A hydrodynamic model of flocking type: BV Weak Solutions and Long-Time Behavior

Cleopatra Christoforou

University of Cyprus

PDE WORKSHOP: Stability Analysis for Nonlinear PDEs
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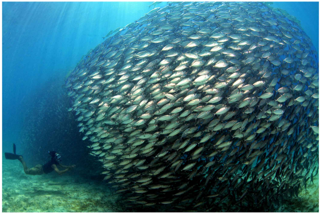
Joint work with Debora Amadori (Univ. of L'Aquila)

Outline

Self-organized systems






Biology: flocking of birds, swarming of insects, fish schools,. . .

Traffic Dynamics: crowds, cosmology. . .

Social science: social networks, opinion formation, linguistics,. . .

from Google-Image

Emergent behavior

Models of self-organized systems describe the dynamics of objects:

$$\mathrm{x}_i(t) \in \Omega \subset \mathbb{R}^d, \quad i = 1, \ldots, N, \quad \mathrm{v}_i = \dot{\mathrm{x}}_i$$

Long-Time dynamics $\Rightarrow$ Time-Asymptotic Flocking

- **alignment:** $\lim_{t\to\infty} \max_{i,j} |\mathrm{v}_i - \mathrm{v}_j| = 0$
- **bounded diameter:** $\sup_{t>0} |\mathrm{x}_i - \mathrm{x}_j| \leq D \quad D > 0, \ \forall\, i, j$

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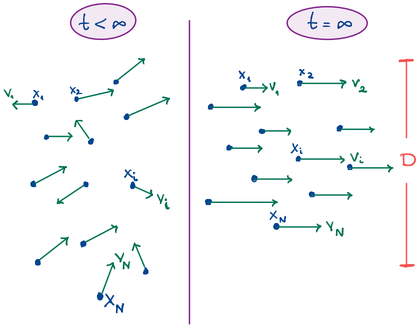

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Mathematical models

- Particle: Cucker–Smale (2007)

$$\begin{cases} \dot{\mathtt{x}}_i = \mathtt{v}_i \\ \dot{\mathtt{v}}_i = \frac{1}{N} \sum_{j=1}^{N} K(\mathtt{x}_i, \mathtt{x}_j)\,(\mathtt{v}_j - \mathtt{v}_i) \end{cases}$$

$K(\mathtt{x}, \mathtt{y}) = \phi(|\mathtt{x} - \mathtt{y}|) > 0$

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- Kinetic:

$$\begin{cases} \partial_t f + v\,\partial_x f + \partial_v \left(f\,L[f\,]\right) = 0 \\ L[f\,](x, v, t) = \iint K(x, x_*)\,(v_* - v)\,f(x_*, v_*, t)\,dx_* dv_* \end{cases}$$

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- Hydrodynamic:

$$\begin{cases} \partial_t \rho + \partial_x (\rho \mathtt{v}) = 0 \\ \partial_t (\rho \mathtt{v}) + \partial_x \left(\rho \mathtt{v}^2 + P\right) = \mathcal{S}^{(1)} \\ \partial_t (\rho E) + \partial_x (\rho \mathtt{v} E + P\mathtt{v} + q) = \mathcal{S}^{(2)} \end{cases}$$

$\mathcal{S}^{(1)}$, $\mathcal{S}^{(2)}$: nonlocal source terms.

Mathematical models/2

[Ha–Tadmor (2008); Ha–Liu; Motsch–Tadmor (2011); Karper–Mellet–Trivisa (2013, 2015); Ha–Kang–Kwon] [Carrillo–Fornasier–Toscani–Vecil; recent review: Shvydkoy]

- Kinetic [KMT 2015]:

$$f_t^\epsilon + \omega \cdot \nabla_x f^\epsilon + \mathrm{div}_\omega(f^\epsilon L[f^\epsilon]) = \frac{1}{\epsilon}\Delta_\omega f^\epsilon + \frac{1}{\epsilon}\mathrm{div}_\omega(f^\epsilon(\omega - \mathrm{v}^\epsilon))$$

$$L[f] = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y) f(y,w)(w-\omega)\,dw\,dy$$

- Hydrodynamic limit $\epsilon \to 0+$ of

$$f^\epsilon \to f(t,x,\omega) = \rho(t,x)\exp\left\{-\frac{|\omega - \mathrm{v}(t,x)|^2}{2}\right\},$$

$$\rho^\epsilon \doteq \int f^\epsilon d\omega, \qquad \rho^\epsilon \mathrm{v}^\epsilon \doteq \int f^\epsilon \omega d\omega$$

$$\rho^\epsilon \to \rho, \qquad \rho^\epsilon \mathrm{v}^\epsilon \to \rho \mathrm{v} \qquad \text{as } \epsilon \to 0+$$

The Euler-type flocking system

$$\begin{cases} \partial_t \rho + \partial_x(\rho \mathtt{v}) = 0 \\ \partial_t(\rho \mathtt{v}) + \partial_x \left(\rho \mathtt{v}^2 + p(\rho)\right) = \rho\, L[(\rho, \mathtt{v})] \end{cases}$$

$$L[(\rho, \mathtt{v})(\cdot, t)](x) = \int_{\mathbb{R}} K(x, x')\rho(x', t) \left(\mathtt{v}(x', t) - \mathtt{v}(x, t)\right)\, dx'$$

- Pressure: $p(\rho) = \alpha^2 \rho$, $\alpha > 0$ const.

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- Pressure: $p(\rho) = \alpha^2 \rho$, $\alpha > 0$ const.
- The Cauchy Problem for **meaningful** initial data:

$$(\rho, \mathtt{m})(x, 0) = (\rho_0(x), \mathtt{m}_0(x)) \qquad \text{in } \mathbb{R}\ .$$

$$\mathtt{m} := \rho\, \mathtt{v} \qquad \text{momentum}$$

Aim

For weak solutions

global in time existence $+$ time-asymptotic flocking

$$\begin{cases} \partial_t \rho + \partial_x \mathtt{m} = 0 \\ \partial_t \mathtt{m} + \partial_x \left(\dfrac{\mathtt{m}^2}{\rho} + \alpha^2 \rho \right) = \displaystyle\int_{\mathbb{R}} K(x,x') \left(\rho(x,t)\mathtt{m}(x',t) - \rho(x',t)\mathtt{m}(x,t) \right) dx' \end{cases}$$

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global in time existence $+$ time-asymptotic flocking

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Our assumptions

- **All-to-all** interaction: $K(x,x') = 1$
- Initial data: $(\rho_0, \mathtt{m}_0)$ compact support $\quad \mathtt{m} := \rho \mathtt{v}$
  with $\operatorname{ess\,inf} \rho_0 > 0$ on a bounded interval $I_0 = [a_0, b_0]$

PART I: Global in time existence of entropy weak solutions

Theorem 1 (Amadori-Chr., 2021)

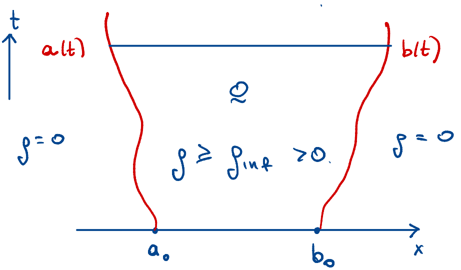
Assume $K \equiv 1$, $(\rho_0, \mathtt{v}_0) \in BV(\mathbb{R})$ and

$$\begin{cases} \operatorname{ess\,inf}_{I_0} \rho_0 > 0 \\ \rho_0(x) = \mathtt{m}_0(x) = 0 \quad \forall\, x \notin I_0 \doteq [a_0, b_0]\,. \end{cases} \qquad (*)$$

Then:

- The Cauchy problem admits an entropy weak solution with concentration $(\rho, \mathtt{m})$ on $\mathbb{R} \times [0, +\infty)$.
- There exist two locally Lipschitz curves $a(t) < b(t)$, $t \geq 0$ and a value $\rho_{inf} > 0$ s.t.:
  - (i) $a(0) = a_0\,, \quad b(0) = b_0\,; \qquad a(t) < b(t)$ for all $t > 0$;
  - (ii) $$\begin{cases} \operatorname{ess\,inf}_{I(t)} \rho(\cdot, t) \geq \rho_{inf} > 0\,, \\ (\rho, \rho\mathtt{v})(x,t) = 0 \qquad \forall\, x \notin I(t)\,. \end{cases}$$
- Conservation of mass and momentum.

Structure (S)

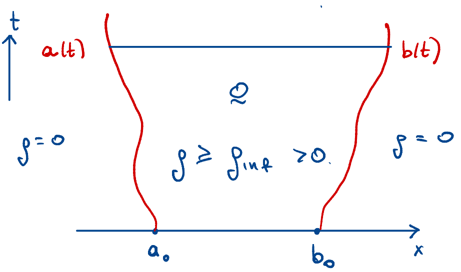


$$\Omega = \{(x,t) : \; x \in I(t), \; t \geq 0\}$$

$$I(t) = [\, a(t), b(t) \,] .$$

(S)

Remarks: • The lower bound $\rho_{inf} > 0$ is **independent of** t.



$$\Omega = \{(x,t) : \; x \in I(t), \;\; t \geq 0\}$$

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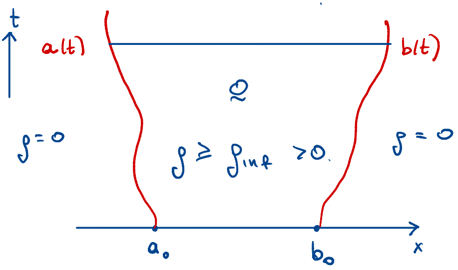
(S)

Remarks: • Assume the *ad-hoc* "boundary" condition:

The vacuum region is connected with the non-vacuum one by a shock

Different from gas dynamics [Liu, Smoller, Yang, Serre, Huang, Pan, Wang]

Structure (S)



$$\Omega = \{(x,t) : \; x \in I(t), \; t \geq 0\}$$

$$I(t) = [a(t), b(t)].$$

(S)

Main difficulties:

(i) large BV data

(ii) loss of strict hyperbolicity around vacuum

Entropy Weak Solution *with Concentration*

Conservation of mass and momentum $+ \; K = const. = 1 \quad \Rightarrow$

$$\Longrightarrow \begin{cases} \partial_t \rho + \partial_x(\rho \mathtt{v}) = 0, \\ \partial_t(\rho \mathtt{v}) + \partial_x \left(\rho \mathtt{v}^2 + \alpha^2 \rho\right) \quad = -M\, \rho(\mathtt{v}(x,t) - \bar{\mathtt{v}}) \\ M = \int_{\mathbb{R}} \rho(\cdot, t) \end{cases}$$

with $\quad \bar{\mathtt{v}} \doteq M_1/M$. Can further Reduce to $\bar{\mathtt{v}} = 0 = M_1$.

$$\begin{cases} \partial_t \rho + \partial_x(\rho \mathtt{v}) = 0, \\ \partial_t(\rho \mathtt{v}) + \partial_x \left(\rho \mathtt{v}^2 + \alpha^2 \rho\right) = -M \rho \mathtt{v} = -M \mathtt{m}\,. \end{cases}$$

- system of balance laws in $(\rho, \mathtt{m})$

$$\begin{cases} \partial_t \rho + \partial_x \mathtt{m} = 0, \\ \partial_t \mathtt{m} + \partial_x \left(\dfrac{(\mathtt{m})^2}{\rho} + \alpha^2 \rho \right) = -M \mathtt{m}\,. \end{cases}$$

Entropy Weak Solution *with Concentration*/2

Definition

$$\begin{cases} \partial_t \rho + \partial_x \mathtt{m} = 0, \\ \partial_t \mathtt{m} + \partial_x \left(\frac{(\mathtt{m})^2}{\rho} + \alpha^2 \rho \right) == -M \mathtt{m} \,. \end{cases} \qquad (1)$$

Let $(\rho, \mathtt{m})$ such that

- the map $t \mapsto (\rho, \mathtt{m})(\cdot, t) \in L^1_{loc} \cap BV$ is continuous in L^1_{loc};
- $\lim_{t \to 0+} (\rho, \mathtt{m})(\cdot, t) = (\rho_0, \mathtt{m}_0)$ in L^1_{loc};
- there exist two locally Lipschitz curves $a(t) < b(t)$, $t \geq 0$ and a value $\rho_{inf} > 0$ s.t. ... we have the structure (S) described.

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- Then $(\rho, \mathtt{m})$ is an entropy weak solution **with concentration** if:

[(a)] $\forall\, \phi \in C_0^\infty(\mathbb{R} \times (0, \infty))$:

- $\displaystyle\iint_\Omega \{\rho\phi_t + \mathtt{m}\phi_x\}\ dxdt = 0$,
- $\displaystyle\iint_\Omega \left\{ \mathtt{m}\phi_t + \left[p(\rho) + \frac{(\mathtt{m})^2}{\rho} \right] \phi_x - M\mathtt{m}\phi \right\} dxdt$
  $\displaystyle - \int_0^\infty \left[p(\rho(b(t)-, t))\phi(b(t), t) - p(\rho(a(t)+, t))\phi(a(t), t) \right] dt = 0$

[(b)] for every convex entropy η with entropy flux q, the following inequality

$$\partial_t \eta(\rho, \mathtt{m}) + \partial_x q(\rho, \mathtt{m}) \leq -\eta_{\mathtt{m}} M\mathtt{m}$$

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The Riemann Problem around Vacuum

Riemann Data at $x = b_0 = 0$:

$$(\rho, \mathtt{m})(x,0) = \begin{cases} (\rho_\ell, \rho_\ell \mathtt{v}_\ell) & x < 0 \\ (\bar{\rho}, \mathtt{m}(\bar{\rho})) & x > 0\,. \end{cases}$$

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The Rankine-Hugoniot conditions are:

$$\begin{cases} \sigma(\rho_\ell - \bar{\rho}) = \rho_\ell \mathtt{v}_\ell - \bar{\rho}\bar{\mathtt{v}} \\ \sigma(\rho_\ell \mathtt{v}_\ell - \bar{\rho}\bar{\mathtt{v}}) = \rho_\ell \mathtt{v}_\ell^2 - \bar{\rho}\bar{\mathtt{v}}^2 + p(\rho_\ell) - p(\bar{\rho}) \end{cases}$$

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- For a 2-shock solution:

$$S_2: \quad \bar{\mathtt{v}} := \mathtt{v}(\bar{\rho}) = \mathtt{v}_\ell - \sqrt{\frac{(p(\bar{\rho}) - p(\rho_\ell))\,(\bar{\rho} - \rho_\ell)}{\bar{\rho}\rho_\ell}} \qquad 0 < \bar{\rho} \le \rho_\ell,$$

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$\Longrightarrow \quad \sigma \rho_\ell \mathtt{v}_\ell = \rho_\ell \mathtt{v}_\ell^2$

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$\Longrightarrow \quad \sigma \rho_\ell \mathtt{v}_\ell = \rho_\ell \mathtt{v}_\ell^2 \quad \Rightarrow$ Rankine-Hugoniot ✓ with $\sigma = \mathtt{v}_\ell$

The Riemann Problem around Vacuum/2

Now the 2-shock $(\rho, \mathtt{m})$ solution for $\bar{\rho} > 0$ satisfies

$$\iint \left\{ \mathtt{m}\phi_t + \left(\frac{\mathtt{m}^2}{\rho} + p(\rho) \right) \phi_x \right\} \, dxdt = 0$$

for all $\phi \in C_0^\infty(\mathbb{R} \times (0, +\infty))$.

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$$(\widetilde{\rho}, \widetilde{\mathtt{m}})(x,t) := \begin{cases} (\rho_\ell, \mathtt{m}_\ell)\,, & x < t\mathtt{v}_\ell \\ (0,0)\,, & x > t\mathtt{v}_\ell \end{cases}.$$

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- $$\iint_{\{x > \sigma t\}} \frac{\mathtt{m}^2}{\rho} \phi_x \, dxdt \quad \xrightarrow{\bar{\rho} \to 0+} \quad - p(\rho_\ell) \int_0^{+\infty} \phi(\mathtt{v}_\ell t, t) \, dt\,.$$

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$$(\widetilde{\rho}, \widetilde{\mathtt{m}})(x,t) := \begin{cases} (\rho_\ell, \mathtt{m}_\ell)\,, & x < t\mathtt{v}_\ell \\ (0,0)\,, & x > t\mathtt{v}_\ell \end{cases}.$$

- $$\iint_{\{x > \sigma t\}} \frac{\mathtt{m}^2}{\rho} \phi_x \, dxdt \quad \xrightarrow{\bar{\rho} \to 0+} \quad -p(\rho_\ell) \int_0^{+\infty} \phi(\mathtt{v}_\ell t, t) \, dt\,.$$
- $$\int_0^{+\infty} \left\{ \int_{\mathbb{R}} \widetilde{\mathtt{m}}\phi_t + p(\widetilde{\rho})\phi_x \, dx + \int_{x < t\mathtt{v}_\ell} \left(\frac{\widetilde{\mathtt{m}}^2}{\widetilde{\rho}} \right) \phi_x \, dx - p(\rho_\ell)\, \phi(v_\ell t, t) \right\} \, dt = 0\,.$$

The Riemann Problem around Vacuum/2

Now the 2-shock $(\rho, \mathtt{m})$ solution for $\bar{\rho} > 0$ satisfies

$$\iint \left\{ \mathtt{m}\phi_t + \left(\frac{\mathtt{m}^2}{\rho} + p(\rho) \right) \phi_x \right\} \, dxdt = 0$$

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- $$\int_0^{+\infty} \left\{ \int_{\mathbb{R}} \widetilde{\mathtt{m}} \phi_t + p(\widetilde{\rho}) \phi_x \, dx + \int_{x < t\mathtt{v}_\ell} \left(\frac{\widetilde{\mathtt{m}}^2}{\widetilde{\rho}} \right) \phi_x \, dx - p(\rho_\ell) \phi(v_\ell t, t) \right\} dt = 0 \, .$$

for $M = 0$, NO source. Motivation: *entropy weak solution with concentration*

The Cauchy Problem: Conservation of mass + momentum

Let $(\rho, \mathtt{m})$ be an entropy weak solution **with concentration**.
Define the total momentum $\widehat{\mathtt{m}}$ to be the distribution

$$\widehat{\mathtt{m}}(\cdot, t) := \mathtt{m}(\cdot, t) + \delta_{b(t)} P_b(t) - \delta_{a(t)} P_a(t)\,, \quad t > 0\,.$$

where

$$P_b(t) := \int_0^t e^{-M(t-s)} p(\rho(b(s)-, s))\, ds\,, \quad P_a(t) := \int_0^t e^{-M(t-s)} p(\rho(a(s)+, s))\, ds\,;$$

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Conservation of mass and momentum:

- $\int_{\mathbb{R}} \rho(x,t)\, dx = \int_{I(t)} \rho(x,t)\, dx = \int_{\mathbb{R}} \rho_0(x)\, dx\,, \qquad \forall\, t \geq 0;$
- $< \widehat{\mathtt{m}}(\cdot, t), \phi_1 >= \int_{I(t)} \mathtt{m}(x,t)\, dx + P_b(t) - P_a(t) = \int_{\mathbb{R}} \mathtt{m}_0(x)\, dx\,, \quad \forall\, t \geq 0\,,$

for any test function $\phi_1 = \phi_1(x)$ that is equal to 1 on $I(t)$.

Existence proof: STEP 1

Conservation of mass and momentum $+ K = const. = 1$

In conclusion:

For $K(x,x') \equiv 1$, the **nonlocal** interaction term becomes **local**!

$$\begin{cases} \partial_t \rho + \partial_x \mathtt{m} = 0, \\ \partial_t \mathtt{m} + \partial_x \left(\frac{(\mathtt{m})^2}{\rho} + \alpha^2 \rho \right) = -M \mathtt{m}\,. \end{cases}$$

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- **Balance laws:** global existence in BV if *dissipative source* [Dafermos–Hsiao, Dafermos. . .; Colombo–Guerra].

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- **Balance laws:** global existence in BV if *dissipative source*

[Dafermos–Hsiao, Dafermos. . .; Colombo–Guerra].

- **Main difficulties:**

(i) loss of strict hyperbolicity around vacuum

[Liu, Liu–Smoller, Liu–Yang, Serre, Huang–Pan, Huang–Marcati–Pan, Huang–Pan–Wang]

(ii) large BV data

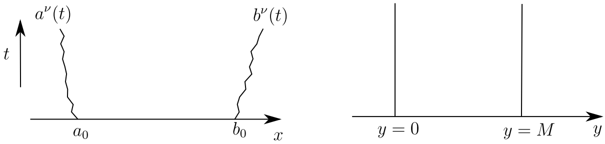
Existence proof: STEP 2

From Eulerian to Lagrangian variables

$(x,t) \mapsto (y,\tau); \quad y = \int_{-\infty}^{x} \rho(x',t)\,dx' \in [0,M], \qquad \tau = t$

$$u \doteq 1/\rho\,, \quad v(y,t) \doteq \mathtt{v}(x,t)\ .$$

$$\begin{cases} \partial_\tau u - \partial_y v = 0, \\ \partial_\tau v + \partial_y(\alpha^2/u) = -Mv \end{cases} \qquad y \in (0,M)$$



Free boundaries $\Rightarrow$ Fixed boundaries

The problem in Lagrangian formulation

$$\begin{cases} \partial_\tau u - \partial_y v = 0, \\ \partial_\tau v + \partial_y(\alpha^2/u) = -Mv \end{cases}, \quad y \in (0, M) \qquad (2)$$

- Initial data:
  $(u_0, v_0) \in BV(0, M)\,, \quad \operatorname{ess\,inf}_{(0,M)} u_0 > 0\,, \quad \int_0^M v_0(y)\, dy = 0$
- *Non-reflecting* boundary conditions:
  all state values are admissible at the boundaries $y = 0$, $y = M$

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– Existence, Cauchy problem: [Nishida ($M = 0$), Dafermos, Luo–Natalini–Yang, Amadori–Guerra; Boundary value pb: Frid (1996), different bdy condition]

– **Equivalence** of weak solutions (Eulerian, Lagrangian): [Wagner (1987) for data with infinite total mass]

<u>STEP 3:</u> Front-tracking approximate solutions (u^ν, v^ν)

- interactions
- time steps $t^n = n\Delta t$
- standby fronts

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- interactions • time steps $t^n = n\Delta t$ • standby fronts

Linear Functionals:

$$L(t) = \sum_{\beta \in J(t)} |\varepsilon_\beta|\,, \qquad L_{in}(t) = \sum_{j=1}^{N(t)} |\varepsilon_j| = \frac{1}{2}\mathrm{TV}\left\{\ln(u)(\cdot, t)\right\}.$$

$$0 < y_1 < y_2 < \ldots < y_{N(t)} < M$$

$$L(t) = L_{in}(t) + L_{0,out}(t) + L_{M,out}(t), \quad \forall t\,.$$

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$$L(t) = L_{in}(t) + L_{0,out}(t) + L_{M,out}(t), \quad \forall t\,.$$

- $L_{in}(t)$, $L(t)$ are non-increasing in time
- $L_{0,out}(t)$, $L_{M,out}(t)$ standby fronts at the bdy $y = 0$, $y = M$.
- bounds on u^ν and v^ν

$0 < u^\nu_{inf} \le u^\nu(y,t) \le u^\nu_{sup}, \quad |v^\nu(y,t)| \le \widetilde{C}_0 \quad \forall\, y \in (0,M), t, \nu\,.$

- finite number of interactions

STEP 4: A weighted functional

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A weighted functional

$$L_\xi(t) = \sum_{j=1,\ \varepsilon_j>0}^{N(t)} |\varepsilon_j| + \xi \sum_{j=1,\ \varepsilon_j<0}^{N(t)} |\varepsilon_j| \qquad \xi \geq 1 .$$

Amadori–Guerra ('01); Amadori–Corli ('08); Amadori–Baiti–Corli–Dal Santo ('15)

Vertical Traces

Given $y \in (0, M)$ and $t > 0$,

$$W_y^\nu(t) = \frac{1}{2}\mathrm{TV}\left\{\ln(u^\nu))(y,\cdot);(0,t)\right\}.$$

STEP 5: Convergence in Lagrangian!

- a subsequence of $(u^\nu, v^\nu) \to (u, v) \in L^1_{loc}((0, M) \times [0, +\infty))$
- the map $t \mapsto (u, v)(\cdot, t) \in L^1(0, M)$ is Lipschitz cts
- $0 < u_{inf} \leq u(y, t) \leq u_{sup}\,, \quad |v(y, t)| \leq \widetilde{C}_0$
- Convergence of horizontal and vertical traces on each $(0, T)$,

in particular

$$\int_0^T v^\nu(0+, s)\, ds \;\to\; \int_0^T v(0+, s)\, ds \qquad \forall\, T > 0\,.$$

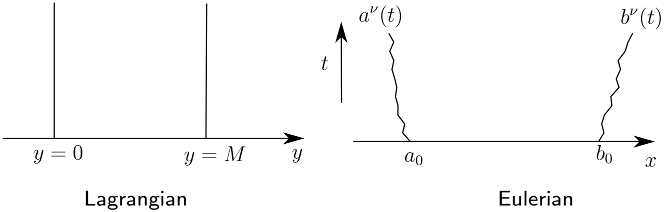
- (u, v) is an entropy weak solution to

$$\begin{cases} \partial_\tau u - \partial_y v = 0, \\ \partial_\tau v + \partial_y(\alpha^2/u) = -Mv \end{cases}$$

STEP 6: From Lagrangian to Eulerian

For each $\nu \in \mathbb{N}$, define the approximate boundaries:

$$a^\nu(t) \doteq a_0 + \int_0^t v^\nu(0+,s)ds\,, \quad b^\nu(t) \doteq a^\nu(t) + \int_0^M u^\nu(y,t)\,dy\,.$$



$$I^\nu(t) = (a^\nu(t), b^\nu(t))\,, \qquad \Omega^\nu = \{(x,t);\ t \geq 0\,,\ x \in I^\nu(t)\}$$

Set $\rho^\nu(x,t) := \dfrac{1}{u^\nu(\chi^\nu(x,t),t)}\,, \qquad \mathtt{v}^\nu(x,t) := v^\nu(\chi^\nu(x,t),t) \qquad x \in I^\nu(t)$

$$\mathtt{m}^\nu(x,t) := \rho^\nu\,\mathtt{v}^\nu \qquad x \in I^\nu(t)$$

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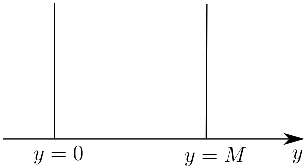
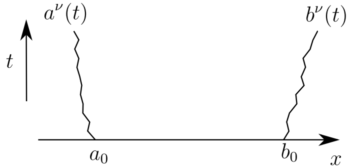


Lagrangian



Eulerian

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Set $\quad \rho^\nu(x,t) = 0 = \mathtt{m}^\nu(x,t) \qquad x \notin I^\nu(t)$.

STEP 7: Conclusion

- As $\nu \to \infty$,

$$a^{\nu}(\cdot) \to a(\cdot)\,, \qquad b^{\nu}(\cdot) \to b(\cdot)$$

uniformly on compact subsets of $[0, +\infty)$.

- Uniform bounds on approximate solutions:

$$\mathrm{TV}\,\{\rho^{\nu}(\cdot, t); \mathbb{R}\}, \qquad \mathrm{TV}\,\{\mathtt{m}^{\nu}(\cdot, t); \mathbb{R}\}$$

- Define

$$x_j^{\nu}(t) = a^{\nu}(t) + \int_0^{y_j^{\nu}(t)} u^{\nu}\left(y', t\right) dy' \qquad j = 1, \ldots, N^{\nu}(t)$$

the Rankine-Hugoniot conditions are approximately satisfied across the piecewise linear curves $x_j^{\nu}(t)$

- Approximate total momentum

$$\mathfrak{I}^{\nu}(t) := \int_{a^{\nu}(t)}^{b^{\nu}(t)} \mathtt{m}^{\nu}(x,t)\,dx + P_b^{\nu}(t) - P_a^{\nu}(t)\,,$$

$$|\mathfrak{I}^{\nu}(t)| \leq e^{-Mt} \cdot e^{M\Delta t_{\nu}} \cdot \frac{M}{\nu} + \widetilde{C}(\eta_{\nu} + \Delta t_{\nu}), \qquad t \geq 0$$

for a suitable constant $\widetilde{C} > 0$, which is independent on t and ν.

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$$\mathtt{m}(x,t) = \begin{cases} \mathtt{m}(x,t) & (x,t) \in \Omega \\ 0 & (x,t) \in \mathbb{R} \times [0,+\infty) \setminus \overline{\Omega}\,. \end{cases}$$

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- Conservation of mass

$$M = \int_{a^{\nu}(t)}^{b^{\nu}(t)} \rho^{\nu}(x,t)\,dx \xrightarrow{\nu\to\infty} \int_{a(t)}^{b(t)} \rho(x,t)\,dx$$

- Conservation of momentum

$$\mathfrak{I}^{\nu}(t) \xrightarrow{\nu\to\infty} \int_{a(t)}^{b(t)} \mathtt{m}(x,t)\,dx + +P_b(t) - P_a(t) = 0 = M_1. \qquad \square$$

PART II: Time-Asymptotic Flocking

Asymptotic flocking as $t \to +\infty$

Definition

- The **support** $I(t) = (a(t), b(t))$ remains bounded for all t:

$$\sup_{0 \le t < \infty} \{b(t) - a(t)\} < \infty \tag{3}$$

- The **oscillation of the velocity**

$$\operatorname{osc}\{\mathrm{v}; I(t)\} = \sup_{x_1,\ x_2 \in I(t)} |\mathrm{v}(x_1, t) - \mathrm{v}(x_2, t)|$$

satisfies

$$\lim_{t \to \infty} \operatorname{osc}\{\mathrm{v}; I(t)\} = 0 \tag{4}$$

Question:

Does time-asymptotic flocking occur for solutions as in Theorem 1?

Remarks

About (3)

After Theorem 1, condition (3) is immediate:

$$\underbrace{\rho_{inf}}_{>0}(b(t) - a(t)) \leq \int_{I(t)} \rho(x,t)\,dx = M \qquad \forall\, t > 0$$

therefore $0 < b(t) - a(t) \leq M/\rho_{inf}$ for all $t > 0$.

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therefore $0 < b(t) - a(t) \leq M/\rho_{inf}$ for all $t > 0$.

About (4)

Condition (4) is equivalent to

$$\sup_{x\in I(t)} |\mathrm{v}(x,t) - \bar{\mathrm{v}}| \to 0 \qquad t\to\infty\,.$$

Time-asympt. flocking $\Rightarrow$ the flock will approach the same velocity $\bar{\mathrm{v}}$

Special solutions

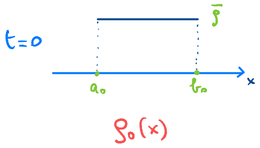
If the initial data are: $\rho_0(x) = \bar{\rho} > 0, \quad \mathrm{v}_0(x) = \bar{\mathrm{v}}, \quad x \in [a_0, b_0]$
with $\bar{\rho}$, $\bar{\mathrm{v}}$ constant values, then from Th. 1

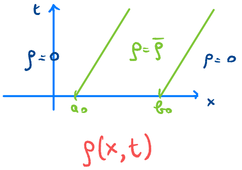
$$\rho(x,t) = \bar{\rho}, \qquad \mathrm{v}(x,t) = \bar{\mathrm{v}}, \quad x \in I(t) = [a_0 + \bar{\mathrm{v}}t, b_0 + \bar{\mathrm{v}}t], \quad t > 0,$$

$$P_a(t) = P_b(t) = p(\bar{\rho})M^{-1}\left(1 - e^{-Mt}\right)$$

$$\widehat{m} = \bar{\rho}\,\bar{\mathrm{v}} + p(\bar{\rho})M^{-1}\left(1 - e^{-Mt}\right)\left(\delta_{b(t)} - \delta_{a(t)}\right)$$

$\Rightarrow$ *time-asymptotic flocking* for every $\bar{\rho} > 0$, $\bar{\mathrm{v}} \in \mathbb{R}$





Long-time behavior

Theorem 2 (Amadori-Chr., 2021)

Let $(\rho_0, \mathtt{m}_0) \in BV(\mathbb{R})$ and
$q := \frac{1}{2}\mathrm{TV}\left\{\ln(\rho_0); I_0\right\} + \frac{1}{2\alpha}\mathrm{TV}\left\{\mathtt{v}_0; I_0\right\} > 0$. Assume that

$$e^{2q}M^2 < \alpha \max\left\{\rho_0(a_0+), \rho_0(b_0-)\right\}$$

Let $(\rho, \mathtt{m})$ be the corresponding entropy weak solution obtained in Theorem 1.

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Let $(\rho, \mathtt{m})$ be the corresponding entropy weak solution obtained in Theorem 1.

$\Rightarrow$ The solution $(\rho, \mathtt{m})$ admits *time-asymptotic flocking*, with an **exponentially fast decay**:

$$\mathrm{osc}\{\mathtt{v}; I(t)\} \leq C_2' e^{-C_1' t}, \qquad \forall\, t \geq t_0$$

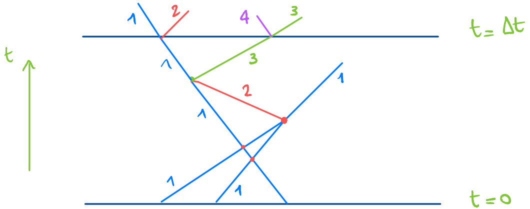
for some $t_0 > 0$ C_1', $C_2' > 0$.

About the proof

Two main ingredients, the first:

[1] A geometric decay depending on the generation order

(# of wave reflections)



- $k \geq 1$ - generation order

$$F_k(t) \doteq \sum_{\varepsilon > 0,\ g_\varepsilon = k} |\varepsilon| + \xi \sum_{\varepsilon < 0,\ g_\varepsilon = k} |\varepsilon| \ , \qquad \widetilde{F}_k(t) \doteq \sum_{j \geq k} F_j(t)$$

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- A functional that assigns a geometric weight ξ^k to waves with gen. order k and that decreases in time, except at time steps.

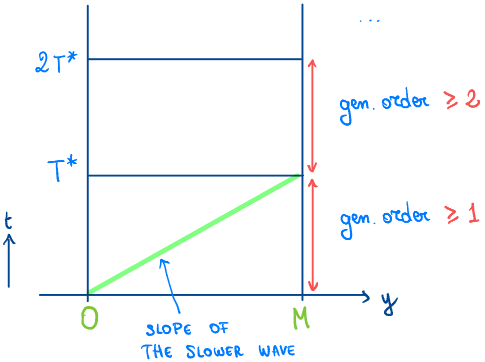
$$V(t) = \sum_{k \geq 1} \xi^k F_k(t)$$ NEW

$$V(t) \leq \left(1 + \frac{(\xi^2 - 1)}{2} M \Delta t\right)^n V(0+) \ ,$$

while $t \in [t^n, t^{n+1})$ and $\xi \in [1, c(q)^{-1/2}]$.

The <u>second</u> main ingredient:

[2] Maximum time length for each generation order



There exists a time $T^* > 0$ such that

waves of generation order k exist up to time kT^*

Combining the two ingredients [1] + [2]:

- Using that $\xi \geq 1$, we have the relation

$$\tilde{F}_k(t) \leq \frac{1}{\xi^k} \sum_{j \geq k} \xi^j F_j(t) \leq \frac{1}{\xi^k} V(t), \quad \forall t$$

- Prove for some $\bar{\xi} > 1$,

$$\lim_{t \to \infty} \tilde{F}_1(t) = 0, \quad \text{since } MT^* < 1$$

$$L_{in}(t) \leq \tilde{F}_1(t) \leq C_2' e^{-C_1' t} q \ ,$$

$\Longrightarrow$ Time-asymptotic flocking $\square$

Conclusion and Perspectives

- Done: for $K(x, x') \equiv 1$ (*all-to-all interaction*),
  - Global existence for BV initial data with structure;
  - Under a sufficient condition on the data, time-asymptotic flocking.

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  - Global existence for BV initial data with structure;
  - Under a sufficient condition on the data, time-asymptotic flocking.

- Current work: **Unconditional flocking** for $K \equiv 1$? that is, does it hold for every BV initial data?

Possibly YES: For Cucker–Smale particle model

$$K(x, x') = \phi(|x - x'|) > 0 \qquad \text{with} \quad \phi \text{ non-increasing,}$$
$$\int^{\infty} \phi = \infty$$

$\Rightarrow$ time-asymptotic flocking for any initial data.

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- Done: for $K(x, x') \equiv 1$ (*all-to-all interaction*),
  - Global existence for BV initial data with structure;
  - Under a sufficient condition on the data, time-asymptotic flocking.

- Current work: **Unconditional flocking** for $K \equiv 1$? that is, does it hold for every BV initial data?

Possibly YES: For Cucker–Smale particle model

$$K(x, x') = \phi(|x - x'|) > 0 \qquad \text{with} \quad \phi \text{ non-increasing,}$$

$$\int^{\infty} \phi = \infty$$

$\Rightarrow$ time-asymptotic flocking for any initial data.

- Future work... Extend the analysis to more general kernels K.

THANK YOU!