Asymptotic behavior of plasmas in Euclidean space

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Joint works with:

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Stability Analysis for Nonlinear PDEs
15-19 August 2022
Oxford University

EPSRC
Engineering and Physical Sciences
Research Council

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Phase-space evolution governed by the Vlasov equation

$$\partial_t f + v(p) \cdot \nabla_x f + F[f] \cdot \nabla_p f = 0$$

where v = velocity   *$v(p) = p$ unless otherwise stated*
F = forcing term (electric field/Lorentz force)

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where $v =$ velocity *$v(p) = p$ unless otherwise stated*
$F =$ forcing term (electric field/Lorentz force)

Main goal: quantitative large-time asymptotics. In particular, behavior of

$$\rho(t, x) = \int_{\mathbb{R}^d} f(t, x, p)\, \mathrm{d}p$$

What's known? Vlasov–Poisson (VP): $F = E$

Existence and uniqueness: Main ingredient is *a priori* estimates for the vector field $p \cdot \nabla_x + E \cdot \nabla_p$ (mid-late 1980s)

- Bardos-Degond (small data)
- Pfaffelmoser, Schaeffer (compact support in (x, p))
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Long-time behavior:

- Bernstein-Greene-Kruskal (1957): explicit inhomogenous traveling wave solutions (BGK waves)
- On $\mathbb{T}^d$: Decay to homogeneous equilibrium ('damping'): Landau (1946), Glassey, Guo, Schaeffer, Strauss (1990s), Mouhot-Villani (2010), Lin-Zeng (2010s) ....
- On $\mathbb{R}^d$: dispersion to 'infinity': Horst, Rein, Pankavich, Ionescu-Pausader-Wang-Widmayer ....

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\begin{aligned}
\|f(t,\cdot,\cdot)\|_{L^\infty_x(L^1_p)} = \|\rho(t)\|_{L^\infty_x} &= \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|f(t,x,p)|\mathrm{d}p \\
&= \sup_x \int_{\mathbb{R}^d}|f_0(x-pt,p)|\mathrm{d}p \\
&\le \sup_x \int_{\mathbb{R}^d}\sup_u |f_0(x-pt,u)|\mathrm{d}p \\
&= \frac{1}{t^d}\int_{\mathbb{R}^d}\sup_u |f_0(w,u)|\mathrm{d}w = \frac{1}{t^d}\|f_0(\cdot,\cdot)\|_{L^1_x(L^\infty_p)}
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\end{aligned}
$$

One would expect even better decay in the plasma (repulsive!!) case

$$\partial_t f + p \cdot \nabla_x f + \boxed{F \cdot \nabla_p f} = 0$$

but that question is still largely open!

Important Known Results for VP ($d = 3$)

Theorem (Bardos-Degond, 1985)

Any small data solution of (VP) satisfies

$$\|\rho(t)\|_\infty \lesssim t^{-3}, \qquad \|E(t)\|_\infty \lesssim t^{-2}.$$

Theorem (Horst, 1990)

Any spherically symmetric solution of (VP) satisfies

$$\|\rho(t)\|_\infty \lesssim t^{-3}, \qquad \|E(t)\|_\infty \lesssim t^{-2}.$$

Theorem (Yang, 2016)

Any solution of (VP) satisfies

$$\|E(t)\|_\infty \lesssim t^{-\frac{1}{6}+}.$$

Plan of the Talk

1. The Good: Decay for Two-Dimensional Symmetric Plasmas
2. The Bad: Arbitrarily Large Solutions Three-Dimensions
3. The Ugly: The Vlasov–Maxwell System

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Two-dimensional radially-symmetric plasmas

Radial-symmetry allows us to replace coordinates $(x, p) \in \mathbb{R}^2 \times \mathbb{R}^2$ by

$$r = |x|, \qquad w = \frac{x \cdot p}{r}, \qquad \ell = |x \times p|^2,$$

(3 deg of freedom instead of 4) and Vlasov–Poisson equations reduce to

$$\partial_t f + w \partial_r f + \left(\frac{\ell}{r^3} + \frac{m(t,r)}{2\pi r} \right) \partial_w f = 0,$$

where

$$m(t,r) = 2\pi \int_0^r \rho(t,q) q \, \mathrm{d}q$$

and

$$\rho(t,r) = \frac{1}{r} \int_0^\infty \int_{-\infty}^\infty f(t,r,w,\ell) \ell^{-1/2} \, \mathrm{d}w \, \mathrm{d}\ell.$$

Characteristics:

$$\begin{cases} \dot{\mathcal{R}}(s) = \mathcal{W}(s) \\ \dot{\mathcal{W}}(s) = \dfrac{\mathcal{L}(s)}{\mathcal{R}(s)^3} + \dfrac{m(s, \mathcal{R}(s))}{2\pi \mathcal{R}(s)} \\ \dot{\mathcal{L}}(s) = 0 \end{cases}$$

Define

$$\mathfrak{R}(t) = \sup_{(r,w,\ell)\in \mathrm{supp}(f_0)} \mathcal{R}(t, 0, r, w, \ell) = \text{“farthest particle”}$$

$$\mathfrak{W}(t) = \sup_{(r,w,\ell)\in \mathrm{supp}(f_0)} |\mathcal{W}(t, 0, r, w, \ell)| = \text{“fastest particle”}$$

and

$$\mathcal{U}(t, r) = -\Delta^{-1}\rho(t, r) = \text{electric potential}$$

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For characteristics satisfying $m(t, \mathcal{R}(t)) \gtrsim 1$, we have

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\begin{aligned}
|\mathcal{W}(t,\tau,r,w,\ell) - w| &= \int_\tau^t \left(\frac{m(s,\mathcal{R}(s))}{\mathcal{R}(s)} + \frac{\ell}{\mathcal{R}(s)^3} \right) \, \mathrm{d}s \\
\text{(this will follow from the theorem)} &\geq C \int_\tau^t \left(s\sqrt{\log(s)} \right)^{-1} \mathrm{d}s \\
&\gtrsim \sqrt{\log(t)}
\end{aligned}
$$

and

$$
|\mathcal{R}(t,\tau,r,w,\ell) - (r + wt)| \gtrsim t\sqrt{\log(t)}.
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$$\gtrsim \sqrt{\log(t)}$$

and

$$|\mathcal{R}(t,\tau,r,w,\ell) - (r + wt)| \gtrsim t\sqrt{\log(t)}.$$

So we cannot hope to converge to the free-streaming solution in any sense.

Theorem (JBA-Morisse-Pankavich, *arXiv:2202.03717*)

Let $f_0 \in C_0^1(\mathbb{R}^4)$ *be radially-symmetric and let* $p \in (2, +\infty]$. *Assume that* $\inf\{\ell : (r, w, \ell) \in \mathrm{supp}(f_0)\} > 0$. *Then we have*

$$\mathfrak{W}(t) \sim \sqrt{\log(t)}, \quad \mathfrak{R}(t) \sim t\sqrt{\log(t)}, \quad \|\mathcal{U}(t)\|_\infty \sim \log(t),$$

as well as the field and density estimates

$$\left(t\sqrt{\log(t)}\right)^{-1+\frac{2}{p}} \lesssim \|E(t)\|_p \lesssim t^{-1+\frac{2}{p}},$$
$$\left(t^2 \log(t)\right)^{-1} \lesssim \|\rho(t)\|_\infty \lesssim t^{-1},$$

and for $(r, w, \ell) \in \mathrm{supp}(f_0)$ *the pointwise estimates*

$$0 \lesssim \mathcal{W}(t, 0, r, w, \ell) \lesssim \sqrt{\log(t)},$$
$$t \lesssim \mathcal{R}(t, 0, r, w, \ell) \lesssim t\sqrt{\log(t)}.$$

Sketch of Proof

1. Characteristics Lower Bound: $\mathcal{R}(t)^2 \geq \ell r^{-2} t^2$ and $\exists T \geq 0$ s.t. $\mathcal{W}(t) > 0 \; \forall t > T$.

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These come from writing out $m(t, r)$ and $m(t, \mathfrak{R}(t))$ and using the lower bound on $\mathcal{R}$.

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5. Improved Characteristics Upper Bounds: $|\mathcal{W}(t)| \lesssim \sqrt{\log(t)}$, $\mathcal{R}(t) \lesssim t\sqrt{\log(t)}$.

This comes from looking at the change in total energy along characteristics: $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left(\mathcal{W}(t)^2 + \ell \mathcal{R}(t)^{-2}\right) + \mathcal{U}(t, \mathcal{R}(t))\right)$

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This uses ideas of Horst to use backwards characteristics to estimate the size of the w support of $f(t, r, w, \ell)$ for fixed $r, \ell > 0$.

The problem: $\mathcal{R} \sim r + wt + \iint E \sim r + wt + t \quad \Rightarrow \quad w \sim 1$

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6. Particle Density: $\|\rho(t)\|_\infty \lesssim t^{-1}$ and trivially $\|\rho(t)\|_\infty \gtrsim \mathfrak{R}(t)^{-2}$.

Plan of the Talk

Now we consider Vlasov–Poisson in three-dimensions.

$$\begin{cases} \partial_t f + v(p) \cdot \nabla_x f + E \cdot \nabla_p f = 0, \\ \nabla \cdot E = \rho, \end{cases}$$

where $\rho(t, x) = \int_{\mathbb{R}^3} f(t, x, p) \, \mathrm{d}p$.

Theorem (JBA-Calogero-Pankavich, *SIMA 2018*)

1) For any constants $C_1, C_2 > 0$ *there exists a smooth, spherically-symmetric solution of (VP) such that*

$$\|\rho(0)\|_\infty, \quad \|E(0)\|_\infty \leq C_1$$

but for some time $T > 0$*,*

$$\|\rho(T)\|_\infty, \quad \|E(T)\|_\infty \geq C_2.$$

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2) For any constants $C_1, C_2 > 0$ *and any* $T > 0$ *there exists a smooth, spherically-symmetric solution of (VP) such that*

$$M = \iint_{\mathbb{R}^6} f_0(x,p)\,\mathrm{d}p\,\mathrm{d}x = C_1$$

and

$$\|\rho(T)\|_\infty, \quad \|E(T)\|_\infty \geq C_2.$$

Reminder:

Theorem (Horst, 1990)

Any spherically symmetric solution of (VP) satisfies

$$\|\rho(t)\|_\infty \leq \frac{C}{t^3}, \qquad \|E(t)\|_\infty \leq \frac{C}{t^2}.$$

Reminder:

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Our results show that in the intermediate regime these quantities can become arbitrarily large and that this may take arbitrarily long.

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1) Design initial data supported on spherical shell in x variable, while p variable supported around $-Cx$ with $C > 0$ to be chosen.

2) Write ODEs for particle trajectories in coordinates adapted to spherical symmetry.

3) Obtain lower & upper bounds for $\mathcal{R}(t)$ uniform in time.

4) Find time T when spherical shell is so small, that density is necessarily very large.

Three-dimensional radially-symmetric plasmas

Spherical-symmetry allows us to replace coordinates $(x, p) \in \mathbb{R}^3 \times \mathbb{R}^3$ by

$$r = |x|, \qquad w = \frac{x \cdot p}{r}, \qquad \ell = |x \times p|^2,$$

(3 deg of freedom instead of 6) and Vlasov–Poisson equations reduce to

$$\partial_t f + w\partial_r f + \left(\frac{\ell}{r^3} + \frac{m(t, r)}{r^2} \right) \partial_w f = 0,$$

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$$m(t, r) = 4\pi \int_0^r \rho(t, q) q^2 \, \mathrm{d}q$$

and

$$\rho(t, r) = \frac{\pi}{r^2} \int_0^\infty \int_{-\infty}^\infty f(t, r, w, \ell) \, \mathrm{d}w \, \mathrm{d}\ell.$$

Characteristics

Particles obey the ODEs:

$$\begin{cases} \dot{\mathcal{R}}(s) = \mathcal{W}(s), \\ \dot{\mathcal{W}}(s) = \dfrac{\mathcal{L}(s)}{\mathcal{R}(s)^3} + \dfrac{m(s, \mathcal{R}(s))}{\mathcal{R}(s)^2}, \\ \dot{\mathcal{L}}(s) = 0, \end{cases}$$

with

$$\mathcal{R}(0) = r, \qquad \mathcal{W}(0) = w, \qquad \mathcal{L}(0) = \ell.$$

Trajectories

A lemma about behavior of characteristics, and in particular the time T_0 when each particle is closest to origin:

Lemma

Let $L > 0$, $P \geq 0$, $y_0 > 0$ and $y_1 < 0$ be given. Assume y satisfies

$$0 \leq \ddot{y}(t) - Ly(t)^{-3} \leq Py(t)^{-2}, \qquad y(0) = y_0, \quad \dot{y}(0) = y_1.$$

1. $\exists!$ *minimum* $T_0 > 0$.
2. $y(T_0) \leq y_*$ *and* $T_0 \geq \frac{y_0 - y_*}{|y_1|}$, *where* $y_* = y_0 \sqrt{\frac{L + Py_0}{y_0^2 y_1^2 + L + Py_0}}$.
3. $y(t)^2 \leq (y_0 + y_1 t)^2 + (Ly_0^{-2} + Py_0^{-1})t^2, \qquad \forall t \in [0, T_0]$.

Initial Data

Require $r \approx a_0$, $w \approx a_1$ (large and negative) and ℓ small:

$$\left(r + \frac{a_0}{|a_1|} w\right)^2 + \frac{\ell}{r^2}\left(\frac{a_0}{a_1}\right)^2 < \frac{\epsilon^2}{a_1^2}$$

and $r \in (a_0 - \delta_r, a_0 + \delta_r)$ with $\delta_r = \epsilon^3$ (ϵ defined appropriately)

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and $r \in (a_0 - \delta_r, a_0 + \delta_r)$ with $\delta_r = \epsilon^3$ (ϵ defined appropriately)

Furthermore, require

$$\rho_0(r) \leq \frac{3}{4\pi a_0^3} \leq C_1, \qquad \forall r > 0,$$

and

$$\rho_0(r) = \frac{3}{4\pi a_0^3}, \qquad \text{for } r \in \left[a_0 - \frac{1}{2}\delta_r, a_0 + \frac{1}{2}\delta_r\right].$$

At Time $t = 0$

Previous conditions imply

$$\frac{3\epsilon^3}{a_0} \leq M \leq \frac{8\epsilon^3}{a_0}$$

and

$$\|\rho(0)\|_\infty, \|E(0)\|_\infty \leq C_1.$$

At Time $T > 0$

From Lemma: minimum of each trajectory satisfies
$T_0 = T_0(r, w, \ell) \geq \frac{y_0 - y_*}{|y_1|}$, allows to find some

$$T \in \left(0, \inf_{(r,w,\ell)\in \mathrm{supp}(f_0)} T_0\right],$$

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so that, from Lemma:

$$\mathcal{R}(T)^2 \leq (r + wT)^2 + (\ell r^{-2} + 8a_0^{-1} r^{-1}) T^2 \leq \cdots \leq 10000\epsilon^4$$

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so that, from Lemma:

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which leads to

$$\|\rho(T)\|_\infty \geq \frac{3M}{4\pi\,(100\epsilon^2)^3} \geq \frac{1}{200^3 a_0 \epsilon^3} \geq C_2$$

and

$$\|E(T)\|_\infty \geq \frac{M}{(100\epsilon^2)^2} \geq \frac{3}{100^2 a_0 \epsilon} \geq C_2.$$

Our theorem was later improved by Zhang to handle C^k norms:

Theorem (Zhang, 2019)

For any constants $C_1, C_2 > 0$ there exists a smooth, spherically-symmetric solution of (VP) such that

$$\|\rho(0)\|_{C^k}, \quad \|E(0)\|_{C^k} \leq C_1$$

but for some time $T > 0$,

$$\|\rho(T)\|_{\infty}, \quad \|E(T)\|_{\infty} \geq C_2.$$

There is also some characterization of how T depends on the initial data.

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2. The Bad: Arbitrarily Large Solutions Three-Dimensions
3. The Ugly: The Vlasov–Maxwell System

The Vlasov–Maxwell system is

$$\begin{cases} \partial_t f + v(p) \cdot \nabla_x f + (E + v(p) \times B) \cdot \nabla_p f = 0, \\ \partial_t E - \nabla \times B = -j, \qquad \nabla \cdot E = \rho, \\ \partial_t B + \nabla \times E = 0, \qquad \nabla \cdot B = 0, \end{cases}$$

where $v(p) = \frac{p}{\sqrt{1+|p|^2}}$ and

$$\rho(t,x) = \int f(t,x,p)\,\mathrm{d}p \qquad \text{and} \qquad j(t,x) = \int v(p) f(t,x,p)\,\mathrm{d}p.$$

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Maxwell's equations are hyperbolic, unlike Poisson which is elliptic
$\Rightarrow$ existence theory is much harder.

Important Existence and Uniqueness Results

Theorem (Glassey-Strauss, 1986)

If momenta are known to be uniformly bounded for all particles on $[0, T]$, then the solution can be continued to $[T, T+h)$ for some $h > 0$.

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If $\|f_0\|_{C^1} + \|E_0\|_{C^2} + \|B_0\|_{C^2} < \epsilon$ then there is a global-in-time classical solution and (inside the light cone)

$$\|E(t)\|_\infty + \|B(t)\|_\infty \lesssim \frac{1}{t^2}.$$

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Theorem (Glassey-Schaeffer, multiple results 1990s)

Global-in-time existence results in various lower dimensional settings.

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Theorem (Klainerman-Staffilani, 2002; Bouchut-Golse-Pallard, 2003)

Reproving the 1986 Glassey-Strauss result by other methods.

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Theorem (Luk-Strain, 2014)

Some improvements of the 1986 Glassey-Strauss result (weaker assumptions).

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Important Existence and Uniqueness Results

We work within the framework of this theorem, to obtain quantitative asymptotic results. One of the main difficulties: we need $f(t, \cdot, \cdot) \in C^2$ and $E(t, \cdot), B(t, \cdot) \in C^3$.

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We want improved asymptotic quantitative results for the fields, which require solving equations like $\Box E^i = -\partial_{x_i}\rho - \partial_t j^i$, and we need these in C^1.

Theorem (JBA-Pankavich, *in preparation*)

Non-neutral / Neutral

- *Every particle has a limiting velocity, achieved at rate* t^{-1} t^{-2}
- $\int f(t,x,p)\,\mathrm{d}x$ *has a limit, convergence at rate* $t^{-1}\log^4(t)$ t^{-2}
- *There exists* $\rho_\infty(p)$ *such that*

$$\sup_{(x,p)}\left|t^3\rho(t,x+v(p)t)-\rho_\infty(p)\right|\lesssim t^{-1}\log^7(t)$$

$$\|\rho(t)\|_\infty\lesssim t^{-4}$$

- *Similarly for* j *and for the derivatives of* ρ, j.
- *There exists* $E_\infty(p)$ *such that inside the light cone (similarly* B_∞*)*

$$\sup_{(x,p)}\left|t^2E(t,x+v(p)t)-E_\infty(p)\right|\lesssim t^{-1}\log^7(t)$$

$$\sup_{(x,p)}|E(t,x+v(p)t)|\lesssim t^{-3}$$

The rates in the neutral case are faster than free-streaming which suggests that there is some damping mechanism.

Main Idea of the Proof

Particle distribution $f(t, x, p)$ satisfies Vlasov:

$$\partial_t f = -v(p) \cdot \nabla_x f - K \cdot \nabla_p f.$$

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We want to integrate in time to establish behavior of trajectories.

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Particle distribution $f(t, x, p)$ satisfies Vlasov:

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Replace f with $g(t, x, p) = f(t, x + v(p)t, p)$, which satisfies:

$$\partial_t g = t\mathbb{A}(p) \underbrace{K}_{\lesssim t^{-2}} \cdot \nabla_x g - \underbrace{K}_{\lesssim t^{-2}} \cdot \underbrace{\nabla_p g}_{\lesssim \log^2(t)} .$$

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First term $\lesssim t^{-1}$, not good enough. Replace g with:
$h(t,x,p) = g(t,x-\log(t)\mathbb{A}(p)K_\infty(p),p)$, Vlasov becomes:

$$\partial_t h = t^{-1}\mathbb{A}(p)\underbrace{(t^2K-K_\infty)}_{\lesssim t^{-\epsilon}}\cdot\nabla_x g + \underbrace{K}_{\lesssim t^{-2}}\cdot\underbrace{\nabla_p g}_{\lesssim \log^2(t)}.$$

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So $\partial_t h$ is integrable in time.

1. Characteristics

$$\begin{cases} \dot{\mathcal{X}}(s) = v(\mathcal{P}(s)), \\ \dot{\mathcal{P}}(s) = E(s, \mathcal{X}(s)) + v(\mathcal{P}(s)) \times B(s, \mathcal{X}(s)) =: K(s, \mathcal{X}(s), \mathcal{P}(s)) \end{cases}$$

Since we know that the fields decay $\lesssim t^{-2}$ we easily find that

$$\begin{aligned} \mathcal{P}_\infty(\tau, x, p) &= \lim_{t \to +\infty} \mathcal{P}(t, \tau, x, p) \\ &= p + \int_\tau^\infty K(s, \mathcal{X}(s), \mathcal{P}(s))\, \mathrm{d}s \end{aligned}$$

exists, and $\mathcal{P} \to \mathcal{P}_\infty$ at rate $\lesssim t^{-1}$.

2. A New Frame

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$$g(t, x, p) = f(t, x + v(p)t, p)$$

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Lemma

We have the estimates

$$|\mathcal{Y}(t)| \lesssim \log(t) \qquad \textit{and} \qquad \mathcal{R}(t) \lesssim \log(t)$$

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Lemma

Derivatives of g grow slower than derivatives of f:

$$\|\nabla_p g(t)\|_\infty \lesssim \log^2(t)$$

(for f it is $\|\nabla_p f(t)\|_\infty \lesssim t$)

3. Decay of the Fields

The fields have decay rates

$$\|E(t)\|_\infty \lesssim t^{-2}$$
$$\|\nabla_x E(t)\|_\infty \lesssim t^{-3}\log(t)$$
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Lemma

As a consequence we have

$$\|\nabla_p^2 g(t)\|_\infty \lesssim \log^4(t)$$

The proof of this lemma involves a lengthy Grönwall argument.

4. Spatial Average

Lemma

Let $F(t,p) = \int f(t,x,p)\,\mathrm{d}x$. Then $F(t,p)$ converges uniformly as $t \to +\infty$ to some $F_\infty(p) \in C_0^2(\mathbb{R}^3)$. Moreover,

$$\|F(t) - F_\infty\|_\infty \lesssim t^{-1}\log^5(t)$$
$$\|\nabla_p F(t) - \nabla_p F_\infty\|_\infty \lesssim t^{-1}\log^7(t).$$

5. Convergence of Macroscopic Densities

Let $\mathcal{D}(p) = |\det(\nabla v(p))|^{-1} = (1 + |p|^2)^{5/2}$
and $\mathbb{B}(q) = \nabla v^{-1}(q)$, and define

$$\rho_\infty(p) = \mathcal{D}(p) F_\infty(p)$$
$$j_\infty(p) = \mathcal{D}(p) v(p) F_\infty(p)$$

Lemma

We have

$$\sup_p \left| t^3 \rho(t, x + v(p)t) - \rho_\infty(p) \right| \lesssim t^{-1} \log^6(t)$$

$$\sup_p \left| t^4 \partial_{x_i} \rho(t, x + v(p)t) - \mathbb{B}_{ik}(v(p)) \partial_{p_k} \rho_\infty(p) \right| \lesssim t^{-1} \log^7(t)$$

and similarly for j (there's also a result for $\partial_t j$).

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Lemma

Let $\Box\psi = \eta$. *Assume* $\exists \eta_\infty$ *s.t.* $|t^4\eta(t, x + v(p)t) - \eta_\infty(p)| \lesssim t^{-1}\log^7(t)$.
Then $\exists \psi_\infty$ *s.t.* $|t^2\psi(t, x + v(p)t) - \psi_\infty(p)| \lesssim t^{-1}\log^7(t)$.

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From this it then follows that $\exists E_\infty(p)$ s.t.

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From this it then follows that $\exists E_\infty(p)$ s.t.

$$\left|t^2 E(t, x + v(p)t) - E_\infty(p)\right| \lesssim t^{-1}\log^7(t)$$

and similarly for B. This was the crucial estimate we needed!

7. Modified Scattering

Let $\mathbb{A}(p) = \nabla v(p)$ and $K_\infty(p) = E_\infty(p) + v(p) \times B_\infty(p)$ and define

$$h(t,x,p) = g(t, x - \log(t)\mathbb{A}(p)K_\infty(p), p)$$
$$= f(t, x + v(p)t - \log(t)\mathbb{A}(p)K_\infty(p), p)$$

Lemma

Then $h(t,x,p)$ converges uniformly as $t \to +\infty$ to some $f_\infty(x,p) \in C(\mathbb{R}^6)$. Moreover,

$$\|h(t) - f_\infty\|_\infty \lesssim t^{-1}\log^7(t)$$

The proof uses the convergence of the fields to integrate $\partial_t h$ in time.

Thank you for your attention!