Oxford Mathematics Alphabet

University of Oxford
Mathematical Institute

L *is for* logic

Logic is a vast and ancient field, and its influence spans not only mathematics but philosophy, computer science, and linguistics. Even in mathematics one cannot pin down exactly where "logic" as a subject lies – it appears as set theory, computability theory, model theory, and proof theory. Since the work of Gottlob Frege and Bertrand Russell in the 19th century, however, mathematical logic has been used as a foundational language underpinning all mathematical discussion.

In the Oxford Mathematical Institute, logicians mainly research in *model theory*, a branch of logic which studies the connection between mathematical semantics (*truth*) and mathematical syntax (*proof*).

$$\models \varphi \quad \Longleftrightarrow \quad \vdash \varphi$$

Illustration of the Completeness and Soundness Theorems, which prove a correspondence between semantic truth of a first-order sentence φ (left) and the existence of a proof of φ within first-order logic (right).

Applications of model theory are rife in geometry, combinatorics, algebra, and number theory (among other fields). One contribution to number theory is work on the *André-Oort Conjecture* (a proof of which Oxford Mathematician Jonathan Pila and collaborators Ananth Shankar, Jacob Tsimmerman, Hélène Esnault, and Michael Groechenig announced in 2021). Pila attacked this problem using *o-minimality*, a subfield of model theory which arose from Alfred Tarski's work on understanding the semialgebraic subsets of real space. Many results in real algebraic geometry have parallels in general o-minimal structures, and can be used to tackle Diophantine problems.

Alfred Tarski in Berkeley

One result of Tarski's exploration was the conclusion that first-order questions about $\mathbb{R}$ as an ordered ring can be answered algorithmically – the structure is *decidable*. This contrasts with (for example) the *existential* questions about $\mathbb{Z}$ as a ring.

A result of Martin Davis, Hilary Putnam, Julia Robinson, and Yuri Matiyasevich shows that no computer program can ever exist that, upon the input of a polynomial $f(x_1, \dots, x_n) \in \mathbb{Z}[x_1, \dots, x_n]$, answers correctly when asked "do there exist $x_1, \dots, x_n \in \mathbb{Z}$ such that $f(x_1, \dots, x_n) = 0$?". This problem is mathematically *undecidable*. Oxford Mathematician Jochen Koenigsmann has contributed to the analogous problem of existential questions about $\mathbb{Q}$ by showing that the non-integer rational numbers exactly satisfy a Diophantine equation (see D is for Diophantine Equations) over $\mathbb{Q}$. This allows us to reduce some questions of the form "$\forall x_1, \dots, \forall x_n \, \exists y_1, \dots, \exists y_m \dots$" about $\mathbb{Q}$ to existential questions about $\mathbb{Z}$, which we know are undecidable. The decidability of existential questions about $\mathbb{Q}$ is one of the many big unsolved problems of logic today – and the more logicians there are, the closer we will get to an answer!

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Content sources: https://en.wikipedia.org/wiki/Mathematical_logic. The Development of Logic by William Kneale, Martha Kneale. Discussion with Jonathan Pila, Ehud Hrushovski, Soinbhe Nic Dhonncha, Arturo Rodriguez Fanlo.
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