Geometry of smooth Gaussian fields

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Confirmation of Status
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10 Oct 2024

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Smooth Gaussian fields

- A C^k*-smooth Gaussian field* is a Gaussian process indexed by $\mathbb{R}^d$ which has C^k-smooth sample paths. [Fields are centered throughout this talk]
- **Kolmogorov theorem**: Suppose that $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a positive definite symmetric function of class $C^{k,k}(V \times V)$ and, in addition, that

$$N := \max_{|\alpha|,|\beta| \leq k} \sup_{x,y \in V} |\partial_x^\alpha \partial_y^\beta K(x,y)| < \infty.$$

  Then there exists a (unique up to an equivalence of distribution) C^{k-1} Gaussian function f on V with the covariance kernel K. Moreover, $\mathbb{E}||f||_{C^{k-1}} \leq C\sqrt{N}$.

Stationary fields

- Call a Gaussian field on $\mathbb{R}^d$ *stationary* or *translation invariant* if its covariance kernel $K(x, y)$ depends only on $x - y$, say $K(x, y) = k(x - y)$.
- **Bochner theorem**: For a continuous k, k is a Fourier transform of a finite symmetric ($\rho(A) = \rho(-A)$) positive Borel measure ρ on $\mathbb{R}^d$, i.e.

$$k(x) = \int_{\mathbb{R}^d} e^{2\pi i(\lambda \cdot x)} d\rho(\lambda).$$

- Call ρ the *spectral measure* of the field.
- The field is a fourier transform of white noise on ρ, i.e.

$$f(x) = W_\rho(e^{2\pi i x \cdot t})$$

The properties of f, K, ρ are closely related.

Examples: Random plane waves

- Spectral measure is (normalised) arc length measure on $S^1 \subset \mathbb{R}^2$. So covariance kernel is $J_0(|x-y|)$, where J_0 is zeroth Bessel function. Here the covariance function oscillates around zero, and decays like $|x-y|^{-1/2}$
- Sample paths are eigenfunctions of Laplacian on $\mathbb{R}^2$ with eigenvalue 1.
- Local scaling limit of a number of other Gaussian fields. E.g. Random spherical harmonics [Wig22].

Picture

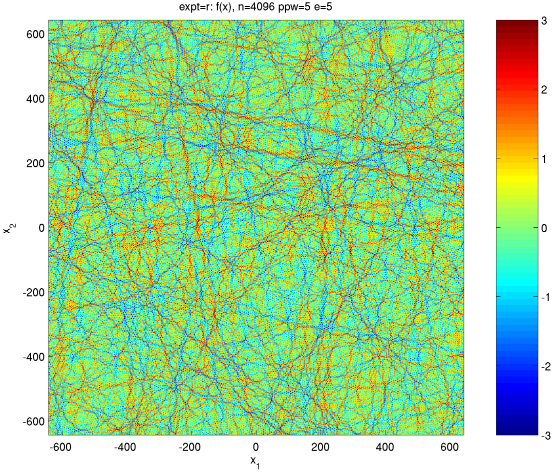

Figure: RPW landscape, around 2000 wavelengths across. Picture: Alex Barnett

Examples: Bargmann-Fock field

- Covariance kernel is $K(x,y) = e^{-|x-y|^2/2}$. Hence the spectral measure has Gaussian-type density.
- The field can be written as

$$f(x) = e^{-|x|^2/2} \sum_{n,m \geq 0} \frac{a_{n,m}}{\sqrt{n!m!}} x_1^n x_2^m$$

- Thought of as a limit of Gaussian ensemble of homogeneous polynomials. So zero sets are "portrait of 'typical' algebraic variety".
- Many percolation theoretic properties are easier to establish in this model because the correlation decay is very fast.

Pictures

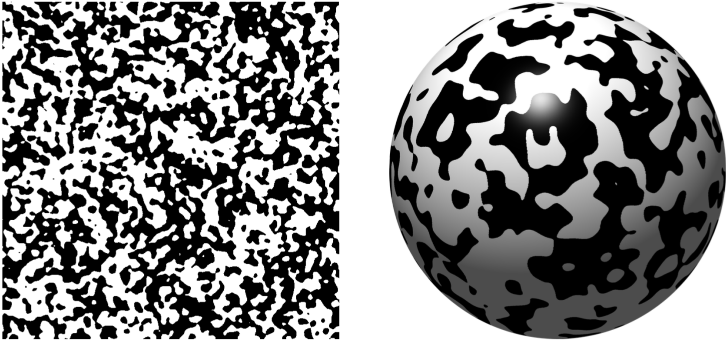

Figure: (Left) Bargman-Fock field sample. (Right) Gaussian ensemble of homogeneous polynomials of degree 300. The scale is $d^{-1/2}$ where d is the degree. Picture: Dmitry Beliaev

Questions

We're interested in large scale geometry of level/excursion sets $\{f = l\}$ or $\{f \geq l\}$ and the landscape of the field f. Quantities that we're interested in:

1. Local
   - Volume of level sets
   - Critical point structure of the field
2. Non-local
   - Number of components of level sets
   - Percolation theoretic probabilities (like box crossing)

Volume of zero sets

- Estimate of difference (in mean) of volume of zero sets of closely coupled stationary Gaussian fields [BH23].
- Proof via mean curvature estimate ('first variation of area' formula).
- Main focus of Transfer of Status viva. Published in *Elec. Comm. Probab.* in 2023.
- Using same idea, Peccati-Stecconi [PS24] studied Malliavin derivative of nodal volume and absolute continuity w.r.t Lebesgue measure.
- Ongoing work with Stecconi for higher moment analogues (technically very challenging).

Critical points of fields

- Critical points structure important in topology of level sets (think Morse theory).
- What is the distribution of total number of critical points of a smooth Gaussian field?
- Expectation, variance by Kac-Rice formula [BCW19].

Point process comparison

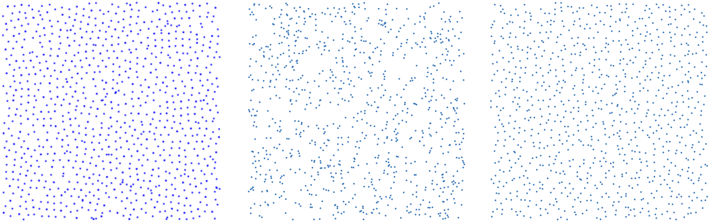

Figure: (Left) Random Plane Waves (RPW) critical points. (Centre) Poisson point process. (Right) Ginibre ensemble Picture: Beliaev, Cammarota, Wigman

- All point processes above scaled to have same intensity.
- Variance asymptotics in $R \times R$ box for $R \to \infty$: $R^2 \log R$ (RPW), R^2 (Poisson), R (Ginibre) [CW17].

Small scale

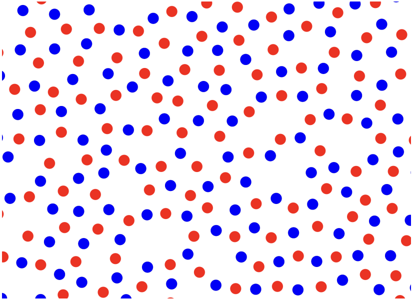

Figure: At small scale, extrema of RPW repel each other. Probability of 2 points existing in a ball of radius r asymptotically r^6, compared to r^4 for Poisson, as $r \to 0$ [BCW19].

High local maxima

Let $f : \mathbb{R}^d \to \mathbb{R}$, $d \geq 2$ be a C^{2+}-smooth stationary Gaussian field. Further assume that,

1. $\text{Var}(f(x)) = 1$ for all x.
2. Decay of correlation: $\mathbb{E}[f(0)f(x)] = o(1/\log \|x\|)$ as $x \to \infty$.
3. Non degeneracy: the vector $(f(0), \nabla f(0))$ has density in $\mathbb{R}^{d+1}$.

Consider a monotone function $u : [0, \infty) \to [0, \infty)$ such that $u(R) \to \infty$ as $R \to \infty$. In a box $[0, R]^d$, we consider the point process of local maxima of the field f above level $u(R)$.

Main result

Let f_R denote a rescaling of the field f. For a Borel set $B \subset \mathbb{R}^d$, define

$$\Phi_R(B) = \text{ number of local maxima above level } u(R) \text{ of } f_R \text{ in } B.$$

Rescaling is done so that intensity measure of Φ_R is lebesgue measure on $\mathbb{R}^d$ for all R.

Theorem (Belyaev-H. 2024)

Let Φ be a Poisson point process on $\mathbb{R}^d$ with lebesgue measure as intensity measure. With assumptions and notations as above, we have

$$\Phi_R \to \Phi \qquad \text{weakly as} \quad R \to \infty.$$

Comments

"Extremal sets of weakly correlated or Markovian Gaussian processes are independent in the limit"

1. Both Bargmann-Fock field and RPW models satisfy the assumptions of the theorem.
2. This is the first time arbitrary level going to infinity considered. Previously for Gaussian processes, only levels comparable to $\sqrt{\log R}$ in box $[0, R]^d$ considered. Number of upcrossing in $d = 1$ [LLR83], exit points of excursion sets [Pit96] etc.
3. For DGFF, [ST20] considered TV distance between high points of the field and indep Bernoulli process on lattice (for levels $\simeq c\sqrt{\log R}$).
4. Thm implies, no filament structure for RPW at high levels.

Proof sketch

- Using Kallenberg theorem [Kal17, Thm 4.18], for simple point processes, enough to show for boxes B

$$\lim_{R\to\infty} \mathbb{P}(\Phi_R(B) = 0) = \mathbb{P}(\Phi(B) = 0)$$

and

$$\limsup_{R\to\infty} \mathbb{E}[\Phi(B)] \leq \mathbb{E}[\Phi(B)].$$

- Then approximate with excursion probabilities,

$$\mathbb{P}(\Phi_R(B) = 0) \simeq \mathbb{P}\left(\max_{x\in R\cdot B} f_R(x) > u(R)\right)$$

Current/ future work

(Not written in Confirmation report!) We are able to give upper bound on rate of convergence also.

Theorem (Belyaev-H. 2024)

Fix a finite box $D \subset \mathbb{R}^d$, let Φ_R^D denote restriction of Φ_R to D. Then, for large R

$$W_1(\Phi_R^D, Pois(D)) \leq u(R) \max_{\|x\| \geq e^{u(R)^2/2}} |\nabla K(x)|$$

here K is the covariance kernel, W_1 is Wasserstein 1-distance w.r.t a metric on finite point measures on D.

Current/ future work

Proof using Palm distribution for point processes. Entirely different from above previous theorem.

1. Consider similar problem when dimension $d \to \infty$. Applications in spin glass theory in stat mech, machine learning.
2. Full scaling limit of critical point explaining filament structure for RPW.

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