A limit theorem for high local maxima of stationary smooth Gaussian fields

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Abstract

We consider the point process of local maxima of stationary smooth Gaussian fields in a box $[-R, R]^d \subset \mathbb{R}^d$ exceeding a level $u(R)$. We show that this point process converges weakly, after suitable rescaling, to homogeneous Poisson point process when $u(R) \to \infty$ as $R \to \infty$. Previously, it was shown only at the level $u(R)$ of expected maxima of the field in the box $[-R, R]^d$. We follow [Pit96, section 15], where he showed Poisson limit for a different point process (called '*A-exit points*') derived from these fields. We also comment on the structure of high critical points of these fields.

1 Introduction

Local maxima / high points of Gaussian fields is an important geometric observable in probability theory, mathematical physics and in natural sciences. Analysis of critical points of smooth fields is crucial in the understanding of of landscape of the field. For example, it plays an important role in computing topological quantities like number of connected components of level sets [BMM22].

In statistics, extreme values of Gaussian processes are vital to real-world applications and are studied well. Limit theorems for extrema of these processes were proved in 1960's & 70's cf. [LLR83]. Then later in 1990's, Piterbarg [Pit96] showed Poisson process convergence for so-called 'A-exit points' over a high level of a smooth Gaussian field of dimension $d \geq 2$.

The following are some of the results pertaining to Poisson convergence of point processes of smooth Gaussian fields (including dimension one).

1. In 1-dim, number of upcrossings at level $u(T) \simeq \sqrt{2 \log T}$ over the interval $[0, T]$, as $T \to \infty$. [LLR83, Chapter 9]

2. In dimension 2 or more, "A-exit points" over level $u(R) \simeq \sqrt{2d \log R}$ in growing region $[0, R]^d$, as $R \to \infty$. [Pit96, Section 15]

3. In dimension 2 or more, for local maxima over level $u(R) \simeq \sqrt{2d \log R}$ in growing region $[0, R]^d$, as $R \to \infty$. [Qi22, Chapter 3]

In all of the examples above, the decay of correlation of the field at infinity is around $\log^{-1}$ of the distance. A somewhat related set of results include limit theorems for extremal processes for class processes with Markovian property like Gaussian free field, branching Brownian motion. Arguin et al. [ABK13] showed that extremal process of branching Brownian motion

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converges weakly to clustered Poisson process. Oleskar-Taylor, Sousi [ST20] showed that high points (level above $\alpha\mathbb{E}[\text{maxima}], 0 < \alpha_0 < \alpha$) of discrete GFF in $d \geq 3$ converges in total variation distance to independent Bernoulli process on the lattice. In essence, we can expect some Poisson limit for extremal process if either the covariance decays fast enough at infinity or there's some Markov property.

Our contribution is to consider limits for local maxima over arbitrary levels $u(R) \to \infty$ as $R \to \infty$. As far as we know, this is the first time a lower level than the expected maxima in a domain is studied in this context. Also, our result includes monochromatic random waves (MRW) model, which is not covered in [Qi22]. Surprisingly, lowering the rate of threshold level does not impose any additional condition on the decay rate of correlations.

Let us remark about the landscape of random planes waves in the context of Thm 2.2. Numerical simulations by A. Barnett (See the webpage) suggests that there's apparent filament structure of extrema above a level (say above three std deviations of the field). Our result indicates that these patterns disappear at high levels.

2 Setup and statement

Consider a C^{2+}-smooth Gaussian field $f : \mathbb{R}^d \to \mathbb{R}$, with $d \geq 2$, $\mathbb{P}$ being the associated probability measure. Let $\mathbb{E}$ denote the expectation with respect to $\mathbb{P}$ and let $r(x, y) = \mathbb{E}[f(x)f(y)]$ be the covariance kernel [see [NS16, Appendix A] for more details]. For $R > 0$, $L = [0, 1]^d \subset \mathbb{R}^d$ and let $L_R = [0, R]^d = R \cdot L$. Let $u : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function such that $u(R) \leq \sqrt{2d \log R} \to \infty$ as $R \to \infty$. By abuse of notation, we often write $u = u(R)$.

Consider the following scaling of the field f,

$$f_R(x) = f(\mu(R) \cdot x) \quad \text{ for } x \in \mathbb{R}^d,$$

where

$$\mu(R)^{-1} = \kappa^{1/d} R u^{\frac{d-1}{d}} \exp\left(-\frac{u^2}{2d}\right) \text{ and } \kappa = 1/(2\pi)^{(d+1)/2}.$$

Now we define a sequence of point process indexed by R as follows. Let

$$\eta_R(B) = \text{ number of local maxima above level } u(R) \text{ of the field } f_R \text{ in } B$$

where B is a Borel set in $\mathbb{R}^d$. Let

$$\Phi_R(B) = \eta_R(R \cdot B)$$

for Borel sets B in $\mathbb{R}^d$. Our goal is to show that $\Phi_R \to \Phi$ *weakly* as point processes where Φ is a homogeneous Poisson point process, given that the field f satisfies some mild regularity and correlation decay conditions (see [Kal17, Chapter 4]).

Assumptions 2.1. *Throughout the article, we impose the following conditions on the Gaussian field f.*

1. *Centred ($\mathbb{E}[f(x)] = 0$), stationary ($r(x, y) = r(x - y)$), normalised ($\mathbb{E}[f(x)^2] = 1$) for all $x, y \in \mathbb{R}^d$.*
2. *Decay of correlation: $r(x) = o((\log \|x\|)^{1-d})$ as $x \to \infty$.*
3. *The vector $(f(0), \nabla f(0))$ has density in $\mathbb{R}^{d+1}$. In addition, either the vector $(f(0), \nabla f(0), \nabla^2 f(0))$ has density in $\mathbb{R}^{(d+1)+d(d+1)/2}$ or f is isotropic field (i.e. $r(x) =$ "$r(\|x\|)$").*

4. Local structure: $r(x) = 1 - \|x\|^2 + o(\|x\|^2)$ *as* $x \to 0$*. Note that* $\exists$ *invertible matrix* M *such that* $r(M \cdot x) = 1 - \|x\|^2 + o(\|x\|^2)$ *as* $x \to 0$ *for any* C^3*-smooth field* f*.*

One observation regarding the covariance structure r is that

$$r(x, y) < 1 \quad \forall x \neq y.$$

This follows from stationarity of the field and the fact that $r(x) \to 0$ as $x \to \infty$. This is helpful when estimating exceedance probability of the field over a large given threshold.

Theorem 2.2. *With the setup above and with the Assumptions 2.1 on the Gaussian field* $f : \mathbb{R}^d \to \mathbb{R}$*, we have*

$$\Phi_R \to \Phi \quad \text{in distribution as } R \to \infty$$

where Φ *is Poisson point process with intensity measure as Lebesgue measure on* $\mathbb{R}^d$*.*

First, note that invertible linear transform T of a Poisson point process (with intensity measure λ) is again a Poisson point process with new intensity measure $|\det(T)|\lambda$. So rescaling the field as in 4. of the assumption above is just for convenience. Next, Bargmann-Fock field and monochromatic random waves for dimension $d \geq 2$ satisfy the assumptions. Indeed, the covariance kernels have decay rates $\exp(-\|x\|^2/2)$ and $O(\|x\|^{-1/2})$ for Bargmann-Fock and monochromatic random waves respectively.

Now, some comments on the scaling of the field. By Appendix A, expected maximum of the field in the region $[0, R]^d$ is asymptotically $\sqrt{2d \log R}$ as $R \to \infty$. Now by super-concentration of maximum for smooth Gaussian field result [Tan15], variance of maxima behaves like $1/\log R$. Hence, for levels above $\alpha\sqrt{2d \log R}$ with $\alpha > 1$, we don't expect to see any point in $[0, R]^d$. So we assume $u \leq \sqrt{2d \log R}$ (supercritical case of $\alpha > 1$ is taken care in 'Chen-Stein method' section anyway).

Let us illustrate our scaling procedure by taking the level to be $u = \sqrt{2d\alpha \log R}$. To compare it to a homogeneous Poisson process, we need to rescale the local maxima point process to, say, unit density. Let $M_u(f, S)$ denote the number of local maxima of f in $S \subset \mathbb{R}^d$ with $f > u$. Then,

$$\mathbb{E}[M_u(f, [0, R]^d)] \simeq (\log R)^{(d-1)/2} R^{(1-\alpha)d}.$$

Rescaling the point process in $[0, R]^d$ by factor $R^{-\alpha}$ (ignoring log factors), we get a unit density process, which corresponds to Φ. Note that we've defined Φ above by reversing this procedure.

Plan of proof

It is well known at least since 1970's that avoidance probabilities (i.e. $\mathbb{P}(\eta(B) = 0)$ for Borel sets B) characterise simple point process (i.e. point processes with mass concentrated only on atoms). Now, weak convergence of these point processes can be studied by scrutinising avoidance probabilities and intensity measures.

Definition 2.3 (DC-ring)**.** *Let* $\mathcal{B}$ *be the Borel* σ*-algebra on* $\mathbb{R}^d$*. A ring* $\mathcal{L} \subset \mathcal{B}$ *is called a DC-ring ('dissecting covering' ring) if for any compact set* K *from* $\mathcal{B}$*, and arbitrary* $\epsilon > 0$*, there exists a finite covering of* K *by some sets* $l \in \mathcal{L}$ *such that diam* $L \leq \epsilon$*.*

Let $\mathcal{L}$ be a ring generated by rectangles

$$\prod_{i=1}^{d} [t_i, t_i + s_i), \quad s_i \geq 0, i = 1, 2, \ldots, d$$

which will be a DC-ring with the property that $\Phi(\partial l) = 0$ a.s. for any $l \in \mathcal{L}$. Then by [Kal17, Theorem 4.18], it is enough to show that

$$\lim_{R\to\infty} \mathbb{P}(\Phi_R(l) = 0) = \mathbb{P}(\Phi(l) = 0), \quad \limsup_{R\to\infty} \mathbb{E}\Phi_R(l) \leq \mathbb{E}\Phi(l) \tag{1}$$

for all $l \in \mathcal{L}$. In the proof below, we'll show this for $L = [0,1]^d$ but the argument works for any $l \in \mathcal{L}$.

First, we approximate avoidance probabilities of the sequence Φ_R by the excursion probabilities of the field f_R (Lemma 3.1). Then we approximate excursion probabilities on rectangles $\mathbb{P}(\sup_{R\cdot L} f_R > u)$ by that on a grid which is fine enough (Lemma 3.2). By standard theory, we know that for a regular enough field f with unit variance, the excursion set $\{f > u\}$ is captured by a grid with width u^{-1} for large u. Now we compare the excursion probabilities of the field f to that of the field f_0 which is an i.i.d copy of f on a each fixed box. This is done by comparison method for Gaussian vectors [Pit96, Thm 1.1] and is the same as proof of [Pit96, Thm15.2]. Lastly, from Lemma 3.4 we show that excursion probabilities of the field f_0 converges to avoidance probabilities of Poisson point process, which proves the first part of eq. (1).

We consider the second part of eq. (1). Computing expected number of critical points of given index of smooth Gaussian fields is classical problem in this field [Adl10]. Thanks to Kac-Rice formulas, we know precise estimates of these quantities, even explicit result in some cases. Using these estimates, we'll show that

$$\lim_{R\to\infty} \mathbb{E}[\Phi_R(L)] = \mathbb{E}[\Phi(L)].$$

These two parts conclude the proof of the theorem 2.2.

3 Proof

Recall that L is a unit box in $\mathbb{R}^d$ and let $L_R := R \cdot L$. Define

$$P_f(u,S) = \mathbb{P}\left(\sup_{t\in S} f(t) \leq u\right) \quad \text{and} \quad \bar{P}_f(u,S) = \mathbb{P}\left(\sup_{t\in S} f(t) \geq u\right).$$

Let A be a ball centred at origin in $\mathbb{R}^d$. We define Minkowski sum of two subsets A, B of $\mathbb{R}^d$ as

$$A \oplus B = \{x + y : x \in A, y \in B\}.$$

Now we approximate the avoidance probability of point process with excursion probabilities.

Lemma 3.1. *With the above setup, we have*

$$\mathbb{P}(\Phi_R(L) = 0) = P_{f_R}(u, L_R) + o(1) \quad \text{as } R \to \infty.$$

Proof. First, observe that $\mathbb{P}(\Phi_R(L) = 0) \geq P_{f_R}(u, L_R)$. From the fact that each connected component of $\{f(x) \geq u\}$ must have a local maximum, we have

$$\{\Phi_R(L) > 0\} \supseteq \left\{\sup_{L_R} f_R \geq u, \sup_{(L_R\oplus A)\setminus L_R} f_R < u\right\}.$$

Note that the RHS just makes sure that L_R has at least one component of $\{f_R(x) \geq u\}$ lying completely inside it. Hence,

$$\mathbb{P}(\Phi_R(L) = 0) \leq P_{f_R}(u, L_R) + \mathbb{P}\left(\sup_{L_R} f_R \geq u, \sup_{(L_R\oplus A)\setminus L_R} f_R \geq u\right).$$

Now,

$$\mathbb{P}\left(\sup_{L_R} f_R \geq u, \sup_{(L_R\oplus A)\setminus L_R} f_R \geq u\right) \leq \bar{P}_{f_R}(u, (L_R \oplus A) \setminus L_R).$$

Noting that $\mathrm{vol}((L_R \oplus A) \setminus L_R) = O(R^{d-1})$ for large R and that $\bar{P}_{f_R}(u, S) = \bar{P}_f(u, \mu(R) \cdot S)$ and applying [Pit96, Thm 7.1], using homogeneity of the field, we have

$$\begin{aligned}\bar{P}_{f_R}(u, (L_R \oplus A) \setminus L_R) \leq & C \cdot \mathrm{vol}(\mu(R) \cdot (L_R \oplus A) \setminus L_R) u^{d-1} \exp(-u^2/2) \\ =& O(R^{-1}) \quad \text{as } R \to \infty.\end{aligned}$$

□

Now, we discretise the domain and approximate the excursion probabilities on this grid as explained before. Let g_R be some scaling (to be determined in the course of the proof). Fixing $b > 0$, define $\mathcal{R}_b = bg_R\mathbb{Z}^d$.

Lemma 3.2. *For any $\epsilon > 0$, there exists $b, R_0 > 0$ such that for all $R > R_0$,*

$$P_{f_R}(u, L_R \cap \mathcal{R}_b) - P_{f_R}(u, L_R) \leq \epsilon.$$

Proof. We have

$$P_{f_R}(u, L_R \cap \mathcal{R}_b) - P_{f_R}(u, L_R) = \mathbb{P}\left(\sup_{L_R\cap\mathcal{R}_b} f_R \leq u, \sup_{L_R} f_R > u\right).$$

By homogeneity of the field f_R, we have (calling $\mu(R)\mathcal{R}_b = \mathcal{R}_b'$)

$$\mathbb{P}\left(\sup_{L_R\cap\mathcal{R}_b} f_R \leq u, \sup_{L_R} f_R > u\right) \leq (R\mu(R))^d \mathbb{P}\left(\sup_{L\cap\mathcal{R}_b'} f \leq u, \sup_{L} f > u\right)$$

Now by the standard theory of excursion approximation (see [Pit96, Lemma 15.3]), when $g_R = (u\mu(R))^{-1}$ and $b > 0$ is small enough, we have

$$\mathbb{P}\left(\sup_{[0,1]^d\cap\mathcal{R}_b'} f \leq u, \sup_{[0,1]^d} f > u\right) \leq \epsilon \quad , R > R_0.$$

□

We define $\lambda_{a,R}$ given numbers $a > \delta > 0$. Divide the rectangle $\mu(R)R\cdot L$ into smaller ones by following construction. Divide each edge of $\mu(R)R \cdot L$ into segments of length 'a' alternated by that of δ. Call $\lambda_{a,R}$ the union of cubes of side length a. Note that the distance between the cubes are greater than δ. The following lemma says that if gap between the cubes of $\lambda_{a,R}$ are small enough, then the excursion probabilities are close to that the discretisation of $R\mu(R) \cdot L$.

Lemma 3.3. *For any $a, \epsilon > 0$ given, there exists $\delta > 0$, such that, for all R large enough we have,*

$$P_f(u, \lambda_{a,R} \cap \mathcal{R}_b') - P_f(u, \mu(R)R \cdot L \cap \mathcal{R}_b') \leq \epsilon.$$

Proof. We have that

$$P_f(u, \lambda_{a,R} \cap \mathcal{R}_b') - P_f(u, \mu(R)R \cdot L \cap \mathcal{R}_b') \leq \mathbb{P}\left(\sup_{\lambda_{a,R}\cap\mathcal{R}_b'} f \leq u, \sup_{\mu(R)R\cdot L\cap\mathcal{R}_b'} f > u\right).$$

Now using homogeneity of the field,

$$\mathbb{P}\left(\sup_{\lambda_{a,R}\cap\mathcal{R}'_b} f \le u, \sup_{\mu(R)R\cdot L\cap\mathcal{R}'_b} f > u\right) \le \bar{P}_f(u, \mu(R)R\cdot L\setminus\lambda_{a,R})$$
$$\le \mathrm{vol}(\mu(R)R\cdot L\setminus\lambda_{a,R})\bar{P}_f(u,L)$$
$$\le \delta\frac{(\mu(R)R)^d}{(a+\delta)}\bar{P}_f(u,L)$$
$$\le C\delta((\mu(R)R)^d)u^{d-1}\exp(-u^2/2)$$

Now, we get that the expression is bounded by $c\cdot\delta$ where c is a constant which doesn't depend on R. □

Let f_0 be a field defined on $\lambda_{a,R}$ such that on the cubes of side length a, the field is made up of i.i.d copies of f. We now show that the excursion probability of f_0 converges to avoidance probability of Poisson point process.

Lemma 3.4. *We have*

$$P_{f_0}(u,\lambda_{a,R}) \to \exp(-vol(L)) \qquad \text{as } R\to\infty.$$

Proof. Let N be the number of cubes of side length a in $\lambda_{a,R}$. Then,

$$P_{f_0}(u,\lambda_{a,R}) = (1-\bar{P}_f(u,[0,a]^d))^N$$

by independence of the field on these cubes. Taking logarithm, it is enough to estimate

$$N\log(1-\bar{P}_f(u,[0,a]^d)) = -N\bar{P}_f(u,[0,a]^d) + O(N\bar{P}_f(u,[0,a]^d)^2).$$

Now,

$$\bar{P}_f(u,[0,a]^d) = \kappa a^d u^{d-1}\exp(-u^2/2)(1+o(1))$$

and

$$N = \left(\frac{R\mu(R)}{a+\delta}\right)^d + O((R\mu(R))^{d-1}).$$

Hence,

$$N\bar{P}_f(u,[0,a]^d) = \left(\frac{a}{a+\delta}\right)^d + o(1) \text{ and } N\bar{P}_f(u,[0,a]^d)^2 = o(1).$$

Since L is a unit box and we can take δ arbitrarily small, we have the result. □

Proof of Theorem 2.2. First, observe that all the proof of lemmas goes through even when L is a finite union of finite rectangles. For any given $\epsilon>0$, there exists a,b,δ,R_0 such that for all $R>R_0$,

$$|\mathbb{P}(\Phi_R(L)=0) - P_f(u,\lambda_{a,R}\cap\mathcal{R}'_b)| \le \epsilon.$$

We show that $|P_f(u,\lambda_{a,R}\cap\mathcal{R}'_b) - P_{f_0}(u,\lambda_{a,R}\cap\mathcal{R}'_b)| \to 0$ as $R\to\infty$ then by Kallenberg's theorem (see [Pit96, Section 13]) we're done. This is done by method of comparison for Gaussian vectors as in Theorem 1.1 of Piterbarg, which is a generalisation of the classical Berman inequality. For the rest of the proof, we follow argument of proof of Thm 15.2 of [Pit96].

Let K_i be a renumbering of cubes with edges of length a which comprise $\lambda_{a,R}$, $i=1,2,\dots,N$. Let covariance of the field f_0 on $\lambda_{a,R}$ be denoted by $r_0(t,s)$. Define $\lambda'_{a,R,b} = \lambda_{a,R}\cap\mathcal{R}'_b$ Then by Theorem 1.1 of [Pit96], we have

$$|P_f(u,\lambda'_{a,R,b}) - P_{f_0}(u,\lambda'_{a,R,b})| \le \frac{1}{\pi} \sum_{t,s\in\lambda'_{a,R,b}} |r(t-s) - r_0(t-s)| \\ \times \int_0^1 (1-(hr(t,s))^2)^{-1/2} \exp\left(-\frac{u^2}{1+hr(t,s)}\right) dh. \tag{2}$$

Denote the summand on RHS of above equation by $\beta(t,s)$. If $t,s \in K_i$ for some i, then $r(t,s) = r_0(t,s)$, hence $\beta(t,s) = 0$.
Now consider the case that t,s belong to different K_i and K_j such that $|t-s| \le R^{\gamma_1}$, where $\gamma_1 > 0$ is a constant chosen later. Since t,s belong different cubes, we have $|t-s| > \delta$, hence $|1-r(t,s)| > \gamma_2 > 0$. So,

$$\frac{1}{1+r(t,s)} > \frac{1}{2} + \frac{\gamma_2}{4}.$$

Now,

$$\begin{aligned} \sum_{\substack{t\in K_i, s\in K_j, i\ne j,\\ |t-s|<R^{\gamma_1}}} \beta(t,s) &\le C_1 \sum |r(t,s)| \exp\left(-\frac{u^2}{1+r(t,s)}\right) \\ &\le C_2(\mu(R)R)^d R^{\gamma_1 d} \exp(-u^2/2(1+\gamma_2/2)) \\ &\le C_3 u^{d-1} e^{u^2/2} e^{\gamma_1 u^2/2} \exp(-u^2/2(1+\gamma_2/2)) \\ &\to 0 \quad \text{as} \quad R\to\infty \quad \text{if} \quad 0<\gamma_1<\gamma_2. \end{aligned} \tag{3}$$

Here we've used that $u \le \sqrt{2d\log R}$, and C_i's are different constants not depending on R.

Lastly, we consider the case where $|t-s| \ge R^{\gamma_1}$. We have,

$$\begin{aligned} \sum_{\substack{t\in K_i, s\in K_j, i\ne j,\\ |t-s|\ge R^{\gamma_1}}} \beta(t,s) &\le C_1 \sum |r(t,s)| \exp\left(-\frac{u^2}{1+r(t,s)}\right) \\ &\le C_2(\mu(R)R)^{2d} r'(R^{\gamma_1}) \exp\left(-\frac{u^2}{1+r'(R^{\gamma_1})}\right) \\ &\le C_3 u^{2d-2} r'(R^{\gamma_1}) \exp\left(\frac{r'(R^{\gamma_1})u^2}{1+r'(R^{\gamma_1})}\right) \end{aligned} \tag{4}$$

where

$$r'(h) := \max_{|t|\ge h} |r(t,0)|, \quad h\in(0,\infty).$$

Observing that the assumption on the decay of correlation (point 2 of Assumption 2.1) implies that $u^{2d-2}r'(R^{\gamma_1}) \to 0$, since we have $u \le \sqrt{2d\log R}$ and $d \ge 2$. In particular, $u^2 r'(R^{\gamma_1}) \to 0$. Hence,

$$\sum_{\substack{t\in K_i, s\in K_j, i\ne j,\\ |t-s|\ge R^{\gamma_1}}} \beta(t,s) \to 0 \quad \text{as} \quad R\to\infty.$$

□

Computation of expectation

Our next goal is to show the following

$$\lim_{R\to\infty} \mathbb{E}[\Phi_R(L)] = \mathbb{E}[\Phi(L)].$$

For the case that $(f(0), \nabla f(0), \nabla^2 f(0))$ having density in $\mathbb{R}^{(d+1)+d(d+1)/2}$, [Adl10, Thm 6.3.1] suffices. If the field is isotropic and if $(f(0), \nabla f(0), \nabla^2 f(0))$ is degenerate then the field has to be monochromatic random wave (MRW) (see [CS18, Prop 3.10]). From Example 3.15 of [CS18], we can calculate the limit of $\mathbb{E}[\Phi_R(L)]$ for the case $d = 2$. But explicit expressions for height densities are hard to get for $d \geq 3$ directly. So we shift the MRW field by an independent normal random variable, so that the joint vector of the field, its gradient, and hessian has density. Then we use the explicit asymptotic as in [Adl10, Thm 6.3.1].

Let us first consider the case that $(f(0), \nabla f(0), \nabla^2 f(0))$ having density in $\mathbb{R}^{(d+1)+d(d+1)/2}$. As mentioned, we'll use the following theorem by Adler

Theorem 3.5 (c.f. [Adl10] Theorem 6.3.1)**.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a stationary, C^2-smooth Gaussian field such that $(f(x), \nabla f(x), \nabla^2 f(x))$ is non-degenerate for all $x \in \mathbb{R}^d$. Further assume that $f(x)$ has zero mean, unit variance. Let $M_u(f, S)$ denote the number of local maxima of f in $S \subset \mathbb{R}^d$ with $f > u$. Then,*

$$\mathbb{E}[M_u(f, S)] = \frac{vol(S) \det(\Lambda_f)^{1/2} u^{d-1}}{(2\pi)^{(d+1)/2}} \exp\left(-u^2/2\right)(1 + O(u^{-1}))$$

where Λ_f is the covariance matrix of ∇f and $O(u^{-1})$ is independent of choice of S.

Then by above theorem, for any Borel set $B \subset \mathbb{R}^d$

$$\begin{aligned}
\mathbb{E}[\Phi_R(B)] &= \mathbb{E}[\eta_R(\mu(R) \cdot B)] \\
&= \frac{\text{vol}(R\mu(R) \cdot B)}{(2\pi)^{(d+1)/2}} u^{d-1} \exp(-u^2/2)(1 + O(u^{-1})) \\
&= \text{vol}(B)(1 + O(u^{-1})) \\
&\to \mathbb{E}[\Phi(B)] \quad \text{as } R \to \infty.
\end{aligned} \tag{5}$$

Here, we've used the fact that determinant of covariance matrix of ∇f is 1, which follows from point 4 of Assumption 2.1.

Now we consider the monochromatic random waves (MRW) case. Let $f : \mathbb{R}^d \to \mathbb{R}$ be an MRW field. Let $\epsilon > 0$ and consider a random variable N, independent of the field f, which is standard normal variate. Define,

$$f_\epsilon(x) := f(x) + \epsilon N, \quad x \in \mathbb{R}^d.$$

Observe that f_ϵ is still a centred, stationary field and that $f_\epsilon(0), \nabla f_\epsilon(0), \nabla^2 f_\epsilon(0)$ is a Gaussian vector with density. Define $M(u, g)$ to be the number of local maxima of a Gaussian field g in $[0, 1]^d$.
Now we have, by an application of Kac-Rice formula,

$$\mathbb{E}[M(u, f_\epsilon)] = \int_{\mathbb{R}} \mathbb{E}[M(u - \epsilon b, f) | N = b] \phi(b) d\text{b}.$$

where ϕ is the pdf of standard normal variate. Also,

$$M(u - \epsilon b, f) \longrightarrow M(u, f) \quad \text{a.s.} \quad \text{as} \quad \epsilon \to 0.$$

We know that $M(u, f)$ is integrable, and monotonic w.r.t. u, so using dominated convergence theorem,

$$\mathbb{E}[M(u - \epsilon b, f)] \to \mathbb{E}[M(u, f)], \quad \epsilon \to 0.$$

Since $\mathbb{E}[M(u, f)]$ is uniformly bounded in u, apply DCT for $\mathbb{E}[M(u - \epsilon b, f)]\phi(b)$ to get,

$$\lim_{\epsilon \to 0} \mathbb{E}[M(u, f_\epsilon)] = \mathbb{E}[M(u, f)].$$

Computing $\mathbb{E}[M(u, f_\epsilon)]$ is handled again by [Adl10, Thm 6.3.1] as eq (5).

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