

## Versatility – Solutions

### Warm-up

- Let's see what happens if we continue the student's method. They've got to

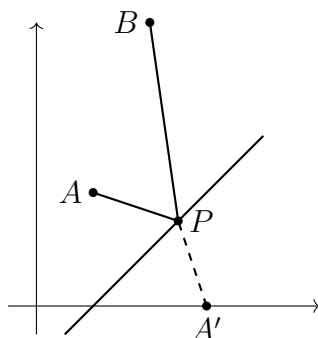
$$\frac{x-2}{\sqrt{(x-1)^2 + (x-3)^2}} + \frac{x-4}{\sqrt{(x-2)^2 + (x-6)^2}} = 0.$$

- So now it would probably make sense to move a term to the right-hand side, square both sides, and multiply up. But let's be careful, because squaring both sides of an equation can introduce additional solutions.

- This gives

$$(x-2)^2 ((x-2)^2 + (x-6)^2) = (x-4)^2 ((x-1)^2 + (x-3)^2)$$

- This looks like a mess, but I can see that the  $x^4$  term will cancel. Multiplying everything out and bringing the remaining terms to the left reveals that in fact the  $x^3$  term cancels too, and it all simplifies to the quadratic  $3x^2 - 8x = 0$ . The value  $x = 0$  isn't the one I want (it doesn't satisfy the equation that we had before we squared), so I'll take  $x = \frac{8}{3}$ . Then  $y = \frac{5}{3}$  and the minimum value of the original function is  $\sqrt{\frac{26}{9}} + \sqrt{\frac{4}{9} + \frac{100}{9}} = \sqrt{26}$ .
- Alternatively, we could interpret the question geometrically. The question wants the point on the line  $y = x - 1$  that minimises the sum of the distances to  $(1, 2)$  and  $(2, 5)$ . Let's write  $A$  for the point  $(1, 2)$ ,  $B$  for the point  $(2, 5)$ , and  $P$  for the unknown point on the line.



- This problem is not easy! Let's use an idea from geometry; imagine reflecting the point  $A$  in the line  $y = x - 1$ , as if the line  $y = x - 1$  is a mirror, to give a new point  $A'$  on the other side of the line. Then the distance  $|A'P|$  will be the same as the distance  $|AP|$ . Then the key idea is that the sum  $|A'P| + |BP|$  will be minimised when the segments  $A'P$  and  $BP$  form a straight line; the quickest route from  $A'$  to  $B$  is a straight line.
- So we need to find the mirror image  $A'$ , and then we need to find the straight line through  $A'$  and  $B$ . I find that  $A'$  is at  $(3, 0)$  and the line is  $y = 15 - 5x$ . This intersects  $y = x - 1$  at  $x = \frac{8}{3}$ ,  $y = \frac{5}{3}$ . As above, the original function has value  $\sqrt{26}$ .

## Questions

### TMUA 2020 Paper 1 Question 19

- It's tempting to just evaluate the quadratic at each of the six values and find out whether it's positive.
- This is lots of work though;
  - $26^2 - 52 \times 26 - 52 = -728$
  - $27^2 - 52 \times 27 - 52 = -727$
  - $51^2 - 52 \times 51 - 52 = -103$
  - $52^2 - 52 \times 52 - 52 = -52$
  - $53^2 - 52 \times 53 - 52 = 1$
  - I can stop now!
- Instead of persisting through all that, perhaps we should rewrite the quadratic to make it easier to evaluate.
- One way to do this is to complete the square for  $(x - 26)^2 - 26^2 - 52$ .
- This is a bit of a pain, because I don't know  $26^2$  and I don't want to work it out.
- Perhaps I should use the difference of two squares to simplify those first two terms. Said differently, I could have just factored the  $x$  out of the first two terms of the original quadratic.
- The expression is  $x(x - 52) - 52$ .
- This is much easier to work with!
- I can see that the expression will definitely be negative for  $0 \leq x \leq 52$  because  $x(x - 52)$  will be less than or equal to zero. When  $x = 53$ , the expression will be  $53 \times (1) - 52 = 1$ , which is positive.
- The answer is E.

### Extension

- It's a good thing that we didn't try to find the roots of the quadratic  $x^2 - 52x - 52 = 0$ , because they're not very nice numbers. As a challenge, find the exact value of the larger root of the quadratic and prove that it's between  $x = 52$  and  $x = 53$ .
- Suppose that we have some polynomial of degree  $n$  and we would like to evaluate it at some horrible value of  $x$ . Explain how you can do this in a way that involves multiplying by  $x$  exactly  $n$  times (so you cannot work out  $x^n$  and then  $x^{n-1}$  and so on).  
If you're interested in this, you can look up "Horner's method" online, but you've probably just invented it yourself.

**TMUA 2021 Paper 1 Question 16**

- We're told the third term in ascending powers is  $105x^2$ . That looks a little odd to me, because the exponent is 2 not 3, but this is just because the first term in ascending powers of  $x$  would be the term that doesn't involve  $x$  at all (or, said differently, the term involves  $x^0$ ). Careful not to be out-by-one when counting!
- The third term would be  $\binom{n}{2}a^{n-2}b^2x^2$ .
- Similarly, the fourth term would be  $\binom{n}{3}a^{n-3}b^3x^3$ .
- Now we're told about the fourth term in descending powers of  $x$ . Before I write out an equation for this, I'm struck by the fact that it's  $210x^3$  again. What's going on here?
- The fourth term in ascending powers of  $x$  has the same coefficient as the fourth term in descending powers of  $x$ . But, more to the point, it has the same power of  $x$ .
- We must be looking at the middle term in the expansion.
- So there are, in total, seven different powers of  $x$  in this expansion. So  $n = 6$ ... careful not to be out-by-one when counting!
- This is really helpful for my equations above. I need to know expressions for the binomial coefficients

$$\binom{n}{2} = \frac{n(n-1)}{2} \quad \text{and} \quad \binom{n}{3} = \frac{n(n-1)(n-2)}{6}$$

so that I can work out  $\binom{6}{2} = 15$  and  $\binom{6}{3} = 20$ .

- I have

$$a^4b^2 = 7 \quad \text{and} \quad a^3b^3 = \frac{21}{2}$$

- We're asked for  $\left(\frac{a}{b}\right)^2$ . That's a bit unusual, but we can work it out from the equations. We could try to solve for  $a$  and  $b$ , but I don't think that we need to. Instead, let's divide the first by the second to get  $\frac{a}{b}$  on the left, and then we'll be almost done. I have

$$\frac{a^4b^2}{a^3b^3} = \frac{7}{\frac{21}{2}}, \quad \text{so} \quad \frac{a}{b} = \frac{2}{3}$$

- Then the square of this is  $\frac{4}{9}$ .
- The answer is B.

**Extension**

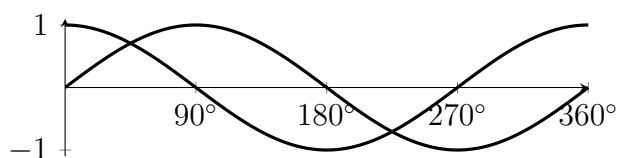
- Using your solution, find the third term, in **descending** powers of  $x$ , of the expansion.
- We have three unknowns ( $a$  and  $b$  and  $n$ ) and we've been given three bits of information, so we might hope that we can solve any version of this question. Investigate this by choosing your own values of  $a$  and  $b$  and  $n$  (not too large!), and calculating the expansion. Then select three terms from your expansion, try to forget  $a$  and  $b$  and  $n$ , and see if you can deduce them from your selected terms. Can you always do this? What might go wrong?

**TMUA 2021 Paper 2 Question 19**

- I could do some algebra, perhaps moving one term to the other side and squaring both sides. I know that's dangerous though, because an equation like  $x = 1 - 2x$  squares to the same thing as  $x = 2x - 1$ , even though they have totally different solutions.
- Instead, let's try to simplify first. It's odd that the question has two square-roots multiplied together. You don't see that very much. Why not? Well, because we can simplify it. If  $a \geq 0$  and  $b \geq 0$  then  $\sqrt{a}\sqrt{b} = \sqrt{ab}$ , and that's normally a nicer thing to write down.
- So I can simplify  $\sqrt{1 + \sin \theta}\sqrt{1 - \sin \theta}$  to  $\sqrt{1 - \sin^2 \theta}$ . The next step is where I might make a mistake.
- I might spot that  $1 - \sin^2 \theta = \cos^2 \theta$  and then I might want to write  $\sqrt{1 - \sin^2 \theta}$  as  $\cos \theta$ . Not so fast!
- Remember that  $\sqrt{x^2}$  is not equal to  $x$  if  $x < 0$ . Instead of writing  $\cos \theta$ , I must write  $|\cos \theta|$ .
- So, remembering this fact about square-roots, the equation simplifies to

$$\sin \theta |\cos \theta| + \cos \theta |\sin \theta| = 0$$

- Whenever I have absolute value signs, I like to think about separate cases depending on whether the expression(s) inside are positive or negative.
- In this case, if  $\sin \theta$  and  $\cos \theta$  are both positive, then I'll have the sum of two positive terms. That can't be zero.
- Similarly, if  $\sin \theta$  and  $\cos \theta$  are both negative, then I'll have the sum of two negative terms. That can't be zero.
- I want exactly one of  $\sin \theta$  and  $\cos \theta$  to be positive, and the other to be negative. Or for one of them to be zero, I suppose (because then both terms will be zero).
- Time to draw graphs of  $\sin \theta$  and  $\cos \theta$  to find out when this happens.



- I want the range  $90^\circ$  to  $180^\circ$  inclusive, and the range  $270^\circ$  to  $360^\circ$  inclusive. That's 182 integer values in total.
- The answer is F.

**Extension**

- Check your understanding; replace the  $+$  in the question with a  $-$ . Which values of  $\theta$  satisfy this new equation?

**TMUA 2022 Paper 1 Question 16**

- The quadratic equation gives

$$x^2 = \frac{3 \pm \sqrt{2}}{7}$$

- If  $x = \cos \theta$  then we're looking at  $\cos^2 \theta$ , so  $\sin^2 \theta$  will be 1 minus this. Similarly for  $\beta$  instead of  $\theta$ .
- So we're looking for a new polynomial where the roots are

$$x^2 = 1 - \frac{3 \pm \sqrt{2}}{7} = \frac{4 \mp \sqrt{2}}{7}$$

- I've used the  $\mp$  symbol there to keep track of the fact that the root with the  $+$  sign corresponds to the root for  $x^2$  with the  $-$  sign in the original polynomial, but this is not very important.
- The key observation is that these would be the roots of some quadratic for  $x^2$ , and if we can work out the coefficients then we're done.
- All the options have leading coefficient 7, so let's work with that. The denominator for the quadratic formula is supposed to be  $2a$ , so I'll multiply both numerator and denominator by 2 to get  $\frac{8 \mp \sqrt{2}}{14}$ . Can I just take  $b = -8$  and then work out  $c$ ?
- I'd need  $b^2 - 4ac = 8$ , with  $b = -8$  and  $a = 7$ . I can do this if I take  $c = 2$ . So my quadratic for  $x^2$  would be  $7x^4 - 8x^2 + 2 = 0$ .
- Alternatively, finish solving the original polynomial for  $x = \pm \sqrt{\frac{3+\sqrt{2}}{7}}$ , or  $x = \pm \sqrt{\frac{3-\sqrt{2}}{7}}$ , then calculate  $\sin \theta$  and  $\sin \beta$  from these values of  $\cos \theta$  and  $\cos \beta$ , and then multiply out  $(x - \sin \theta)(x + \sin \theta)(x - \sin \beta)(x + \sin \beta) = 0$ . You'll get  $x^4 - \frac{8}{7}x^2 + \frac{2}{7} = 0$ .
- The answer is B.

**Extension**

- Explain why a polynomial of degree four with roots  $\pm \cos \theta$  and  $\pm \cos \beta$  will never have an  $x^3$  term or an  $x^1$  term.

Explain why replacing  $x^2$  with  $1 - x^2$  in such a polynomial will always give a new polynomial with roots  $\pm \sin \theta$  and  $\pm \sin \beta$ .

Check that replacing  $x^2$  with  $1 - x^2$  in  $7x^4 - 6x^2 + 1 = 0$  gives  $7x^4 - 8x^2 + 2 = 0$ .

## MAT 2019 Q11

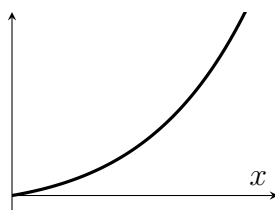
- I might try to rearrange this for

$$\frac{x}{y} = 2^{y-x}.$$

- I could even take logarithms of both sides for

$$\log_2 x - \log_2 y = y - x.$$

- But now what?
- Instead, let's think about the function  $f(x) = x2^x$ . Perhaps that's not the best choice of variable, because I'd like to say that I'm evaluating that function at  $x$  and at  $y$ , as two different inputs. If it helps, imagine rewriting the question to use  $a$  and  $b$  with  $0 < a < b$  and  $a2^a = b2^b$ . Then I'm looking at  $f(a) = f(b)$ .
- Does this ever happen? I should probably try to sketch  $f(x)$  and see if it repeats any values.



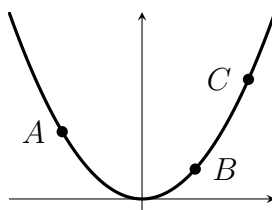
- Doing this, I realise that the function does not repeat any values, because  $f(x) = x2^x$  is an increasing function for  $x > 0$ . It's the product of two positive increasing functions.
- Since we're told that  $x < y$  (no possibility of  $x = y$ ), we conclude that  $x2^x = y2^y$  simply never happens.
- The answer is (a).

## Extension

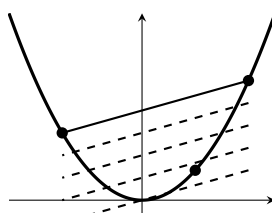
- True or false? "If  $f(x)$  is increasing and  $g(x)$  is increasing, then  $f(x)g(x)$  is increasing."
- True or false? "If  $f(x)$  is decreasing and  $g(x)$  is decreasing, then  $f(x)g(x)$  is decreasing."
- Compare and contrast with the similar Extension question on the Graphs worksheet following TMUA 2020 Paper 2 Question 5.

## Part of an Interview

- It's helpful to draw a diagram.



- It's tempting to write out an expression for the area in terms of  $a$  and  $b$  and  $c$ , perhaps with the hope of differentiating it with respect to  $b$ . This is really messy if you use Heron's formula, it's a bit of a pain if you use the shoelace formula, and it's more or less a non-starter if you want to work out any angles. If you want to check your answer, the area simplifies down to  $\frac{1}{2}(b-a)(c-a)(c-b)$  provided  $a < b < c$ .
- Instead of doing all that, let's think about how moving the point  $B$  changes the area of the triangle. The points  $A$  and  $C$  stay fixed, so it makes sense to think of that as the "base" of the triangle, with the different positions for  $B$  changing the "perpendicular height" of the triangle. It's perhaps a little odd to call  $AC$  the base, because it's the top of the triangle, but the formula  $\frac{1}{2}bh$  still works!
- At this point, you might like to do some algebra to find the perpendicular distance from  $B$  to the line  $AC$ , aiming to maximise this by differentiating. That's still a bit messy, so let's think more about where this will happen.
- I'll draw on some lines that show the "height" of the triangle  $ABC$ , like a height chart. I'll draw dashed lines that are parallel to  $AC$ .



- I want to put  $B$  as far from the line  $AC$  as possible, so I should put it at a point where one of those dashed lines is tangent to the curve. So I should find the point where the gradient of the curve matches the gradient of the line  $AC$ .
- This algebra is easy, just set  $b$  to be the value of  $x$  where  $2x = \frac{c^2 - a^2}{c - a}$ . This simplifies to

$$b = \frac{a + c}{2}$$

- The answer turned out to be the arithmetic mean of  $a$  and  $c$ ; perhaps that's what you would have guessed!

- Depending on how the interview is going, we might repeat the calculation with  $y = x^4$  instead of  $y = x^2$ . The general formula for the area is horrible in this case, so you should probably use this gradient method. The answer is not the arithmetic mean of  $a$  and  $c$  in this case. Perhaps we could say that this second example proves that we can't just guess (sadly).
- The approach we've used above is foreshadowing for the method of Lagrange multipliers, where you maximise a target function subject to some constraint function by choosing a point where the gradients align. This is something we teach to first-year students on the Oxford Mathematics degree, and I wouldn't expect people to have seen it before an interview.
- The first year of the Oxford Mathematics degree also contains something called the Mean Value Theorem, which guarantees that for any two points on a smooth curve, there's a point in between where the gradient of the curve matches the gradient of the chord between the two points. (Note: "smooth" has a technical definition that is a bit stronger than I need here, but I'm hoping that you'll read it simply as a condition on the sorts of curve that we're talking about. Curves like  $y = |x|$  are not covered by the Mean Value Theorem).