distance has to be positive.
- So my condition is $\left|\frac{a}{2}\right|=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}-c}$.
- That's not one of the options, but if I square both sides then I get $\frac{a^2}{4}=\frac{a^2}{4}+\frac{b^2}{4}-c$ which simplifies to $b^2=4c$.
- I should pause a moment to think about the effect of squaring both sides. Yes, it's OK because both terms are positive before I square.
- Alternatively, thinking algebraically, the circle is tangent to the y-axis if and only if the line $x=0$ meets the circle at exactly one point. That happens if the simultaneous equations $x=0$ and $x^2+ax+y^2+by+c=0$ have exactly one solution.
- Substituting the first into the second gives $y^2+by+c=0$, a quadratic with a single solution precisely when the discriminant b^2-4c is equal to zero.
- The answer is B.

Extension

- Without repeating all the work, find a necessary and sufficient condition for the circle to be tangent to the x-axis.
- Prove or disprove the following claim "If the circle is tangent to both axes, then $a=b$". Prove or disprove the converse of that claim.

TMUA 2020 Paper 2 Question 7

- This is a fluency test for the geometry content on various kinds of quadrilaterals.
- To show that a condition is individually sufficient, we must show that whenever that condition is true, the parallelogram is a square (ignoring the other conditions).
- This is quite tricky.
- On the other hand, to show that a condition is **not** individually sufficient, we just need to find a case where that condition holds and the parallelogram is not a square. One counter-example will do! That sounds like more fun.
- Let's look at condition I first.
- If the length of PQ is equal to the length of QR then, because opposite sides of a parallelogram are equal in length, all four sides will be equal in length. Are we done?
- No, because there are parallelograms that have all four sides the same length but that are not squares; I'm thinking of a rhombus. As a precise example, I'm thinking of the rhombus with 72° and 108° angles and side length 1. That exists, it's a parallelogram, it satisfies the condition, and it's not a square. So it's a counter-example.
- Condition II next. My rhombus has diagonals that intersect at right angles, and it's (still) not a square. It's a counter-example again!
- Condition III might catch me out, because my rhombus is not a counter-example here (it does not have equal angles). When you fail to find a counter-example, it's tempting to think that no counter-example exists.
- So let's try to prove that condition III **is** individually sufficient instead. What can we say? Opposite angles of a parallelogram are equal, so if $\angle PQR = \angle QRS$ then all four angles will be equal, so all the angles will be 90°. It's looking like a square.
- But wait a minute, we don't have any information about the side lengths. I can imagine another shape that has right-angles and pairs of opposite sides parallel. It's a rectangle! So although my first counter-example failed, I've now found a working counter-example.
- We've found counter-examples for each of the claims in bold along the top of the table.
- The answer is H.

Extension

- We've seen that none of the conditions are individually sufficient for the parallelogram to be a square. On the other hand, if all three of the conditions hold simultaneously, then it really is the case that the shape must be a square.

  So what happens if we assume exactly two of the conditions? Which pairs of the conditions is/are together sufficient for the parallelogram $PQRS$ to be a square?

TMUA 2022 Paper 2 Question 1

- This is a fluency test for differentiation and algebra. Before you differentiate, can you "see" what is going to happen? Can you spot the x-coordinate of one of the stationary points before you do the differentiation?
- We have
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 12x^3 + 12x^2 + 12x$$
- That's zero when $x = 0$, and it's zero if $x^2 + x + 1 = 0$. I've taken out the factor of 12 to make my life easier.
- The discriminant of that quadratic is negative, so there are no solutions.
- Alternatively, we could complete the square for $\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$, which is always positive.
- So the stationary point at $(0, -5)$ is the only one.
- The answer is B.

Extension

- Is the stationary point at $(0, -5)$ a local maximum or a local minimum?
- Sketch the curve.
- Explain why changing the leading coefficient from 3 to -3 would change the number of stationary points.

  Changing the sign of one of the other coefficients instead would change the number of stationary points. Which one?

  See if you can follow these sign changes through your work without re-doing the algebra.

MAT 2010 Q1F

- This is a fluency test for the trapezium rule. Do you know where the function will be evaluated when the trapezium rule is applied with n equal intervals?
- We know a rule for whether the trapezium rule produces an overestimate or an underestimate, but it's based on the second derivative, which is not something that I can work out for this function (and in fact, it is not defined at the "corners").
- Instead, let's think about any special cases we know of where the trapezium rule produces exactly the correct answer; perhaps we remember that this happens when the function is a straight line (note: that's sufficient but not necessary, see the Extension below)
- This function is made of straight line segments. Can we get the trapezium rule to produce exactly the correct answer in each section?
- The key idea for me is that the trapezium rule will treat each section of the function separately if the list of values where the function is evaluated includes all the values $\frac{1}{3}$, $\frac{1}{2}$, and $\frac{3}{4}$, because then we will have a separate set of trapeziums for each section of the function. They will line up perfectly with the line segment in that section, and give the correct answer for that section.
- Perhaps I should note that I haven't analysed the trapezium rule formula to get that insight; I know that the formula comes from the areas of a row of trapeziums, so I haven't had to worry about, for example, the factors of 2 in the formula. It'll just work.
- The equally-spaced points will include those fractions if the differences between them are all multiples of $\frac{1}{n}$. This requires n to be a multiple of 3 and a multiple of 4, so n must be a multiple of 12 for this perfect-alignment thing to happen.
- Technically I should now argue the converse, to show that if n is not a multiple of 12 then the trapezium rule does not give the correct answer, but...
  - ...in cases where $x = \frac{1}{3}$ is not one of the values used for the trapezium rule, I can visualise the trapeziums "cutting the corner" of the function, producing an overestimate, which is convincing enough for me.
  - ...if it works when n is a multiple of 4, then it works when n is a multiple of 12, and this question is only allowed to have one correct answer! This is a cheeky approach avoiding the spirit of the question, and it's why questions like this tend to have language like "which of these conditions is necessary and sufficient".
- The answer is (d).

Extension

- "If the function is a straight line then the trapezium rule gives the correct answer."

  Prove that the converse of this claim is not correct, by thinking about $\int_0^{2\pi} \sin x \, \mathrm{d}x$.

  Invent another counter-example, not involving $\sin x$.

MAT 2020 Q1I

- This is a fluency test for geometric series. Can you remember the convergence condition? Did you remember that there is such a thing as a convergence condition?
- The common ratio r is equal to $\dfrac{1}{\tan x}$ but that only satisfies the condition $-1 < r < 1$ if $\tan x < -1$ or $\tan x > 1$. So the sum doesn't converge (and can't equal $\tan x$) if we don't satisfy that condition.
- Note that $\tan x$ is not the ratio, and (confusingly) if $-1 < \tan x < 1$ then the series does **not** converge. This might surprise us later.
- If we satisfy that condition that either $\tan x < -1$ or $\tan x > 1$, then the sum of the geometric series to infinity is

$$\frac{1/\tan x}{1 - 1/\tan x} = \frac{1}{\tan x - 1}$$

- When I see something like that, I get a little worried about the possibility that $\tan x$ might be 1. In this case, that little worry helps me to remember the convergence condition, because the case where the formula is undefined ($r = 1$) is excluded by the convergence condition ($|r| < 1$).
- We must now try to solve the equation

$$1 = (\tan x)(\tan x - 1)$$

- This is a quadratic equation for $\tan x$, so it's tempting to write $u = \tan x$, rearrange to $u^2 - u - 1 = 0$ and conclude that $\tan x = \frac{1}{2}(1 \pm \sqrt{5})$.
- But we must go back and check our solution. One of those values has $-1 < \tan x < 1$, so the sum above would not converge.
- There is only one value of $\tan x$ that works. In the range $-90° < x < 90°$, that value for $\tan x$ corresponds to a single value of x (no tricks there!).
- The answer is (b).

Extension

- Sketch the graph of $y = \tan x$ and the graph of $y = \frac{1}{\tan x - 1}$ for $-90° < x < 90°$.
- Use Desmos to plot $y = \displaystyle\sum_{n=1}^{4} (\tan x)^{-n}$ and $y = \displaystyle\sum_{n=1}^{5} (\tan x)^{-n}$.

  [If you type "sum" then Desmos will create a big Σ, with fields above and below for you to enter the limits of summation.]

  Comment on the similarities and differences between those plots.

Part of an Interview

- Let's work out the derivative and see what we get.
- With $f(x) = ax^5 + bx^3 + cx$, the derivative is $f'(x) = 5ax^4 + 3bx^2 + c$.
- This is a quadratic in x^2, so I can complete the square. It's a little fiddly, but I get

$$f'(x) = 5a\left(x^2 + \frac{3b}{10a}\right)^2 + c - \frac{9b^2}{20a}.$$

- In what follows, I'm going to refer to $5a\left(x^2 + \frac{3b}{10a}\right)^2$ as the "first term" of this expression.
- This is a bit of a mess, and I don't know anything about the coefficients, so perhaps it's a good idea to look back at the original function $f(x)$. What happens when x is very large?
- If x is very large then the function will behave, more or less, like ax^5, because that term will be the largest by a long way, for large enough values of x. If a is negative then that's a decreasing function. So for the function to be increasing, we definitely need to have $a > 0$.
- This is helpful when we read the expression for $f'(x)$ above, because we have a square (positive) multiplied by a (positive) multiplied by 5 (positive). So that first part is definitely positive.
- Therefore, if $c - \frac{9b^2}{20a}$ is positive, then the whole expression will be positive and we have an increasing function.
- Notice that this condition is sufficient, but we haven't yet thought about whether it's necessary.
- In particular, we might go back to the original problem and spot that if a and b and c are all positive then the function is obviously increasing, and that includes cases like $y = x^5 + 1000x^3 + x$ where c is definitely not larger than $\frac{9b^2}{20a}$. What's going on here?
- Here's the key idea to make further progress; if $\frac{3b}{10a}$ is positive then the first term does not get as small as zero, so the rest of the terms can be a little bit negative and we will still have a positive derivative overall.
- How can we make that precise?
- We can calculate the minimum value of the first term, in the case that $\frac{3b}{10a} > 0$, and our condition will be that we want the sum of this minimum value and the rest of the terms to be positive.

- The minimum value of the first term turns out (rather nicely) to be $\frac{9b^2}{20a}$, so this condition on the sum of the terms is just simply $c > 0$.
- Just for completeness, let's note that if $\frac{3b}{10a}$ is not positive, then the first term could be zero for some value of x. That shows that, in this case, it is in fact necessary for the rest of the terms to be positive. So, if $\frac{3b}{10a} \leqslant 0$, then it is necessary and sufficient for $c > \frac{9b^2}{20a}$. Note that, if $a > 0$, then we can simplify the inequality $\frac{3b}{10a} \leqslant 0$ to $b \leqslant 0$. So this is just a condition on the sign of b.
- So we have two cases to think about; if $\frac{3b}{10a} > 0$ then we need $c > 0$, and in all other cases we need $c > \frac{9b^2}{20a}$.
- Our final answer is that the function is increasing if and only if both $a > 0$ and also either
  - $b > 0$ and $c > 0$, or
  - $b \leqslant 0$ and $c > \frac{9b^2}{20a}$.
- If, like me, you find it difficult to keep track of all these cases, you might find it helpful to draw a diagram. Let's assume that $a > 0$. Draw axes with b on the horizontal axis and c on the vertical axis. Identify the regions associated with:
  - the first condition we found; $c - \frac{9b^2}{20a}$ is positive.
  - the condition we found later by considering a case where the first term is never zero.
  - the conditions in the final answer.
- Depending on how the interview is going, we might think about functions with $f'(x) \geqslant 0$. This involves thinking carefully about the boundaries of various regions in your diagram.