

SymTFT Approach for Mixed States with Non-Invertible Symmetries

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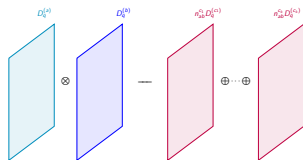
14th May 2026

[arXiv:2507.05350 – S. Schafer-Nameki, A. Tiwari, AW, C. Zhang]

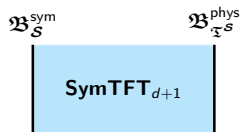
- Aim: classifying phases of matter for pure and mixed-state phases

$$\rho = \sum_n p_n |\psi_n\rangle \langle \psi_n|$$

- Symmetries are key: but they go far beyond groups and can be non-invertible



- Symmetry Topological Field Theory (SymTFT)



- Result: systematic classification and novel phases protected by strong and weak generalized symmetries, including strong-to-weak symmetry breaking and strong-weak SPT phases

Related works: [Luo, Wang, Bi, '25], [Qi, Sohal, Chen, Stephen, Prem, '25]

Introduction to Generalized Symmetries

Symmetries as topological operators

- Ordinary symmetry operator: $U(t)$ that commutes with the Hamiltonian

$$\Rightarrow U(t) = U(t'), \quad \forall t, t'$$

$U(t)$ acts on any local operator at any point $\Rightarrow$ acts on all of space

$$U(t) O_R(\vec{x}, t) U^{-1}(t) = R(g) \cdot O_R(\vec{x}, t)$$

Generalized global symmetries



Topological operators/defects in a QFT

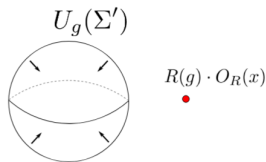
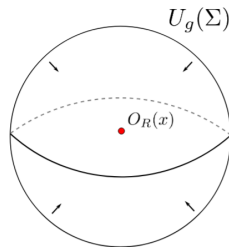
[Gaiotto, Kapustin,
Seiberg, Willett, '14]

- Reviews:

[Schäfer-Nameki '23], [Bhardwaj et al. '23]

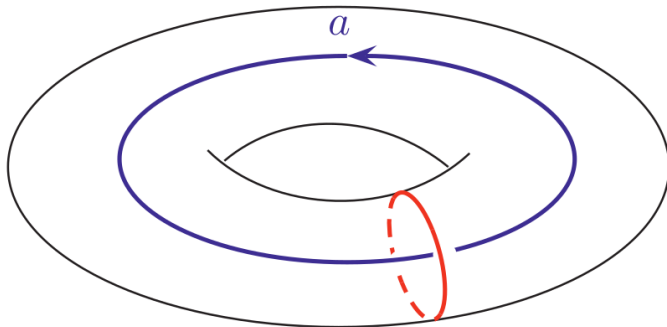
[Brennan, Hong '23], [Luo, Wang², '23]

[Shao '23], [McGreevy, '22]



Example: 1-form symmetries on a torus

- $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form symmetries on a torus with -1 phase from unlinking



- Toric code: topological quantum error correcting code [A. Kitaev, '97]
Topological lines (anyons) encode Pauli matrices $XZ = -ZX$

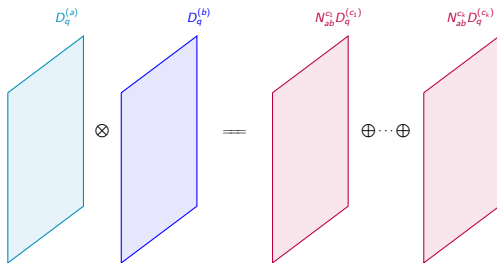
Non-Invertible Symmetries

- Generalized symmetries $\leftrightarrow$ topological defects
- Topological defects need not admit inverses

Non-invertible symmetries:

$$D_q^{(a)} \otimes D_q^{(b)} = \sum_c \mathcal{N}_{ab}^c D_q^{(c)}$$

- Long history in 2d CFTs
- Mathematically described by fusion categories



[Fröhlich, Fuchs, Runkel, Schweigert, '06], [Bhardwaj, Tachikawa, '17], [Thorngren, Wang, '19–'21], [Bhardwaj, Bottini, Pajer, Schafer-Nameki, '23], [Bhardwaj, Bottini, Pajer, Schafer-Nameki, **AW**, '24],...

Pure state gapped phases

What is a gapped (topological) phase of matter?

- Ground states of (symmetric) local Hamiltonians with energy gap $\Delta > 0$

- Equivalence relation $|\psi\rangle \sim |\psi'\rangle$

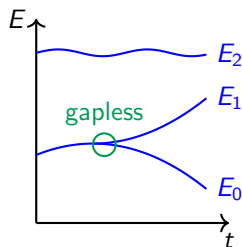
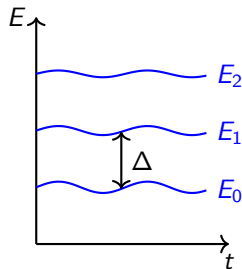
if there exists a path of (symmetric) gapped $H(t)$

between H and $H' \Rightarrow U = \mathcal{T}\left[e^{-i \int_0^1 dt H(t)}\right]$

such that $U|\psi\rangle = |\psi'\rangle$

[X. Chen, Z.-C. Gu, X.-G. Wen, '10]

- If $|\psi\rangle \approx |\psi'\rangle$ they are in different phases
 $\Rightarrow$ gapless critical point at the phase transition



Example: Ising symmetry in (1+1)d

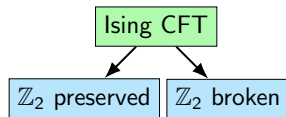
- Transverse field Ising Hamiltonian (with Pauli matrices X, Z)

$$H = - \sum_j Z_j Z_{j+1} - g \sum_j X_j$$

- $\mathbb{Z}_2$ spin flip symmetry $\eta = \prod_j X_j$

- Phases:

- $g \gg 1$: single ground state $\otimes_j \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle)$
- $g = 0$: two ground states $\otimes_j |\uparrow\rangle$ and $\otimes_j |\downarrow\rangle$
- $g = 1$: critical Ising CFT $c = 1/2$

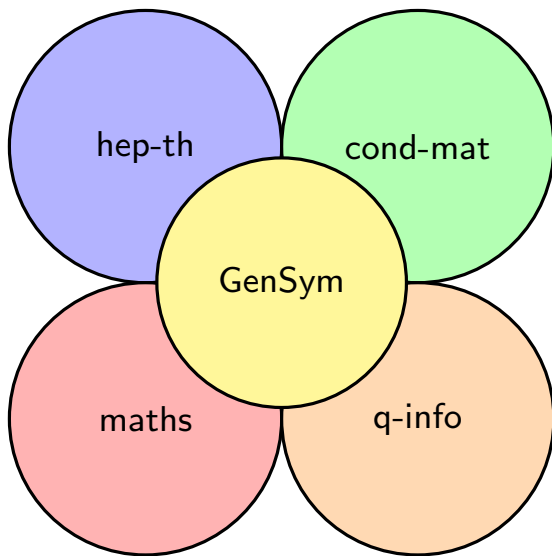


- Kramers-Wannier duality:

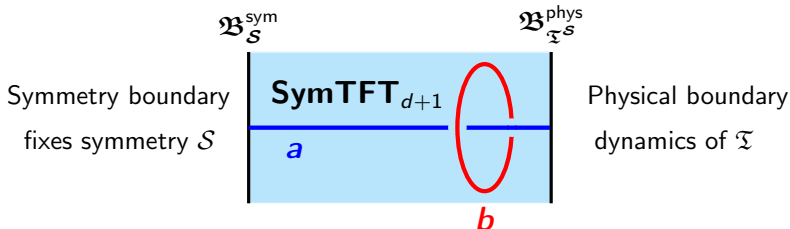
$X_j \rightarrow Z_j Z_{j+1}$ and $Z_j Z_{j+1} \rightarrow X_{j+1}$, corresponds to $g \rightarrow g^{-1}$

$g = 1$: non-invertible symmetry $\mathcal{N}$ of the critical Ising CFT

$$\mathcal{N} \otimes \mathcal{N} = 1 \oplus \eta$$



Symmetry Topological Field Theory



- Generalized charges of operators: from linking in the SymTFT bulk
- Example: $\mathbb{Z}_N$ SymTFT from $U(1)$ BF-theory:

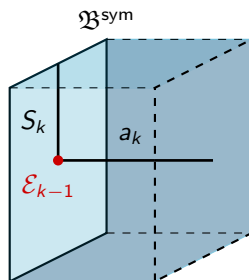
$$S_{BF} = \frac{N}{2\pi} \int_{d+1} b \wedge da$$

Bulk topological line operators with commutation relation:

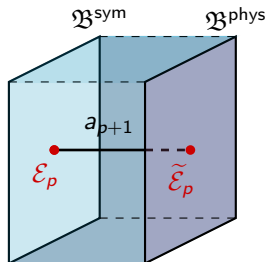
$$e^{i \int_{\Sigma} b} e^{i \int_M a} = e^{\frac{2\pi i \text{Link}(\Sigma, M)}{N}} e^{i \int_M a} e^{i \int_{\Sigma} b}$$

- [D. Freed, G. Moore, C. Teleman, '22], [J. Kaidi, K. Ohmori, Y. Zheng, '22]
[S. Schäfer-Nameki et al.], [X.G. Wen et al.]

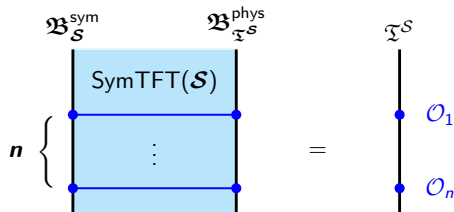
- Symmetry generators S_k : topological defects in the SymTFT that cannot end on $\mathfrak{B}^{\text{sym}}$



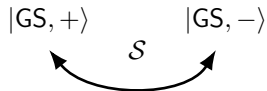
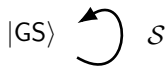
- Charged order parameters come from defects ending on $\mathfrak{B}^{\text{phys}}$

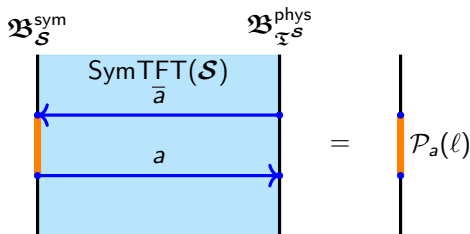


Pure-state symmetric gapped phases



- **SPT**: symmetry protected topological phase
 $n = 1$ symmetric ground state
- **SSB**: spontaneous symmetry
 breaking phase $n > 1$ ground states related by $\mathcal{S}$
- **Topological order**: SSB for higher-form symmetry





- Order parameters: anyons a anyons that end on $\mathfrak{B}^{\text{phys}}$ and $\mathfrak{B}^{\text{sym}}$
 $\Rightarrow$ local operators which detect (partial) SSB

$$\langle \psi | \mathcal{O}_a(x) \mathcal{O}_a^\dagger(y) | \psi \rangle \neq 0$$

- Disorder parameters: anyons that end on $\mathfrak{B}^{\text{phys}}$ but not $\mathfrak{B}^{\text{sym}}$

$$\langle \psi | \mathcal{P}_a(x, y) | \psi \rangle \neq 0$$

twisted sector operators that detect different SPTs

Mixed-state phases

Mixed Quantum States

- Mixed states arise naturally in open quantum systems, noisy quantum devices, and quantum computation setups, where the system is not described by an isolated pure state due to noise or decoherence
[Ma, Turzillo, '24], [Ellison, Cheng, '24] etc.

- They are statistical ensembles of pure states, described by a density matrix ρ on a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$:

$$\rho = \sum_n p_n |\psi_n\rangle \langle \psi_n|, \quad 0 \leq p_n \leq 1, \quad \sum_n p_n = 1$$

- The density matrix is Hermitian, $\rho = \rho^\dagger$, and encodes expectation values by

$$\langle \mathcal{O} \rangle_\rho = \text{Tr}(\rho \mathcal{O})$$

- For fixed-point (gapped) mixed-states, one has

$$\rho^2 \propto \rho$$

with equality holding iff ρ is a pure state i.e. $\rho = |\psi\rangle \langle \psi|$

- Symmetries of mixed states can be **strong** or **weak**, and their interplay can lead to new types of symmetric phases
- Goal: classify the possible symmetric phases of mixed quantum states

Let D_s be a representation of $\mathcal{S}$ on $\mathcal{H}$, with $D_s^\dagger = D_{s^\dagger}$:

- Strong Symmetry:

$$D_s \rho = c_s \rho$$

all states in the average have same symmetry charge c_s

- Weak Symmetry:

$$D_s \rho D_s^\dagger = |c_s|^2 \rho$$

states in the average can have different complex phase in charge c_s

- Consider finite group symmetries G at first:

$$D_s = U_g \quad \text{and} \quad c_s = e^{i\theta_s}$$

- For non-invertible symmetries we require that for all $s \in \mathcal{S}$ there is $s^\dagger \in \mathcal{S}$

$$s \otimes s^\dagger = 1 \oplus \bigoplus_i m_i a_i \equiv 1 \oplus a$$

D_s is generally non-unitary and $|c_s|$ can be different to 1 (also zero)

- From $\rho = \sum_n p_n |\psi_n\rangle \langle \psi_n|$ define a pure state on $\mathcal{H}_L \otimes \mathcal{H}_R$ (flip bra to ket)

$$|\rho\rangle\rangle \equiv |\rho^{\text{Choi}}\rangle\rangle = \frac{1}{\text{Tr}(\rho^2)} \sum_n p_n |\psi_n\rangle |\overline{\psi_n}\rangle \in \mathcal{H}_L \otimes \mathcal{H}_R$$

- $\rho = \rho^\dagger \Rightarrow |\rho\rangle\rangle$ has an anti-unitary $T = \text{SWAP}^*$ symmetry acting as

$$T |\psi_L\rangle |\overline{\psi_R}\rangle = |\overline{\psi_R}\rangle |\psi_L\rangle$$

$$T |\rho\rangle\rangle = |\rho\rangle\rangle$$

- The strong symmetry action becomes

$$D_s \rho = c_s \rho \quad \rightarrow \quad (D_{s,L} \otimes \mathbf{1}_R) |\rho\rangle\rangle = c_s |\rho\rangle\rangle$$

$$\rho D_s^\dagger = \overline{c}_s \rho \quad \rightarrow \quad (\mathbf{1}_L \otimes \overline{D}_{s,R}) |\rho\rangle\rangle = \overline{c}_s |\rho\rangle\rangle$$

- The weak symmetry action becomes

$$D_s \rho D_s^\dagger = |c_s|^2 \rho \quad \rightarrow \quad D_{s,L} \otimes \overline{D}_{s,R} |\rho\rangle\rangle = |c_s|^2 |\rho\rangle\rangle$$

- Explicit breaking of weak (and also strong) symmetry:
charged operator $\mathcal{O}(x)$ satisfying

$$\mathrm{Tr}(\mathcal{O}(x)\rho) \propto \mathrm{Tr}(\mathcal{O}(x)\rho^2) = \langle\langle \rho | \mathcal{O}_L(x) | \rho \rangle\rangle \neq 0$$

- Explicit breaking of strong symmetry:

$$\mathrm{Tr}(\rho \mathcal{O}(x) \rho \mathcal{O}^\dagger(x)) = \langle\langle \rho | \mathcal{O}_L(x) \overline{\mathcal{O}}_R(x) | \rho \rangle\rangle \neq 0$$

- Weak (and also strong) spontaneous symmetry breaking:

$$\mathrm{Tr}(\rho \mathcal{O}(x) \mathcal{O}^\dagger(y)) \propto \mathrm{Tr}(\rho^2 \mathcal{O}(x) \mathcal{O}^\dagger(y)) \neq 0$$

- Strong-to-weak spontaneous symmetry breaking (SWSSB):

$$\mathrm{Tr}(\rho \mathcal{O}(x) \mathcal{O}(y) \rho \mathcal{O}(y)^\dagger \mathcal{O}(x)^\dagger) = \langle\langle \rho | \mathcal{O}_L(x) \mathcal{O}_L(y) \overline{\mathcal{O}}_R(x) \overline{\mathcal{O}}_R(y) | \rho \rangle\rangle \neq 0$$

- Consider the SymTFT for the Choi state. The symmetry is $\mathcal{S}_L \boxtimes \mathcal{S}_R$, satisfying the SWAP* invariance

$$\mathcal{S}_R = \overline{\mathcal{S}_L}$$

- The SymTFT we consider is

$$\mathcal{Z}(\mathcal{S} \boxtimes \overline{\mathcal{S}})$$

- Which boundary conditions give rise to strong symmetries?
- Which boundary conditions correspond to gapped mixed phases, i.e. after tracing out one Hilbert space, give rise to a consistent density matrix?
- Similar questions for weak symmetries
- First: start with strong symmetries.

- Not all Lagrangians in $\mathcal{Z}(\mathcal{S} \boxtimes \overline{\mathcal{S}})$ are admissible. Let

$$\mathcal{L} = \bigoplus_{a_L, b_R} n_{a_L, b_R} a_L b_R$$

- What are the strong symmetry boundary conditions?

$$\mathcal{L}_S^{\text{strong}} = (\mathcal{L}^{\mathcal{S}})_L \otimes (\mathcal{L}^{\overline{\mathcal{S}}})_R$$

- Conditions on gapped boundary conditions $\mathcal{L}$:

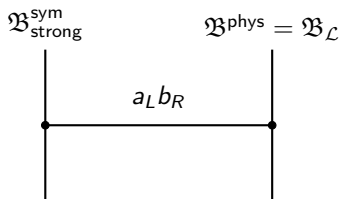
1. $T = \text{SWAP}^*$ invariance: This condition follows from the Hermiticity of ρ and implies

$$T(\mathcal{L}) = \mathcal{L}, \quad T(a_L b_R) = \overline{b_L} \overline{a_R},$$

which for any anyons a_L, b_R swaps the two copies and performs complex conjugation (i.e. $\theta_{a_L} = \overline{\theta_{\overline{a_R}}}$). This implies

$$\boxed{n_{a_L, b_R} = n_{\overline{b_L}, \overline{a_R}}} \tag{1}$$

- **Positivity:** The expectation value of patch operators needs to be positive definite in the singlet Choi state. From the SymTFT



- Each anyon of this type corresponds to

$$a_L b_R \longleftrightarrow n_{a_L, b_R} \mathcal{P}_{a_L}^i(x, y) \otimes \mathcal{P}_{b_R}^i(x, y)$$

with the patch operators having non-zero expectation value in $|\rho_{\mathcal{L}}\rangle\rangle$ (the density matrix associated to this physical boundary)

- Positivity implies

$$\boxed{n_{a_L, b_R}^2 \leq n_{a_L, \bar{a}_R} n_{\bar{b}_L, b_R}} \quad (2)$$

- We call such algebras mixed state (Lagrangian) algebras

- Depending on the properties of $\mathcal{L}$ we get different types of gapped phases.
- **Pure state phases:** These correspond to particular factorized Lagrangians

$$\mathcal{L}_{\text{phys}}^{\text{strong}} = \mathcal{L}_L \otimes \mathcal{L}_R$$

Then the Choi state is

$$|\rho_{\mathcal{L}}\rangle\rangle = |\psi_{\mathcal{L}_L}\rangle \otimes |\psi_{\mathcal{L}_R}\rangle$$

which gives a pure state density matrix.

- **Non-factorized $\mathcal{L}$:** algebra with condensed diagonal charges:

$$n_{a_L, b_R} \neq 0, \quad n_{a_L, 1} = n_{1, b_R} = 0,$$

i.e. $\mathcal{L}$ contains only the diagonal gauge charge, without the off-diagonal terms. Then the resulting gapped phase will have a weak symmetry.

- $\Rightarrow$ **Strong-Weak SSB (SWSSB) with possible strong-weak SPT**

Examples: Phases with Strong $\mathbb{Z}_2$ Symmetry

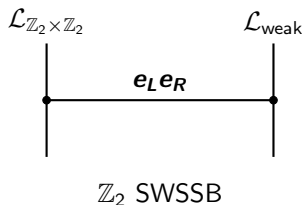
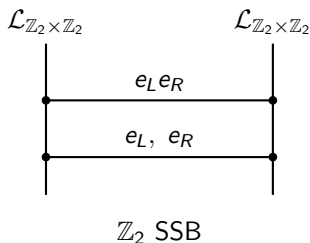
- The mixed state Lagrangians for strong $\mathbb{Z}_2$ symmetry are

$$\mathcal{L}_{\text{sym}} = \mathcal{L}_{\mathbb{Z}_2 \times \mathbb{Z}_2} = (1 \oplus e_L) \otimes (1 \oplus e_R)$$

$$\mathcal{L}_{\text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2)} = (1 \oplus m_L) \otimes (1 \oplus m_R),$$

$$\mathcal{L}_{\text{weak}} = 1 \oplus e_L e_R \oplus m_L m_R \oplus e_L m_L e_R m_R$$

- The first two factorize, while the last is a proper mixed phase:



- In the SWSSB phase the combination $m_L m_R$ (weak symmetry) braids trivially with the order parameter while the strong symmetries m_L, m_R are SSB

SWSSB with strong-weak SPTs

- For $\mathcal{S} = \mathbb{Z}_2^A \times \mathbb{Z}_2^B$ there are both trivial and SPT phases for pure states. Likewise, we find mixed state SPTs from this symmetry.
- The SymTFT is

$$\mathcal{Z}(\mathbb{Z}_2^A \times \mathbb{Z}_2^B) \boxtimes \mathcal{Z}(\mathbb{Z}_2^A \times \mathbb{Z}_2^B)$$

- We take the $\text{Vec}_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ canonical boundary as the symmetry:

$$\mathcal{L}_{\mathbb{Z}_2 \times \mathbb{Z}_2}^{\text{strong}} = (1 \oplus e_L^A \oplus e_L^B \oplus e_L^A e_L^B)(1 \oplus e_R^A \oplus e_R^B \oplus e_R^A e_R^B)$$

- Then there is a particular mixed state Lagrangian algebra

$$\mathcal{L}_{\text{phys}} = 1 \oplus \underline{e_L^A e_R^A} \oplus e_L^A m_L^B \oplus e_L^A m_R^B \oplus e_R^A m_L^B \oplus e_R^A m_R^B \oplus \dots$$

- The charged local operator detects $\mathbb{Z}_2^A$ SWSSB

$$\mathcal{O}_{e_L^A e_R^A} \Rightarrow \mathbb{Z}_2^A \text{ SWSSB}$$

- The **operators** give rise to a string-like order parameter for the weak $\mathbb{Z}_2^A$ together with a string attached from the strong $\mathbb{Z}_2^B$ symmetry.
- Therefore this is a $\mathbb{Z}_2^A$ SWSSB with a **mixed strong-weak SPT**

- The main advantage of the SymTFT approach is that it can tackle non-invertible symmetries and higher dimensions. We consider the former here.
- **Example: Strong Rep(S_3) Symmetry.** Consider the irreducible representations of the permutation group

$$S_3 = \langle a, b \mid a^3 = 1, \quad b^2 = 1, \quad bab = a^2 \rangle$$

$$\text{Rep}(S_3) = \{1, P, E\} : \quad P \otimes P = 1, \quad E \otimes E = 1 \oplus P \oplus E$$

- The mixed state SymTFT is $\mathcal{Z}(S_3) \boxtimes \overline{\mathcal{Z}(S_3)}$, whose anyons are labeled by ℓ_k for $k \in \{L, R\}$:

$$\ell \in \{1, P, E, a, a^\omega, a^{\bar{\omega}}, b, b^-\}, \quad \omega = e^{2\pi i/3}$$

Here $a \equiv [a]$ and $b \equiv [b]$. The symmetry Lagrangian is

$$\mathcal{L}_{\text{Rep}(S_3)}^{\text{strong}} = (1 \oplus a \oplus b)_L \otimes (1 \oplus a \oplus b)_R$$

Non-Invertible Symmetric Mixed Phases

- The non-factorized mixed state Lagrangians lead to different patterns of symmetry breaking:

$$\begin{array}{c}
 \mathcal{L}_{\text{Rep}(S_3)}^{\text{strong}} \quad \mathcal{L}_1 \\
 \begin{array}{|c|} \hline a_L a_R \\ \hline b_L b_R \\ \hline \end{array} \\
 \text{Rep}(S_3) \text{ SWSSB}
 \end{array}$$

$$\begin{array}{c}
 \mathcal{L}_{\text{Rep}(S_3)}^{\text{strong}} \quad \mathcal{L}_2 \\
 \begin{array}{|c|} \hline 2a_L a_R \\ \hline \end{array} \\
 (\text{Rep}(S_3)/\mathbb{Z}_2) \text{ SWSSB}
 \end{array}$$

$$\begin{array}{c}
 \mathcal{L}_{\text{Rep}(S_3)}^{\text{strong}} \quad \mathcal{L}_3 \\
 \begin{array}{|c|} \hline b_L b_R \\ \hline \end{array} \\
 \mathbb{Z}_2 \text{ SWSSB}
 \end{array}$$

$$\begin{array}{c}
 \mathcal{L}_{\text{Rep}(S_3)}^{\text{strong}} \quad \mathcal{L}_4 \\
 \begin{array}{|c|} \hline 2a_L a_R, a_L, a_R \\ \hline b_L b_R \\ \hline \end{array} \\
 (\text{Rep}(S_3)/\mathbb{Z}_2) \text{ SSB}, \mathbb{Z}_2 \text{ SWSSB}
 \end{array}$$

- These results are corroborated with $\text{Rep}(S_3)$ lattice models and by explicit constructions of the density matrices from doubled pure states

- Hamiltonian on doubled Hilbert space $\mathcal{H}_L \otimes \mathcal{H}_R$

$$\tilde{H}_{(\tilde{F}, \beta)}^{\text{Rep}(G)} = -\frac{1}{|\tilde{F}|} \sum_{j, f} (R_{\beta}^{\bar{f}})_j (L_{\beta}^f)_{j+1} - \sum_j P_j^{(\tilde{F})}$$

- Density matrix for its (pure-state) ground state on

$$\tilde{\rho}_{(\tilde{F}, \beta)} \propto \prod_j \left(\sum_{f \in \tilde{F}} (R_{\beta}^{\bar{f}})_j (L_{\beta}^f)_{j+1} \right) P_j^{(\tilde{F})}$$

- The density matrix $\rho_{(\tilde{F}, \beta)}^{\text{Rep}(G)}$ for the state invariant under $\text{Rep}(G)$ symmetry is obtained with the projector

$$P^{\text{Rep}(G)} = \sum_{\Gamma \in \text{Rep}(\tilde{G})} \frac{\dim(R)}{|\tilde{G}|} S_{\Gamma}$$

- $\rho_{(\tilde{F}, \beta)}$ on $\mathcal{H}$ is obtained by tracing out $\mathcal{H}_R$

$$\rho_{(\tilde{F}, \beta)} = \text{Tr}_{\mathcal{H}_R} \left[\rho_{(\tilde{F}, \beta)}^{\text{Rep}(G)} \right]$$

- Rep(S_3) SWSSB:

$$\rho_1 \propto P_{\text{sym}}^{(1)} \propto \sum_{\{g_n: \prod_n g_n = 1\}} |g_1 \cdots g_N\rangle \langle g_1 \cdots g_N|$$

- Rep(S_3)/ $\mathbb{Z}_2$ SWSSB:

$$\rho_2 \propto P_{\text{sym}}^{(1)} \prod_j P_j^{(\mathbb{Z}_3)} \propto \sum_{\{l_n: \sum_n l_n \equiv 0 \pmod{3}\}} |a^{l_1} \cdots a^{l_N}\rangle \langle a^{l_1} \cdots a^{l_N}|$$

- $\mathbb{Z}_2$ SWSSB:

$$\rho_3 \propto P_{\text{sym}}^{(1)} \prod_j P_j^{(\mathbb{Z}_2)} \propto \sum_{\{l_n: \sum_n l_n \equiv 0 \pmod{2}\}} |b^{l_1} \cdots b^{l_N}\rangle \langle b^{l_1} \cdots b^{l_N}|$$

- Rep(S_3)/ $\mathbb{Z}_2$ SSB, $\mathbb{Z}_2$ SWSSB:

$$\rho_4 \propto P_{\text{sym}}^{(1)} \prod_j \left(\mathbb{I}_j \mathbb{I}_{j+1} + R_j^a L_{j+1}^{a^2} + R_j^{a^2} L_{j+1}^a \right)$$

Weak and Strong Symmetric Phases

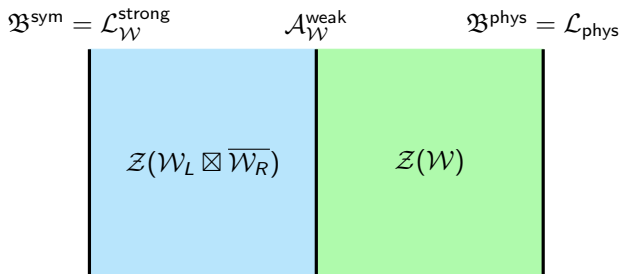
- So far we started with a strong symmetry. How do we get a weak symmetry $\mathcal{W}$ as a starting point?
- Start with $\mathcal{W}_L \boxtimes \mathcal{W}_R$ symmetry, and condense to a diagonal $\mathcal{W}$ symmetry:

$$\mathcal{Z}(\mathcal{W}_L \boxtimes \mathcal{W}_R) \rightarrow \mathcal{Z}(\mathcal{W})$$

i.e. input the strong symmetry $\mathcal{L}_{\mathcal{W}}^{\text{strong}} = \mathcal{L}_{\mathcal{W},L} \otimes \mathcal{L}_{\mathcal{W},R}$, then condense

$$\mathcal{A}_{\mathcal{W}}^{\text{weak}} = \bigoplus_w n_w w_L \overline{w_R}$$

- Club-sandwich setup:



Example: Invertible Weak Symmetries

- For weak $\mathcal{W} = \mathbb{Z}_2$ symmetry, start with $\mathcal{Z}(\mathbb{Z}_2^L \boxtimes \mathbb{Z}_2^R)$ and condense

$$\mathcal{A}_{\mathbb{Z}_2}^{\text{weak}} = 1 \oplus e_L e_R$$

$\mathcal{A}_{\mathcal{W}}^{\text{weak}}$	
$1 \oplus e_L e_R$	1
e_L, e_R	e
$m_L m_R$	m
m_L, m_R	

The reduced topological order is $\mathcal{Z}(\mathcal{W}) = \mathcal{Z}(\mathbb{Z}_2)$ with generators $m_L m_R \sim m$ acting both from the left and the right

$$\mathcal{L}' = 1 \oplus e$$



Inserting $\mathfrak{B}^{\text{sym}} = (1 \oplus e_L)(1 \oplus e_R)$ on the left-hand side and performing the left sandwich collapse:

$$\mathcal{L}' = 1 \oplus e$$

This is precisely a weak $\mathbb{Z}_2$ symmetry

Not all Symmetries can be Weak

- Consider the general fusion

$$D_a \otimes D_b = \bigoplus_c N_{ab}^c D_c$$

- Let them satisfy the weak symmetry condition

$$D_w \rho D_w^\dagger = |c_w|^2 \rho$$

- Consistency with the fusion rules requires

$$|c_a|^2 |c_b|^2 \rho = D_a (D_b \rho D_b^\dagger) D_a^\dagger = \left(\bigoplus_c N_{ab}^c D_c \right) \rho \left(\bigoplus_d N_{ab}^d D_d^\dagger \right)$$

- If all D_a are weak, then

$$\text{RHS} = \bigoplus_f m_f D_f \rho D_f^\dagger = \sum_f |c_f|^2 \rho$$

- Therefore the fusion must be invertible, i.e. there can be no non-trivial sum over f .

Example: Weak non-invertible and strong $\mathbb{Z}_2 \times \mathbb{Z}_2$

- Start with $\text{Rep}(D_8) = \{1, 1_a, 1_c, 1_{ca}, E\}$ strong symmetry where $D_8 = (\mathbb{Z}_2^a \times \mathbb{Z}_2^b) \rtimes \mathbb{Z}_2^c$ is mapped to 3-qubit Hilbert space as

$$|a\rangle \mapsto |010\rangle, \quad |b\rangle \mapsto |001\rangle, \quad |c\rangle \mapsto |100\rangle$$

- Condense $\mathcal{A} = 1 \oplus [ab]_L [ab]_R$ to make the non-invertible symmetry weak
- Consider

$$\begin{aligned} \mathcal{L}_{\text{phys}} = & (1 \oplus 1_{a_L} 1_{a_R})(1 \oplus 1_{c_L} \oplus 1_{c_R} \oplus 1_{c_L} 1_{c_R}) \\ & \oplus [ab](1_{a_L} \oplus 1_{c_L a_L} \oplus 1_{c_L a_R} \oplus 1_{c_L a_L c_R}) \\ & \oplus [ab](1_{a_R} \oplus 1_{c_R a_R} \oplus 1_{a_L c_R} \oplus 1_{c_L c_R a_R}) \\ & \oplus 4[c_L c_R]_2, \end{aligned}$$

- Lattice realization

$$[ab] \mapsto X^{\text{II}} X^{\text{III}}, \quad 1_a \mapsto Z^{\text{I}}, \quad 1_c \mapsto Z^{\text{II}} Z^{\text{III}}$$

Based on $\text{Rep}(D_8)$ lattice model for pure-state phases

[AW, Yang, Tiwari, Pichler, Schäfer-Nameki, *PRL* '25]

- The corresponding mixed-state phase is SWSSB with strong-weak SPT

$$\rho \propto \prod_i [\mathbb{I} + \prod_i (Z_i^{\text{I}} Z_i^{\text{III}})] [\mathbb{I} + (Z^{\text{I}} X^{\text{II}} X^{\text{III}})_i (X^{\text{II}} X^{\text{III}})_{i+1}]$$

- ρ is strongly symmetric under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ generators

$$\mathcal{S}_{1_a} = \prod_i Z_i^{\text{I}}, \quad \mathcal{S}_{1_c} = \prod_i Z_i^{\text{II}} Z_i^{\text{III}}$$

- ρ is only weakly symmetric under the non-invertible duality symmetry $\mathcal{S}_E$

$$\mathcal{S}_E \rho \not\propto \rho, \quad \mathcal{S}_E \rho \mathcal{S}_E^\dagger = 4 \rho$$

- ρ has a non-trivial strong-weak SPT between the strong $\mathcal{S}_{1_a}$ and weak $\mathcal{S}_E$

- The SymTFT provides a systematic framework for classifying mixed-state phases and symmetry breaking patterns.
- The key ingredient is the doubled-Hilbert-space formulation, obtained by mapping the density matrix into a Choi state.
- Imposing SWAP* invariance and positivity makes the SymTFT analysis closely parallel to the pure-state case.
- Physical density matrices can be constructed from the doubled picture, e.g. using lattice realizations inspired by the SymTFT construction.
- Non-invertible symmetries exhibit important constraints: not all symmetries can be consistently realized as weak symmetries.
- Interesting future directions include higher-dimensional generalizations and applications to mixed states in holography and high-energy theory.

Thank you!