Precision – Solutions

Warm-up

- The number a might be zero. As a specific counterexample, take $a = 0$, $b = 1$, $c = 2$. Then $ab = ac$ but $b \neq c$.
- The function $\sin x$ is not one-to-one. As a specific counterexample, take $A = \dfrac{\pi}{3}$ and $B = \dfrac{2\pi}{3}$. Then $\sin A = \sin B$ but $A \neq B$.
- Negative values of x square to the same values as positive values of x. As a specific counterexample, take $x = -1$ and $y = 1$. Then $\sqrt{x^2} = \sqrt{y^2}$ but $x \neq y$.
- Squaring a negative number gives something positive. As a specific counterexample, take $x = -1$. Then $x^2 > x$ but $x < 1$.
- The roots p and q might be equal (note that the statement is about solutions to the equation, and not about roots of a polynomial counted with multiplicity). As a specific counterexample, take $a = 2$ and $p = 3$ and $q = 3$. Then the equation $2(x-3)^2 = 0$ has only one real solution for x.
- This looks like a quadratic equation, except if $a = 0$. As a specific counterexample, take $a = 0$ and $b = 1$ and $c = -1$. Then $ax^2 + bx + c = 0$ is the equation $x - 1 = 0$ and this has only one real solution for x.
- N might be a multiple of 10 but not a multiple of 100. As a specific counterexample, take $N = 30$. Then $N^2 = 900$ is a multiple of 100, but N is not a multiple of 100.
- The integer $n = 41$ is a counterexample because $41^2 + 41 + 41$ is 41×43.
- The number a might be 1. As a specific counterexample, take $a = 1$ and $x = 2$ and $y = 3$. Then $a^x = a^y$ but $x \neq y$.

Questions

TMUA 2021 Paper 2 Question 10

- A counterexample is a value of n such that both n is a prime and the conclusion (u_n is a multiple of 3 or u_n is a multiple of 5) is false.
- Note that the conclusion is false if and only if both u_n is not a multiple of 3 and u_n is not a multiple of 5.
- We're looking for the smallest value of n that's a counterexample, so let's start with the smallest option and work our way up.
- 1 is not prime, so it is not a counterexample to this claim (note that the actual value of u_1 is simply irrelevant).
- 2 is prime, so we look at $u_2 = 21$ and we look at the conclusion of the statement (u_2 is a multiple of 3 or u_2 is a multiple of 5). In this case the conclusion is true, so $n = 2$ is not a counterexample.
- 3 is prime, so we look at $u_3 = 30$ and we look at the conclusion of the statement (u_3 is a multiple of 3 or u_3 is a multiple of 5). In this case the conclusion is true; remember that the word "or" is inclusive for TMUA, so it's not a problem that 30 is both a multiple of 3 and a multiple of 5.
- 4 is not prime, so we move on.
- 5 is prime, so we look at $u_5 = 44$ and we look at the conclusion of the statement (u_5 is a multiple of 3 or u_5 is a multiple of 5). In this case the conclusion is not true. We've found a counterexample, and we know it's the smallest one because we've been working systematically, so we're done.
- In case you care, 6 is not a counterexample (because 6 is not prime) and 7 is a counterexample (but not the smallest).
- The answer is E.

Extension

- I've read the statement as

  "If (n is a prime number), then (u_n is a multiple of 3 or u_n is a multiple of 5)".

  An alternative way to read the statement could be

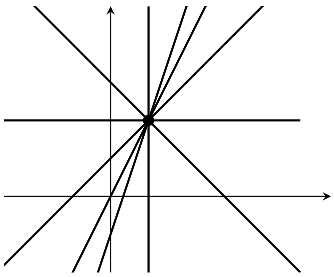
  "(If n is a prime number, then u_n is a multiple of 3) or (u_n is a multiple of 5)."

  Prove that these two interpretations of the statement are equivalent.

  You might find it helpful to write A for the statement "n is a prime number", B for the statement "u_n is a multiple of 3", and C for the statement "u_n is a multiple of 5". You could work out, in terms of A, B, and C, exactly when each statement is false.

TMUA 2022 Paper 2 Question 5

- The converse of P is "**if** the x-intercept of L is positive, **then** the y-intercept of L is negative".
- The contrapositive of P is "**if** the x-intercept of L is not positive or does not exist, **then** the y-intercept of L is not negative or does not exist".
- I've been pedantic there. The way the question is written makes it sound like all lines have an x-intercept (because it says "the" x-intercept as if there's always one), but that's not true. The x-intercept of a horizontal line does not exist.
- Remember that the contrapositive of P is true if and only if the original statement P is true. This is good news, because it means that we don't actually have to think about the statement of the contrapositive above, unless we really want to.
- Here's a diagram of some lines through $(1, 2)$. I've tried to draw lines with various x-intercepts and various y-intercepts, because I'm trying not to miss any cases.



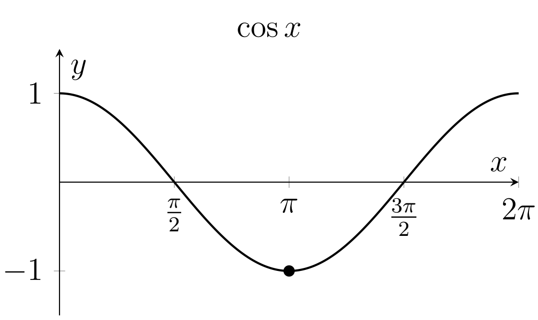
- First let's think about the statement P. I've got a line on my diagram with a negative y-intercept. That one does have a positive x-intercept, but will this always happen?
- Looking at the diagram, I can say a little more than just "positive x-intercept"; it looks like the x-intercept is between 0 and 1. What happens between 0 and 1? The line changes from a negative value at the y-intercept to a positive value at $(1, 2)$. So there's a point in between where the line has zero value, and that's the x-intercept. I've convinced myself that statement P is true.
- Now let's think about the converse of P. I've got a few lines on my diagram with positive x-intercept; the one from before, a vertical one, and one with negative gradient. The one with negative gradient doesn't have negative y-intercept, so it's a counterexample to the converse of P. To be precise, that line is $y = 3 - x$.
- The contrapositive of P is true because the statement P is true.
- The answer is F.

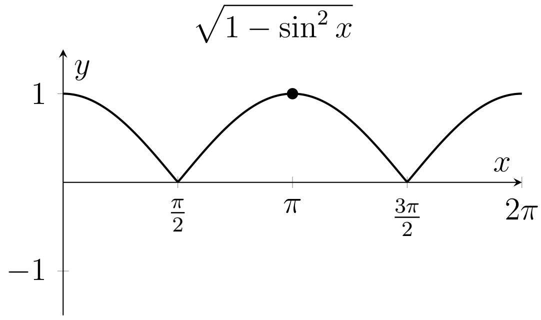
Extension

- The point $(1, 2)$ is fixed for this question. Consider what would happen if it was a different point (for example, perhaps $(1, 3)$ or $(-2, -3)$). Would that change your answer to the question?

TMUA 2021 Paper 2 Question 5

- The first line looks fine, and the last line is definitely nonsense, so something's gone wrong.
- We could work from the first line downwards until we find a mistake. I thought it would be fun to work backwards instead.
- Line D makes a claim for all values of x. Substituting $x = \pi$ gives $(1-1)^2 = \left(1 + \sqrt{1-0^2}\right)^2$, which is nonsense, but it is precisely the nonsense in line E.
- What happens if we substitute $x = \pi$ in the other lines?
- Line C would read $1 - 1 = 1 + \sqrt{1-0^2}$. That's also wrong, so I think there's a mistake somewhere between line A and line C (inclusive).
- Line B would read $-1 = \sqrt{1-0^2}$. That's wrong, and strongly suggests that some funny business with square roots has been going on.
- Line A is obviously fine; for $x = \pi$ it would be $0^2 + (-1)^2 = 1$.
- So I suspect that the square roots are the problem. The first one appears in line B. Is that the mistake?
- Yes, it looks like someone has rearranged and taken square roots of both sides. But that doesn't work, because $\cos x$ might be negative (for example, when $x = \pi$), whereas the square root is never negative.





- The answer is B.

Extension

- Fix line B by writing $|\cos x| = \sqrt{1-\sin^2 x}$ and follow this correction through the rest of the lines. Convince yourself that there are no further errors. You should find that substituting $x = \pi$ gives $4 = 4$, a result that I'm much happier with.
- Write your own version of this question by writing out an argument with an error, perhaps using one of the statements from the Warm-up for inspiration.

TMUA 2022 Paper 2 Question 6

- We're being asked to think about the statements "if P then Q" and "if Q then P".
- If both are true, we'll select A. If neither are true, we'll select D.
- If we find that "if P then Q" is true but the other one isn't, we'll select C.
- If we find that "if Q then P" is true but the other one isn't, we'll select B.
- They hardly ever come up on the test, but median, mean, and mode are on the TMUA Content Specification, in the "M" section that lists GCSE content. To find the median, we would sort the list and find the item in the middle (if there are an odd number of items) or the two items at the middle (if there are an even number of items). In the first case, the median is the middle item, and in the second case the median is the mean of the two items at the middle.
- So if n is odd, then the median will be one of the items in the list, by the definition of the median.
- What about "if Q then P"?
- I can't think of a way to prove this, so I'm going to look for a counterexample. A counterexample would be any list of n integers where the median is one of the numbers in the list, and where n is even.
- I like simple examples, so I'll think about $n = 2$. A list of two integers has median equal to the mean of those integers, which feels like it will be somewhere in-between. Can I make that in-between number be equal to one of the items in the list? Yes, if they're equal. So my counterexample is the list 1, 1 with median 1.
- I've decided that P is sufficient for Q, but P is not necessary for Q.
- The answer is C.

Extension

- Suppose that the question instead said "A list consists of n integers, not all the same" with no changes to statements P and Q. Does that change the answer?

MAT 2021 Q1J

- There are four sides, so we should write down four expressions for the side lengths and start setting them equal to each other to see what we get.

$$|AB| = \sqrt{(b-a)^2 + (c-b)^2} \qquad |BC| = \sqrt{(c-b)^2 + (d-c)^2}$$
$$|CD| = \sqrt{(d-c)^2 + (a-d)^2} \qquad |DA| = \sqrt{(a-d)^2 + (b-a)^2}$$

- There are six necessary conditions that we could form by setting any two of these equal to each other. The following set of three conditions are (together) also sufficient for all four sides to be equal.

$$|AB| = |BC| \quad \text{and} \quad |BC| = |CD| \quad \text{and} \quad |CD| = |DA|.$$

  That's because we can infer from those three equalities that any pair of sides are equal.

- Those conditions are (after squaring both sides)

$$\begin{aligned}(b-a)^2 + (c-b)^2 &= (c-b)^2 + (d-c)^2 \quad \text{and}\\ (c-b)^2 + (d-c)^2 &= (d-c)^2 + (a-d)^2 \quad \text{and}\\ (d-c)^2 + (a-d)^2 &= (a-d)^2 + (b-a)^2.\end{aligned}$$

- Notice that each of these simplifies (that's why I picked these three), and in fact the first and third simplify to the same thing. So my conditions are now

$$(b-a)^2 = (d-c)^2 \quad \text{and} \quad (c-b)^2 = (a-d)^2.$$

- The first condition is equivalent to "$b - a = d - c$ or $b - a = c - d$".

  The second condition is equivalent to "$c - b = a - d$ or $c - b = d - a$".

- There are four combinations to check.
  - If $b - a = d - c$ and $c - b = a - d$ then $a = c$
  - If $b - a = d - c$ and $c - b = d - a$ then $a = b$
  - If $b - a = c - d$ and $c - b = a - d$ then $b = c$
  - If $b - a = c - d$ and $c - b = d - a$ then... these are the same as each other!

- We're told that the numbers are distinct, so only the last combination is interesting to me.

- The answer is (d).

Extension

- Prove that if $a - b + c - d = 0$ then $ABCD$ is a square.

Part of an Interview

- Perhaps your intuition is that starting with the small number on the left is the best way to go. Let's try that.
- Replacing $1+2$ with 3 costs £6 and the sum is now $3+3+4+5$.
- Replacing $3+3$ with 6 costs £54 and the sum is now $6+4+5$.
- I should say at this point that during the interview I always helped the interviewee by using a calculator myself to do all the sums for them.
- There are two things you might want to do next; keep working from the left, or add $4+5$ because those are now the smallest numbers. I'm going to keep working from the left.
- Replacing $6+4$ with 10 costs £240 and the sum is now $10+5$.
- Finally, replacing $10+5$ with 15 costs £750, and we're done. The total cost was £1050.
- Alternatively, perhaps your intuition is that starting with the large numbers on the right is the best way to go. Let's try that.
- Replacing $5+4$ with 9 costs £180. Replacing $3+9$ with 12 costs £324. Replacing $2+12$ with 14 costs £336. Finally, replacing $1+14$ with 15 costs £210. The total cost was £1050.
- At this point, you might have a hypothesis in mind; maybe it doesn't matter what order you add them in, the sum will always be £1050.
- I might ask whether you think something similar would be true for other sums (on no evidence so far!). Our general hypothesis might be "the order of operations never matters, the total cost just depends on the numbers".
- This is the "real" fact that I wanted to talk about, hiding behind the particular case in the question I originally stated.
- One way to approach this is to go to smaller cases first.
- What happens if my computer adds three numbers; $a+b+c$ perhaps.
- For the sum $a+b+c$, we might add $a+b$ first and then add c, for a total cost of £$(ab(a+b)+(a+b)c(a+b+c))$.
- Or we might add $b+c$ first and then add a, for a total cost £$(bc(b+c)+(b+c)a(a+b+c))$.
- We might even add $a+c$ first and then add b, for a total cost £$(ac(a+c)+(a+c)b(a+b+c))$.
- After some work, we find that these are equal. They're all

$$\pounds\left(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2+2abc\right).$$

- Where have we seen terms like that before?

- Come to think of it, with just two items to add, the cost is $a^2b + ab^2$. Where have we seen terms like that before?
- If you said binomial expansion, good, but make that precise! What's the exponent?
- $a^2b + ab^2 = \frac{1}{3}\left((a+b)^3 - a^3 - b^3\right)$. We might guess that the expression with a and b and c in it simplifies to $\frac{1}{3}\left((a+b+c)^3 - a^3 - b^3 - c^3\right)$ and in fact this is true.
- We might even guess that with five numbers the total cost is something like

  $$\frac{1}{3}\left((a+b+c+d+e)^3 - a^3 - b^3 - c^3 - d^3 - e^3\right)$$

  and we would be pleased to note that this gives the correct number for the case $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$ that we had before (don't worry, I've still got my calculator to help!).
- But what's going on? Why do these formulas work?
- We could prove this with mathematical induction, but it's a little fiddly. Here's a slightly different way to think about the problem.
- My computer "values" numbers using the function $f(x) = \dfrac{x^3}{3}$. To calculate costs, it works out how much the answer is worth, and gives me a discount for each of the numbers that are being added. Adding a and b costs £$(f(a+b) - f(a) - f(b))$.
- Therefore, throughout the process, the sum of the values of the numbers minus the amount I've been charged so far remains constant.
- Whenever I do a sum, it will increase the value of the numbers that I've got, but it will also increase the amount that I've been charged. At the end of the calculation, I can infer what my total bill must be by looking at the value of my final answer and the values of the starting numbers.
- Depending on how the interview is going, we might talk about a less expensive computer; this one only charges me £$a \times b$ to add a and b. What can you say in this case?