Scaling Limits for Stochastic PDEs

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with P. Cardaliaguet and P.E. Souganidis.

Stability Analysis for Nonlinear PDEs
Oxford
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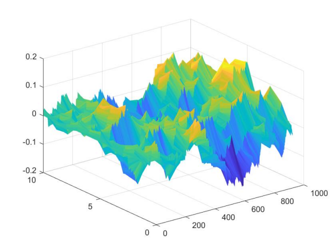
Introduction

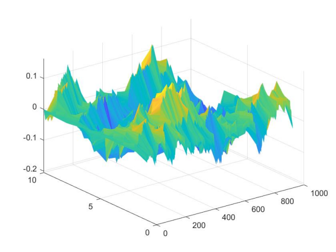
Hydrodynamic Limit for Interacting Diffusions

T. Funaki, H. Spohn *Motion by mean curvature from the Ginzburg-Landau interface model.* CMP, 185(1) (1997), pp. 1-36.

$$d\Phi_t(x) = -\sum_{|x-y|_1=1} V'(\Phi_t(x) - \Phi_t(y))dt + \sqrt{2}dB_t(x) \text{ for } x \in \Lambda \subset \mathbb{Z}^d.$$

The fields Φ live on a discrete lattice and take values in $\mathbb{R}^d$, $B_t(x)$ are i.i.d. Brownian motions, V' is the derivative of a strictly convex symmetric function, and $|\cdot|_1$ is the L^1-norm. ($V(x) = \frac{1}{2}|x|^2$: Laplace)





$$d\Phi_t(x) = -\sum_{|x-y|_1=1} V'(\Phi_t(x) - \Phi_t(y))dt + \sqrt{2}dB_t(x) \text{ for } x \in \Lambda \subset \mathbb{Z}^d.$$

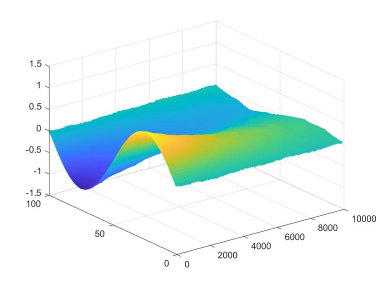
Funaki-Spohn 1999 NONLINEAR

$$\Phi^\varepsilon(r,t) = \varepsilon\Phi_{\varepsilon^{-2}t}(x) \text{ for } r \in [x-\varepsilon/2, x+\varepsilon/2)^d \text{ with } N = [\varepsilon^{-1}]$$

under diffusive rescaling onvergence to the solution h of

$$\partial_t h(r,t) = \mathrm{div}((D\sigma)(\nabla h))$$

(Hydrodynamic limit)



Invariant Measure and limit

Invariant measure on finite lattice: ($\nabla_i \Phi$: Discrete gradients.)

$$\frac{1}{Z} e^{\beta \sum_{x\in\Lambda} V(\nabla_i \Phi(x))} \prod_{x\in\Lambda} d\Phi(x).$$

Limit points as $\Lambda \to \mathbb{Z}^d$: DLR states (Dobrushin-Lanford-Ruelle)
Funaki-Spohn: Unique DLR state μ_P for each "tilt" P : $V(y+P))$, periodic b.c. by coupling and "PDE methods"

Technique for limit: Local equilibrium states (Guo, Papanicolaou Varadhan):
- macroscopic quantity $P \sim \nabla h$ varies slowly on microscale.
- near $\epsilon^{-1}x$, process in law close to invariant measure $\mu_{P(x)}$.
- replace nonlinearities by expectation under $\mu_{P(x)}$
-μ_p stationary (up to tilt) and ergodic
See book by Kipnis and Landim

Important fact: For $d = 1, 2$ no invariant measure for fields Φ exists, only for discrete gradients $\nabla_i \Phi$.

What about scaling SPDEs?

$$d_t U_t^\epsilon = \mathrm{div}\mathcal{A}(DU_t^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega_1)dt + \frac{1}{\epsilon}\sum_{k\in\mathbb{Z}^d} A(\frac{x-k}{\epsilon})dB_t^k \ \text{in} \ \mathbb{R}^d \times (0,+\infty),$$
$$U_0^\epsilon = u_0. \ \text{in} \ \mathbb{R}^d$$

$(B^k)_{k\in\mathbb{Z}^d}$ i.i.d. BM, $\mathcal{A}$ stat. ergodic in space-time (Z^d), ind. of BM's, $A(x)$ smooth, compact support.

Note: Similar to Funaki-Spohn, but different model. (Scale of correlation vs. scale of discretization)

AIM: Use homogenization techniques

Scaled (linear) SPDEs:
A. Dunlap, Y. Gu, L. Rhyzik and O. Zeitouni
M. Hairer, E. Pardoux, A. Piatnitski

Here: **Nonlinear** (divergence form)

Scaling limit for SPDE by homogenization techniques

Homogenization:
Law of microstructure fixed, stationary ergodic.
Scaling limits for interacting particle systems:
Self-organised ergodic structure on microscale,
changing on macroscale (local equilibrium)
invariant measure μ_P corresponds to law of gradient corrector
Theorem (Cardaliaguet-D.-Souganidis 2020):
Solutions of

$$\begin{aligned} d_t U_t^\epsilon &= \mathrm{div}\mathcal{A}(DU_t^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega_1)dt + \frac{1}{\epsilon}\sum_{k\in\mathbb{Z}^d} A(\frac{x-k}{\epsilon})dB_t^k = f(x,t), \\ U_0^\epsilon &= u_0. \end{aligned}$$

converge a.s. and in expectation in suitably weighted L^2 spaces to unique solutions of

$$\overline{u}_t - \mathrm{div}\ \overline{a}(D\overline{u}) = f$$

$\mathcal{A}$ strictly monotone

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Strategy of Proof

$$U_t^\epsilon(x) = \epsilon \tilde{V}_{\frac{t}{\epsilon^2}}(\frac{x}{\epsilon}) + \tilde{W}_t^\epsilon(x),$$

$$d\tilde{V}_t = \Delta \tilde{V}_t dt + \sum_{k\in\mathbb{Z}^d} A(x-k)dB_t^k \text{ in } \mathbb{R}^d \times \mathbb{R}, \quad (1)$$

$$\partial_t \tilde{W}_t^\epsilon = \operatorname{div}\Big(\tilde{a}(D\tilde{W}^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega)\Big) \text{ in } \mathbb{R}^d \times (0,+\infty) \quad (2)$$

$$\tilde{W}_0^\epsilon = u_0 \text{ in } \mathbb{R}^d,$$

with the random nonlinearity

$$a(p,x,t,\omega) = \mathcal{A}(p + D\tilde{V}_t(x,\omega_0), t, x, \omega_1) - D\tilde{V}_t(x,\omega_0)$$

(Simplification under structural assumptions as in F-S)

Tasks:

1. Existence of family of eternal, stationary, attracting solns. for gradient of (1)
$\Rightarrow D\tilde{V}$ space-time stationary ergodic
2. Homogenize nonlinear divergence form PDEs as (2) with space-time stat. erg. coefficients with low time regularity.

Stability considerations

1. Existence of family of eternal, stationary, attracting solns. for gradient of (1)
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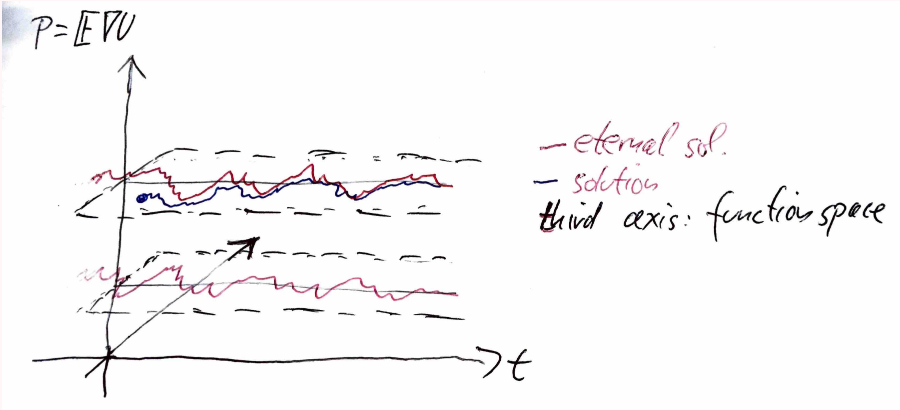
2. Homogenize nonlinear divergence form PDEs as (2) with space-time stat. erg. coefficients with low time regularity



Heat Equation

Eternal solution for heat equation with additive noise

Space time stationary solutions for

$$dV_t = \Delta V_t dt + \sum_{k \in \mathbb{Z}^d} A(x-k) dB_t^k.$$

via

$$\begin{aligned} dV_t &= \Delta V_t dt + \sum_{k \in \mathbb{Z}^d} A(x-k) dB_t^k \quad \text{on } [-n^2, \infty) \times \mathbb{R}^n \\ V_{-n^2}(x) &= 0 \end{aligned}$$

and $n \to \infty$.
$A(x)$ smooth and compact support

- moment bounds

- correlation decay

- convergence as $n \to \infty$

Moment Bounds

$$\mathbb{E}[|V_t(x)|^2] \leq Ct \wedge \begin{cases} 1 & \text{if } d \geq 3, \\ \ln(t+1) & \text{if } d = 2, \\ t^{1/2} & \text{if } d = 1. \end{cases}$$

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}[|DV_t(x)|^2] + \mathbb{E}[|D^2 V_t(x)|^2] \leq C(t \wedge 1),$$

Method: Heat kernel and Ito formula

Space-time stationary limit expected only for gradients!

Decorrelation

For $l \in \mathbb{Z}$ and $R \geq 1$, let $V^{l,R}$ be the solution to

$$\begin{cases} dV_t^{l,R} = \Delta V_t^{l,R} dt + \sum_{k \in \mathbb{Z}^d,\ |k-l| \leq R} A(x-k) dB_t^k \ \text{ in } \ \mathbb{R}^d \times (0, +\infty), \\ V_0^{l,R} = 0 \ \text{ in } \ \mathbb{R}^d. \end{cases}$$

(Switch off noise outside $B_R(l)$)

$$\mathbb{E}\left[\left| DV_t(0) - DV_t^{0,R}(0) \right|^2 \right] \leq C \begin{cases} R^{-d} & \text{if } R^2/t \leq 1, \\ \exp\{-R^2/(5t)\} & \text{otherwise}, \end{cases},$$

Note $\mathbb{Z}^d$ invariance in space.

Note: $V^{l_1,R}$ and $V^{l_2,R}$ are independent if $|l_1 - l_2| \gg R$.

Convergence

$$\begin{aligned} dV_t^n &= \Delta V_t^n + \sum_{k\in\mathbb{Z}^d} A(x-k)dB_t^k \quad \text{on } [-n^2,\infty)\times\mathbb{R}^n \\ V_{-n^2}^n(x) &= 0 \end{aligned}$$

$(DV^n)_{n\in\mathbb{N}}$ is Cauchy sequence in $L^2(B_r \times [-T,T]\times\Omega)$ for any fixed r and T.

Method: Suppose $m \ll n$.
$V^n - V^m$ solves deterministic heat equation.
Solution is heat kernel applied to difference in initial values at time $-m^2$, i.e. to $V^n_{-m^2}$.
Heat kernel: "Averaging"
Replace V by $V^{l,R}$, use $m,n \gg 1$. (Recall $\mathbb{E}\left[|DV_t(0) - DV_t^{0,R}(0)|^2\right] \leq CR^{-d}$)
Attractor Similar principle: Difference of two solutions satisfies deterministic heat equation. Use gradient estimate by averaging for heat equation

Self-organized ergodicity and stability

Lemma

1. There exists a unique process $Z : \Omega_0 \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ (eternal solution) with

$$\mathbb{E}\left[\int_{\tilde{Q}_1} |Z_t(x)|^2 dxdt\right] < \infty, \quad dZ_{i,t}(x) = \Delta Z_{i,t}(x)dt + \sum_{k\in\mathbb{Z}^d} D_{x_i} A(x-k) dB_t^k.$$

2. Z is an attractor in the sense that, if V is a solution of stoch. heat eq. in $\mathbb{R}^d \times (0,\infty)$ such that $V(\cdot, 0) = 0$, then

$$\lim_{t\to+\infty} \mathbb{E}\left[\int_{Q_1} |DV_t(x) - Z_t(x)|^2 dx\right] = 0.$$

3. Higher dimensions: For $d \geq 3$, there exists a unique up to constants space-time stationary adapted process $V : \Omega \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ solving the stoch. heat eq. in $\mathbb{R}^d \times \mathbb{R}$ s.t. $\mathbb{E}\left[\int_{\tilde{Q}_1} |V_t(x)|^2 dxdt\right] < \infty$.

Consequence:

$$a(p,x,t,\omega) = \mathcal{A}(p + D\tilde{V}_t(x,\omega_0), t, x, \omega_1) - D\tilde{V}_t(x,\omega_0)$$

stationary ergodic w.r.t $\mathbb{R} \times \mathbb{Z}^d$.

Nonlinear divergence form

Qualitative homogenization for nonlinear divergence form parabolic PDE

$$u^\epsilon_t - \mathrm{div} a\left(Du^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega\right) = f \text{ in } \mathbb{R}^d \times (0,\infty) \quad u^\epsilon(\cdot, 0) = u_0,$$

$a : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}^d$ is strongly monotone, Lipschitz continuous and space-time stationary and ergodic with respect to $\mathbb{Z}^d \times \mathbb{R}$-action.
Main result: Theorem on existence of corrector. (Cardaliaguet-D.-Souganidis 2020) For each P in $\mathbb{R}^d$ there ex. $\chi^P(x,t,\omega)$ s.t.

$$\int_{\tilde{Q}_1} \chi^P(x,t,\omega) dxdt = 0 \;\; P\text{–a.s.}, \;\; D\chi^P \in \mathbf{L}^2_{\mathbf{pot}}, \;\; \partial_t \chi^P \in \mathbf{H}^{-1}_{\mathbf{x}},$$

$$\partial_\tau \chi^P - \mathrm{div}(a(P + D\chi^P, y, \tau, \omega)) = 0 \text{ in } \mathbb{R}^{d+1},$$

$$\epsilon\chi^P\left(\frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega\right) \to 0$$

in $L^2_{loc}(\mathbb{R}^{d+1})$, $\mathbb{P}$–a.s. and in expectation. (Parabolic Sublinearity)

Effective Nonlinearity

$$u_t^\epsilon - \operatorname{div} a\left(Du^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega\right) = f \text{ in } \mathbb{R}^d \times (0,\infty) \quad u^\epsilon(\cdot,0) = u_0,$$

$a : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}^d$ is strongly monotone, Lipschitz continuous and space-time stationary and ergodic with respect to $\mathbb{Z}^d \times \mathbb{R}$-action.

$$\partial_\tau \chi^P - \operatorname{div}(a(P + D\chi^P, y, \tau, \omega)) = 0 \text{ in } \mathbb{R}^{d+1},$$

Effective nonlinearity

$$\overline{a}(p) = \mathbb{E}\left[\int_{\tilde{Q}_1} a(p + D\chi^p, y, \tau, \omega) dy d\tau\right]$$

is monotone and Lipschitz continuous.

Qualitative homogenization for nonlinear divergence form parabolic PDE: Related work

$$u_t^\epsilon - \mathrm{div} a\left(Du^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega\right) = f \text{ in } \mathbb{R}^d \times (0,\infty) \quad u^\epsilon(\cdot, 0) = u_0,$$

$a : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}^d$ is strongly monotone, Lipschitz continuous and space-time stationary and ergodic with respect to $\mathbb{Z}^d \times \mathbb{R}$-action.

Related work:

- Efendiev and Panov
- Efendiev, Jiang and Pankov
- Zhikov, Kozlov and Oleinik
- Landim, Olla and Yau
- Fannjiang and Komorowski
- Komorowski and Olla
- Rhodes
- Delarue and Rhodes
- Lin, Smart and Lin
- Armstrong, Bordas and Mourrat

Existence of corrector: Strategy

Space-time cube $\tilde{Q}_L$, (finite domain), regularize with $\lambda > 0$.

$$\lambda u_L - \lambda \partial_{tt} u_L + \partial_t u_L - \operatorname{div}(a(Du_L + p, \omega)) = 0 \text{ in } \tilde{Q}_L \quad u_L = 0 \text{ in } \partial \tilde{Q}_L.$$

A-priori estimates independent of L lead to existence of

$$\lambda \chi^{\lambda,p} - \lambda \partial_{tt} \chi^{\lambda,p} + \partial_t \chi^{\lambda,p} - \operatorname{div}(a(D\chi^{\lambda,p} + p, \omega)) = 0 \text{ in } \mathbb{R}^{d+1}.$$

Estimates uniform in λ for $\partial_t \chi^{\lambda,p}$ in $\mathbf{H}^{-1}$ and $D\chi^{\lambda,p}$ in $\mathbf{L}^2$.
Use monotonicity of nonlinearity and lemmas on reconstruction of stationary functions from derivatives.

Lemmas on vector fields

$\theta \in \mathbf{H}_\mathbf{x}^{-1}$, $w \in \mathbf{L}^2_{\mathbf{pot}}$ satisfy for $i = 1, \ldots, d$,

$$\langle \theta, \partial_{x_i} \phi \rangle_{\mathbf{H}_\mathbf{x}^{-1}, \mathbf{H}_\mathbf{x}^1} = \mathbb{E}\left[w_i \partial_t \phi \right]$$

Then there exists a measurable map $u : \mathbb{R}^{d+1} \times \Omega \to \mathbb{R}$ such that $Du = w$ and $\partial_t u = \theta$
If in addition $\xi \in \mathbf{L}^2$ satisfies

$$\theta - \mathrm{div}(\xi) = 0 \ \text{ in } \ \mathbf{H}_\mathbf{x}^{-1},$$

then

$$\mathbb{E}\left[\int_{\tilde{Q}_1} w \cdot \xi \right] = 0.$$

Heuristics

$$\int_{\tilde{Q}_1} Du \cdot \xi \int_{\tilde{Q}_1} = - \int_{\tilde{Q}_1} u \cdot \mathrm{div}(\xi) = - \int_{\tilde{Q}_1} u \partial_t u = - \int_{\tilde{Q}_1} \partial_t (\frac{1}{2} u^2).$$

Further ingredients

Lemma for sublinearity $\mathbb{P}-$a.s. and in expectation,

$$\lim_{R\to\infty} R^{-(d+3)} \int_{\tilde{Q}_R} |u(x,t)|^2 dxdt = \lim_{R\to\infty} R^{-(d+1)} \int_{\tilde{Q}_R} \left|\frac{u(x,t)}{R}\right|^2 dxdt = 0$$

Equivalent: $u^\epsilon(x,t,\omega) = \epsilon u(x/\epsilon, t/\epsilon, \omega)$.

$$\lim_{\epsilon\to 0} \int_{\tilde{Q}_R} |u^\epsilon(x,t)|^2 dxdt = 0.$$

Technical Difficulties:
Time derivative only in H^{-1}, ergodicity along lines (Lemma by Kosygina-Varadhan)

Homogenization: For perturbed test function method, approximate gradient of test function by piecewise constant function

Summary

Cardaliaguet-D.-Souganidis 2020

Result 1 (Scaling limit of SPDE by homogenization techniques)

$$d_t U_t^\epsilon \quad = \quad \mathrm{div}\mathcal{A}(DU_t^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega_1)dt + \frac{1}{\epsilon} \sum_{k \in \mathbb{Z}^d} A(\frac{x-k}{\epsilon})dB_t^k = f(x,t),$$

converge a.s. and in expectation in suitably weighted L^2 spaces to unique solutions of

$$\overline{u}_t - \mathrm{div}\ \overline{a}(D\overline{u}) = f$$

Result 2 (Homogenization/correctors)

$$\text{Problem} \quad u_t^\epsilon - \mathrm{div} a\left(Du^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega\right) = f \ \text{ in } \ \mathbb{R}^d \times (0, \infty)$$

$$\text{Corrector} \quad \partial_\tau \chi^P - \mathrm{div}(a(P + D\chi^P, y, \tau, \omega)) = 0 \ \text{ in } \ \mathbb{R}^{d+1},$$

Effective nonlinearity

$$\overline{a}(p) = \mathbb{E}\left[\int_{\tilde{Q}_1} a(p + D\chi^p, y, \tau, \omega) dy d\tau\right]$$

Outlook

$$d_t U_t^\epsilon = \mathrm{div}\mathcal{A}(DU_t^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega_1)dt + \frac{1}{\epsilon}\sum_{k\in\mathbb{Z}^d} A(\frac{x-k}{\epsilon})dB_t^k = f(x,t),$$

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arxiv:2004.03857

- Precise connection with Funaki-Spohn
- More degenerate operators (MCF approach by v. Renesse, Es-Sarhir)
- Multiplicative noise
- Allen-Cahn type problems
- Kac-type interactions and dynamic Lebowitz-Penrose limit

Thank you for your attention!