Conjectures on Stembridge's Shifted Littlewood-Richardson Coefficients

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Abstract

In this short note, we define partitions $\lambda, \mu \in P(n)$ to be equivalent if the cross-diagonals of their Young diagrams have the same cardinalities, and show that each equivalence class has a unique strict partition. Suppose $\nu, \lambda \in DP(n)$ and $\lambda \sim \mu$. We conjecture that if $g_{\nu\mu} \neq 0$ then $\nu \trianglerighteq \lambda$, and that $g_{\lambda\mu} \neq 0$, where $g_{\nu\mu} = f_{\nu\delta}^{\mu+\delta}$ is Stembridge's shifted Littlewood-Richardson coefficient for $\delta = (l(\mu), l(\mu) - 1, ..., 1)$.

1 An Equivalence on Partitions

Let $\mu = (\mu_1, ..., \mu_k) \in P(n)$. We assume throughout that partitions have decreasing parts, i.e. $\mu_1 \geqslant \mu_2 \geqslant ... \geqslant \mu_k$. It is well-known that partitions admit a Young diagram.

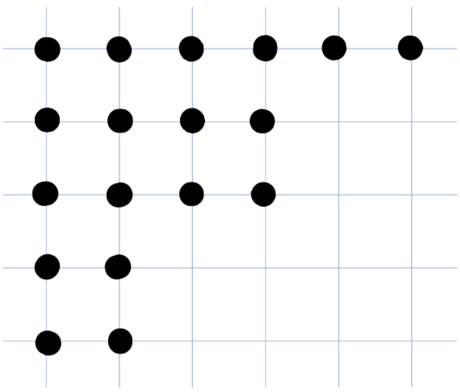

Figure 1: The Young diagram of $\mu = (6, 4, 4, 2, 2)$

We say that the dot in the ith row and jth column is the (i, j)th dot.

Let the *k-th cross-diagonal* of μ be the cross-diagonal of its Young diagram containing only the (i, j)th dots where $k = i + j - 1$.

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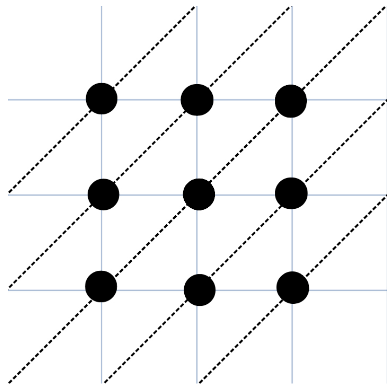

Figure 2: The Young diagram of $\mu = (3,3,3)$ with its cross-diagonals drawn.

Definition 1.1. For $\mu \in P(n)$, let $c(\mu) \in P(n)$ be the partition whose parts are the number of dots in each cross-diagonal of μ. If necessary, we reorder the parts of $c(\mu)$ so that they decrease.

Example 1.2. *If $\mu = (3,3,3)$ then $c(\mu) = (3,2,2,1,1)$ since it has one cross-diagonal of 3 dots, two with 2 dots, and two with 1 dot.*

Definition 1.3. For $\lambda, \mu \in P(n)$ we write $\lambda \sim \mu$ if $c(\lambda) = c(\mu)$.

It is clear that $\sim$ is an equivalence relation. Further details on this relation can also be found in [NV25].

Example 1.4. *We enumerate some equivalence classes.*

n	*equivalence classes of $P(n)$*
2	$\{(2),(1,1)\}$
3	$\{(3),(1^3)\},\ \{(2,1)\}$
4	$\{(4),(1^4)\},\ \{(3,1),(2,2),(2,1,1)\}$
5	$\{(5),(1^5)\},\ \{(4,1),(2,1^3)\},\ \{(3,2),(3,1,1),(2,2,1)\}$
6	$\{(6),(1^6)\},\{(5,1),(2,1^4)\},\{(4,2),(4,1,1),(3,3),(3,1^3),(2^3),(2,2,1,1)\},\{(3,2,1)\}$

Recall that a partition λ is *strict* if it has distinct parts, i.e. $\lambda_i > \lambda_{i+1}$ for all i. $DP(n) \subseteq P(n)$ denotes the strict partitions. Note that, in the examples above, each equivalence class contains exactly one strict partition.

Lemma 1.5. *$c(\mu)^t$ is the unique strict partition equivalent to $\mu \in P(n)$.*

Proof. $c(\mu)^t$ is equivalent to $c(\mu)$, and hence μ, since the transpose preserves the cross-diagonals.

Consider the *shifted* Young diagram of μ obtained by shifting the kth row by $(k-1)$ spaces to the right. This has the effect of straightening the cross-diagonals into lines.

Then, shift the dots as far as possible up their columns.

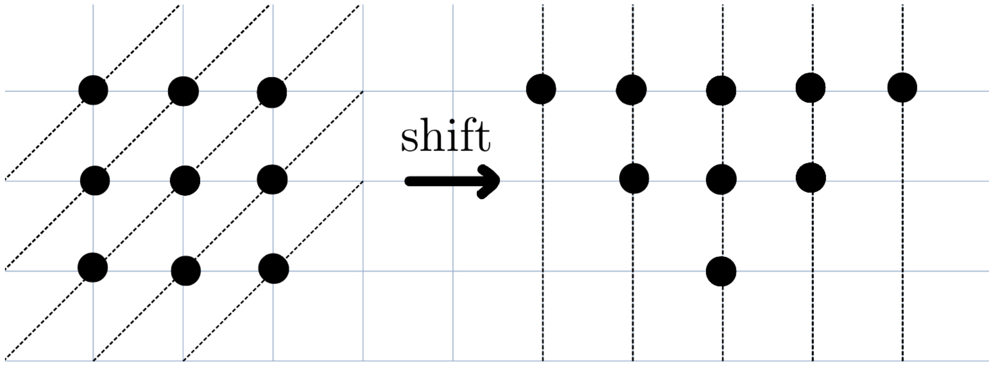

Figure 3: The rows of $\mu = (3, 3, 3)$ are shifted, and dots have been moved up their columns as far as possible.

By definition, counting the columns of this diagram gives $c(\mu)$, and so counting the rows gives $c(\mu)^t$. The shifting of dots up their columns on $c(\mu)$ has the effect on μ of shifting dots up their cross-diagonals. This clearly gives a strict partition, which we see to be $c(\mu)^t$.

Suppose $\lambda \in P(n)$ is strict. Then the dots in λ cannot be shifted up any further along their cross-diagonals, so by the above $\lambda = c(\lambda)^t$. So if $\lambda \sim \mu$ then $\lambda = c(\lambda)^t = c(\mu)^t$, hence $c(\mu)^t$ is the unique strict partition equivalent to μ. □

Let $\lambda = (\lambda_1, ..., \lambda_k)$, $\mu = (\mu_1, ..., \mu_k) \in P(n)$ (adding zero parts if necessary so that they have the same length). Recall that λ *dominates* μ ($\lambda \trianglerighteq \mu$) if $\forall 1 \leqslant r \leqslant k$

$$\sum_{i=0}^{r} \lambda_i \geqslant \sum_{i=0}^{r} \mu_i$$

Lemma 1.6. *If $\lambda \in DP(n)$ is strict, then λ dominates its equivalence class.*

Proof. Suppose $\mu \in P(n)$ is equivalent to λ. Recall that λ is obtained from μ by shifting dots up as far as possible along cross-diagonals. Hence the first i rows of λ contain at least as many dots as that of μ for all $i \in \mathbb{N}$, hence $\lambda \trianglerighteq \mu$. □

2 Conjectures on Shifted Littlewood-Richardson Coefficients

For $\nu \in DP(n)$ and $\mu \in P(n)$, we consider the coefficients $g_{\nu\mu}$ defined by Stembridge in Theorem 9.3 of [Ste89]. They are the coefficients in the decomposition of shifted Schur Q-functions in the basis of Schur polynomials, but they may be described purely in terms of shifted tableaux. They are a special case of the shifted Littlewood-Richardson coefficients: $g_{\nu\mu} = f^{\mu+\delta}_{\nu\delta}$ where $\delta = (l, l-1, ..., 1)$ for $l = l(\mu)$.

We recall Stembridge's description. The coefficient $g_{\nu\mu}$ is the number of tableaux of content ν and shape μ satisfying the following rules:

1. Some entries of the tableaux are marked. For all $k \in \mathbb{N}$
   - each row has at most one marked k
   - each column has at most one unmarked k

   where a marked k is denoted k'.

2. With respect to the ordered alphabet $1' < 1 < 2' < 2 < \ldots$ the tableaux is weakly increasing in rows and columns.

3. (Lattice Property) Recall that the word $w_1 \ldots w_n$ of a tableaux is its sequence of entries read from left to right, bottom row to top row. For $i \geqslant 1$ and $0 \leqslant j \leqslant 2n$ let

$$m_i(j) = \begin{cases} \text{multiplicity of } i \text{ among } w_{n-j+1} \ldots w_n & 0 \leqslant j \leqslant n \\ m_i(n) + \text{multiplicity of } i' \text{ among } w_1 \ldots w_{j-n} & n+1 \leqslant j \leqslant 2n \end{cases}$$

   In other words, $m_i(j)$ is calculated by counting the unmarked i right to left, then the marked i left to right, stopping at step j. If $m_{i-1}(j) = m_i(j)$ then

$$\begin{aligned} w_{n-j} &\neq i, i' && (0 \leqslant j < n) \\ w_{j-n+1} &\neq (i-1), i' && (n \leqslant j < 2n) \end{aligned}$$

4. For all $i \in \mathbb{N}$, the leftmost i in the word is unmarked.

$1'$	1	1	1
1	$2'$	2	2
2	3		
3	4		

Figure 4: A Stembridge tableaux of content $\nu = (5, 4, 2, 1)$ and shape $\mu = (4, 4, 2, 2)$. Its word is $342312'221'111$.

More details are in for example [Ngu22], which also provides alternate models. Also see [EP25] and references therein for recent work on the shifted LR coefficients.

We make two conjectures using the equivalence relation of the previous section.

Conjecture 2.1. *Let λ be the unique strict partition equivalent to $\mu \in P(n)$. Then*

(i) $\forall \nu \in DP(n)$ *if* $g_{\nu\mu} \neq 0$ *then* $\nu \trianglerighteq \lambda$

(ii) $g_{\lambda\mu} \neq 0$

I verified both by hand up to $n = 9$. I was able to prove (i) for $l(\nu) \leqslant 3$ and (ii) for $l(\lambda) \leqslant 3$ case-by-case, although the proof is long and a little messy.

For (i) it follows from the lattice property that $g_{\nu\mu} \neq 0$ implies $\nu \trianglerighteq \mu$, but I'm seeking a sharper lower bound for ν.

Suppose $\mu \sim \mu'$. It is not generally true that $g_{\nu\mu} = g_{\nu\mu'}$ although this does hold if $\mu' = \mu^t$ (Proposition 3.7 of [Ngu22]).

My motivation is the appearance of these coefficients in Lemma 3.2 of [Ciu22], where the equivalence relation appears in the type A cases of Propositions 4.4-4.6. More generally, I would be interested in the (projective) representation-theoretic meaning of the equivalence classes.

References

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