# Examiners' Report:
# Final Honour School of Mathematics Part A
# Michaelmas Term 2022

November 30, 2022

## Part I

### A. STATISTICS

- **Numbers and percentages in each class.**
  See Table 1.

Table 1: Numbers in each class

| Range | Numbers | | | | | Percentages % | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 2022 | 2021 | 2020 | 2019 | 2018 | 2022 | 2021 | 2020 | 2019 | 2018 |
| 70–100 | 59 | 53 | 43 | 57 | 57 | 36.65 | 37.32 | 32.58 | 35.19 | 35.62 |
| 60–69 | 71 | 57 | 65 | 71 | 69 | 44.1 | 40.14 | 49.24 | 43.83 | 43.12 |
| 50–59 | 22 | 29 | 21 | 27 | 22 | 13.66 | 20.42 | 15.91 | 16.67 | 13.75 |
| 40–49 | 6 | 2 | 3 | 5 | 9 | 3.73 | 1.41 | 2.27 | 3.09 | 5.62 |
| 30–39 | 2 | 0 | 0 | 1 | 3 | 1.24 | 0 | 0 | 0.62 | 1.88 |
| 0–29 | 1 | 1 | 0 | 1 | 0 | 0.62 | 0.7 | 0 | 0.62 | 0 |
| Total | 161 | 142 | 132 | 162 | 160 | 100 | 100 | 100 | 100 | 100 |

- **Numbers of vivas and effects of vivas on classes of result.**
  Not applicable.
- **Marking of scripts.**
  All scripts were single marked according to a pre-agreed marking scheme which was strictly adhered to. The raw marks for paper A2 are out of 100, and for the other papers out of 50. For details of the extensive checking process, see Part II, Section A.
- **Numbers taking each paper.**
  All 161 candidates are required to offer the core papers A0, A1, A2 and ASO, and five of the optional papers A3-A11. Eight candidates took six long options. Statistics for these papers are shown in Table 2 on page 2.

Table 2: Numbers taking each paper

| Paper | Number of Candidates | Avg RAW | StDev RAW | Avg USM | StDev USM |
|---|---|---|---|---|---|
| A0 | 161 | 33.59 | 9.7 | 66.67 | 13.79 |
| A1 | 161 | 30.21 | 7.76 | 65.96 | 11.02 |
| A2 | 161 | 57.91 | 17.36 | 66.36 | 10.49 |
| A3 | 89 | 27.61 | 8.78 | 64.34 | 13.2 |
| A4 | 136 | 32.43 | 8.8 | 67.3 | 11.67 |
| A5 | 106 | 28.07 | 9.34 | 66.63 | 10.71 |
| A6 | 88 | 34.26 | 7.6 | 65.6 | 10.69 |
| A7 | 61 | 35.44 | 9.68 | 68.9 | 13 |
| A8 | 145 | 32.21 | 7.77 | 65.59 | 11.18 |
| A9 | 68 | 27.68 | 7.61 | 66.37 | 10.68 |
| A10 | 36 | 37.75 | 8.24 | 68.25 | 15.65 |
| A11 | 74 | 33.65 | 8.15 | 66.42 | 11.89 |
| ASO | 161 | 32.14 | 8.76 | 66.07 | 12.82 |

## B. New examining methods and procedures

Exams returned to their in-person format following two years of online exams during the pandemic.

## C. Changes in examining methods and procedures currently under discussion or contemplated for the future

None.

## D. Notice of examination conventions for candidates

The first notice to candidates was issued on 30th March 2022 and the second notice on the 9th June 2022.

These can be found at https://www.maths.ox.ac.uk/members/students/undergraduate-courses/ba-master-mathematics/examinations-assessments/examination-20, and contain details of the examinations and assessments. The course handbook contains the link to the full examination conventions and all candidates are issued with this at induction in their first year. All notices and examination conventions are on-line at
https://www.maths.ox.ac.uk/members/students/undergraduate-courses/examinations-assessments/examination-conventions.

# Part II

## A. General Comments on the Examination

### Acknowledgements


- Elle Styler for her work in supporting the Part A examinations throughout the year, and for her help with various enquiries throughout the year.
- Waldemar Schlackow for running the database and the algorithms that generate the final marks, without which the process could not operate.
- Clare Sheppard for her help and support, together with the Academic Administration Team, with marks entry, script checking, and much vital behind-the-scenes work.
- The assessors who set their questions promptly, provided clear model solutions, took care with checking and marking them, and met their deadlines, thus making the examiners' jobs that much easier.
- Several members of the Faculty who agreed to help the committee in the work of checking the papers set by the assessors.
- The internal examiners and assessors would like to thank the external examiners, Prof Neil Strickland and Prof John Billingham, for helpful feedback and much hard work throughout the year, and for the important work they did in Oxford in examining scripts and contributing to the decisions of the committee.


### Timetable

The examinations began on Monday 13th June and ended on Friday 24th June.

### Mitigating Circumstances Notices to Examiners

A subset of the examiners (the 'Mitigating Circumstances Panel') attended a pre-board meeting to band the seriousness of the individual notices to examiners. The outcome of this meeting was relayed to the Examiners at the final exam board, who gave careful regard to each case, scrutinised the relevant candidates' marks and agreed actions as appropriate. See Section E for further details.

### Setting and checking of papers and marks processing

As is usual practice, questions for the core papers A0, A1 and A2, were set by the examiners and also marked by them with the assistance of assessors (with the exception that Dr Richard Earl set three of the questions for A2). The papers A3-A11, as well as each individual question on ASO, were set and marked by the course lecturers/assessors. The setters produced model answers and marking schemes led by instructions from Teaching Committee in order to minimize the need for recalibration.

The internal examiners met in December to consider the questions for Michaelmas Term courses (A0, A1, A2 and A11). The course lecturers for the core papers were invited to comment on the notation used and more generally on the appropriateness of the questions. Corrections and modifications were agreed by the internal examiners and the revised questions were sent to the external examiners.

In a second meeting the internal examiners discussed the comments of the external examiners and made further adjustments before finalising the questions. The same cycle was repeated in Hilary term for the Hilary term long option courses and at the end of Hilary and beginning of Trinity term for the short option courses. *Papers A8 and A9 are prepared by the Department of Statistics and jointly considered in Trinity term.* Before questions were submitted to the Examination Schools, setters were required to sign off a camera-ready copy of their questions.

The whole process of setting and checking the papers was managed digitally on SharePoint. Examiners adopted specific and detailed conventions to help with version checking and record keeping. This has worked very well.

Examination scripts were collected by the markers from Exam Schools or delivered to the Mathematical Institute for collection by the markers and returned there after marking. A team of graduate checkers under the supervision of Clare Sheppard and Elle Styler sorted all the scripts for each paper, cross-checking against the mark scheme to spot any unmarked questions or part of questions, addition errors or wrongly recorded marks. Also sub-totals for each part were checked against the marks scheme, noting any incorrect addition.

### Determination of University Standardised Marks

The examiners followed the standard procedure for converting raw marks to University Standardized Marks (USM). The raw marks are totals of marks on each question, the USMs are statements of the quality of marks on a standard scale. The Part A examination is not classified but notionally 70 corresponds to 'first class', 50 to 'second class' and 40 to 'third class'. In order to map the raw marks to USMs in a way that respects the qualitative descriptors of each class the standard procedure has been to use a piecewise linear map. It starts from the assumption that the majority of scripts for a paper will fall in the USM range 57-72, which is just below the II(i)/II(ii) borderline and just above the I/II(i) borderline respectively. In this range the map is taken to have a constant gradient and is determined by the corners $C_1$ and $C_2$, which encode the raw marks corresponding to a USM of 72 and 57 respectively. The guidance requires that the examiners should use the entire range of USMs. Our procedure interpolates the map linearly from $C_1$ to $(M, 100)$ where $M$ is the maximum possible raw mark. In order to allow for judging the position of the II(i)/III borderline on each paper, which corresponds to a USM of 40, the map is interpolated linearly between $C_3$ and $C_2$ and then again between $(0, 0)$ and $C_3$. Thus, the conversion of raw marks to USMs is fixed by the choice of the three corners $C_1, C_2$ and $C_3$. While the default $y$-values for these corners were given above and are not on the class borderlines, the examiners may opt to change those default values, e.g., to avoid distorting marks around class boundaries. The final choice of the scaling parameters is made by the examiners, guided by the advice from the Teaching Committee, considering the distribution of the raw marks and examining individuals on each paper around the borderlines.

The final resulting values of the parameters that the examiners chose are listed in Table 3.

Table 4 gives the resulting final rank and percentage of candidates with this overall average USM (or greater).

Table 3: Parameter Values

| Paper | C1 | C2 | C3 |
|---|---|---|---|
| A0 | (41.6,72) | (22.1,57) | (12.69,37) |
| A1 | (35.8,72) | (22.3,57) | (12.81,37) |
| A2 | (73,72) | (35.5,57) | (20.39,37) |
| A3 | (35.2,72) | (18.7,57) | (10.74,37) |
| A4 | (39.5,72) | (21.1,57) | (12.12,37) |
| A5 | (38.8,72) | (15.8,57) | (9.076,51) |
| A6 | (40,70) | (25.7,57) | (14.76,37) |
| A7 | (42,70) | (22.5,57) | (12.92,37) |
| A8 | (39,72) | (24,57) | (13.78,37) |
| A9 | (32.8,72) | (19.3,57) | (11.08,37) |
| A10 | (42,70) | (31.6,57) | (18.15,37) |
| A11 | (41.2,72) | (24.7,57) | (14.18,37) |
| ASO | (39.2,72) | (22.7,57) | (13.04,37) |

Table 4: Rank and percentage of candidates with this overall average USM (or greater)

| Av USM | Rank | Candidates with this USM or above | % |
|---|---|---|---|
| 94.7 | 1 | 1 | 0.62 |
| 85 | 2 | 2 | 1.24 |
| 84.5 | 3 | 3 | 1.86 |
| 82.3 | 4 | 4 | 2.48 |
| 81.7 | 5 | 5 | 3.11 |
| 81.1 | 6 | 6 | 3.73 |
| 80.7 | 7 | 7 | 4.35 |
| 80.6 | 8 | 8 | 4.97 |
| 80.1 | 9 | 9 | 5.59 |
| 79.9 | 10 | 10 | 6.21 |
| 79.1 | 11 | 11 | 6.83 |
| 78.9 | 12 | 12 | 7.45 |
| 78.7 | 13 | 13 | 8.07 |
| 77.9 | 14 | 14 | 8.7 |
| 77.85 | 15 | 15 | 9.32 |
| 77.5 | 16 | 17 | 10.56 |
| 77.4 | 18 | 18 | 11.18 |
| 77.2 | 19 | 19 | 11.8 |
| 76.3 | 20 | 20 | 12.42 |
| 76 | 21 | 21 | 13.04 |
| 75.7 | 22 | 22 | 13.66 |
| 75.1 | 23 | 23 | 14.29 |
| 75 | 24 | 24 | 14.91 |
| 74.9 | 25 | 26 | 16.15 |
| 74.6 | 27 | 27 | 16.77 |

Table 4: Rank and percentage of candidates with this overall average USM (or greater) [continued]

| Av USM | Rank | Candidates with this USM or above | % |
|---|---|---|---|
| 74.2 | 28 | 29 | 18.01 |
| 74 | 30 | 30 | 18.63 |
| 73.7 | 31 | 31 | 19.25 |
| 73.6 | 32 | 32 | 19.88 |
| 73.5 | 33 | 34 | 21.12 |
| 73.2 | 35 | 35 | 21.74 |
| 73.1 | 36 | 36 | 22.36 |
| 73 | 37 | 37 | 22.98 |
| 72.9 | 38 | 38 | 23.6 |
| 72.7 | 39 | 39 | 24.22 |
| 72.4 | 40 | 40 | 24.84 |
| 71.7 | 41 | 42 | 26.09 |
| 71.5 | 43 | 43 | 26.71 |
| 71.38 | 44 | 44 | 27.33 |
| 71.2 | 45 | 45 | 27.95 |
| 71.15 | 46 | 46 | 28.57 |
| 71.1 | 47 | 47 | 29.19 |
| 71 | 48 | 48 | 29.81 |
| 70.7 | 49 | 49 | 30.43 |
| 70.5 | 50 | 51 | 31.68 |
| 70.2 | 52 | 53 | 32.92 |
| 70.1 | 54 | 54 | 33.54 |
| 70 | 55 | 55 | 34.16 |
| 69.9 | 56 | 57 | 35.4 |
| 69.8 | 58 | 58 | 36.02 |
| 69.7 | 59 | 59 | 36.65 |
| 69.3 | 60 | 60 | 37.27 |
| 69.1 | 61 | 61 | 37.89 |
| 69 | 62 | 62 | 38.51 |
| 68.5 | 63 | 63 | 39.13 |
| 68.3 | 64 | 64 | 39.75 |
| 68.2 | 65 | 66 | 40.99 |
| 68.1 | 67 | 68 | 42.24 |
| 68 | 69 | 69 | 42.86 |
| 67.9 | 70 | 70 | 43.48 |
| 67.8 | 71 | 71 | 44.1 |
| 67.6 | 72 | 72 | 44.72 |
| 67.4 | 73 | 74 | 45.96 |
| 67.3 | 75 | 75 | 46.58 |
| 67.22 | 76 | 76 | 47.2 |
| 67.2 | 77 | 77 | 47.83 |
| 67.15 | 78 | 78 | 48.45 |

Table 4: Rank and percentage of candidates with this overall average USM (or greater) [continued]

| Av USM | Rank | Candidates with this USM or above | % |
|---|---|---|---|
| 67.1 | 79 | 79 | 49.07 |
| 67 | 80 | 80 | 49.69 |
| 66.9 | 81 | 81 | 50.31 |
| 66.7 | 82 | 82 | 50.93 |
| 66.65 | 83 | 83 | 51.55 |
| 66.6 | 84 | 85 | 52.8 |
| 66.6 | 84 | 85 | 52.8 |
| 66.5 | 86 | 86 | 53.42 |
| 66.4 | 87 | 88 | 54.66 |
| 66.4 | 87 | 88 | 54.66 |
| 66.3 | 89 | 90 | 55.9 |
| 66.3 | 89 | 90 | 55.9 |
| 66 | 91 | 92 | 57.14 |
| 66 | 91 | 92 | 57.14 |
| 65.6 | 93 | 93 | 57.76 |
| 65.5 | 94 | 96 | 59.63 |
| 65.1 | 97 | 98 | 60.87 |
| 65 | 99 | 99 | 61.49 |
| 64.9 | 100 | 100 | 62.11 |
| 64.8 | 101 | 101 | 62.73 |
| 64.7 | 102 | 102 | 63.35 |
| 64.4 | 103 | 103 | 63.98 |
| 63.9 | 104 | 104 | 64.6 |
| 63.8 | 105 | 105 | 65.22 |
| 63.7 | 106 | 106 | 65.84 |
| 63.6 | 107 | 109 | 67.7 |
| 63.6 | 107 | 109 | 67.7 |
| 63.6 | 107 | 109 | 67.7 |
| 63.4 | 110 | 111 | 68.94 |
| 63.4 | 110 | 111 | 68.94 |
| 63.3 | 112 | 112 | 69.57 |
| 63.2 | 113 | 113 | 70.19 |
| 63 | 114 | 116 | 72.05 |
| 62.6 | 117 | 117 | 72.67 |
| 61.8 | 118 | 118 | 73.29 |
| 61.6 | 119 | 119 | 73.91 |
| 61.4 | 120 | 121 | 75.16 |
| 61.2 | 122 | 122 | 75.78 |
| 60.6 | 123 | 123 | 76.4 |
| 60.4 | 124 | 124 | 77.02 |
| 60.2 | 125 | 125 | 77.64 |
| 60 | 126 | 126 | 78.26 |

Table 4: Rank and percentage of candidates with this overall average USM (or greater) [continued]

| Av USM | Rank | Candidates with this USM or above | % |
|---|---|---|---|
| 59.8 | 127 | 127 | 78.88 |
| 59.7 | 128 | 128 | 79.5 |
| 59.6 | 129 | 129 | 80.12 |
| 59.5 | 130 | 130 | 80.75 |
| 59.3 | 131 | 131 | 81.37 |
| 58.8 | 132 | 132 | 81.99 |
| 58.1 | 133 | 133 | 82.61 |
| 57.8 | 134 | 134 | 83.23 |
| 57.7 | 135 | 135 | 83.85 |
| 57.56 | 136 | 136 | 84.47 |
| 56.6 | 137 | 137 | 85.09 |
| 56.33 | 138 | 138 | 85.71 |
| 56.3 | 139 | 139 | 86.34 |
| 56.1 | 140 | 140 | 86.96 |
| 55.9 | 141 | 141 | 87.58 |
| 55.8 | 142 | 142 | 88.2 |
| 55.5 | 143 | 143 | 88.82 |
| 55.1 | 144 | 144 | 89.44 |
| 55 | 145 | 145 | 90.06 |
| 54.7 | 146 | 146 | 90.68 |
| 54.6 | 147 | 147 | 91.3 |
| 53.5 | 148 | 148 | 91.93 |
| 53.1 | 149 | 149 | 92.55 |
| 52.7 | 150 | 150 | 93.17 |
| 52.5 | 151 | 151 | 93.79 |
| 50.7 | 152 | 152 | 94.41 |
| 45.9 | 153 | 153 | 95.03 |
| 45.44 | 154 | 154 | 95.65 |
| 45.3 | 155 | 155 | 96.27 |
| 44.88 | 156 | 156 | 96.89 |
| 44.8 | 157 | 157 | 97.52 |
| 43.9 | 158 | 158 | 98.14 |
| 38.1 | 159 | 159 | 98.76 |
| 37.5 | 160 | 160 | 99.38 |
| 27.33 | 161 | 161 | 100 |

### Recommendations for Next Year's Examiners and Teaching Committee

The examiners were strongly in favour of having assessors present for an initial period in the examinations. (The two typographical errors which caused some confusion in A10 and A11 could have been immediately remedied had this been the case this year.) They also recommend paper checking procedures be reviewed, to ensure that someone has attempted in advance the *final version* of each paper *without solutions*, rather than just checked the paper and solutions through its various iterations. And further that Teaching Committee ensure Examination Schools has up-to-date software for large fonts on its printers, to avoid any repeat of the errors in printing these papers we unfortunately encountered this year. Finally the Chairman of Examiners suggest that Teaching Committee requests IT support write code which would allow one to immediately generate latex tables on examination data for inclusion in these reports from the source database (at present all the data is transcribed by hand).

## B. Equality and Diversity issues and breakdown of the results by gender

Table 5, page 10 shows percentages of male and female candidates for each class of the degree.

Table 5: Breakdown of results by gender

| Class | Number | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 2022 | | | 2021 | | | 2020 | | |
| | Female | Male | Total | Female | Male | Total | Female | Male | Total |
| 70–100 | 7 | 52 | 59 | 5 | 48 | 53 | 10 | 33 | 43 |
| 60–69 | 23 | 48 | 71 | 21 | 36 | 57 | 22 | 43 | 65 |
| 50–59 | 8 | 14 | 22 | 15 | 14 | 29 | 7 | 14 | 21 |
| 40–49 | 4 | 2 | 6 | 1 | 1 | 2 | 2 | 1 | 3 |
| 30–39 | 0 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0–29 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 |
| Total | 43 | 118 | 161 | 42 | 100 | 142 | 41 | 91 | 132 |
| Class | Percentage | | | | | | | | |
| | 2022 | | | 2021 | | | 2020 | | |
| | Female | Male | Total | Female | Male | Total | Female | Male | Total |
| 70–100 | 16.28 | 44.07 | 36.65 | 11.9 | 48 | 37.32 | 24.39 | 36.26 | 30.32 |
| 60–69 | 53.49 | 40.68 | 44.10 | 50 | 36 | 40.14 | 53.66 | 47.25 | 50.45 |
| 50–59 | 18.6 | 11.86 | 13.66 | 35.71 | 14 | 20.42 | 17.07 | 15.38 | 16.22 |
| 40–49 | 9.3 | 1.69 | 3.73 | 2.38 | 1 | 1.41 | 4.88 | 1.1 | 2.99 |
| 30–39 | 0 | 1.69 | 1.24 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0–29 | 2.33 | 0 | 0.62 | 0 | 1 | 0.7 | 0 | 0 | 0 |
| Total | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |

## C. Detailed numbers on candidates' performance in each part of the exam

Individual question statistics for Mathematics candidates are shown in the tables below.

### Paper A0: Linear Algebra

| Question | Mean Mark | | Std Dev | Number of attempts | |
|---|---|---|---|---|---|
| | All | Used | | Used | Unused |
| Q1 | 17.91 | 18.04 | 6.12 | 136.00 | 1 |
| Q2 | 12.49 | 13.14 | 6.01 | 44.00 | 3 |
| Q3 | 16.86 | 16.86 | 4.56 | 137.00 | 0 |

**Paper A1: Differential Equations 1**

| Question | Mean Mark | | Std Dev | Number of attempts | |
|---|---|---|---|---|---|
| | All | Used | | Used | Unused |
| Q1 | 11.88 | 12.31 | 4.92 | 64.00 | 3 |
| Q2 | 15.83 | 15.94 | 4.65 | 105.00 | 1 |
| Q3 | 15.62 | 15.71 | 4.14 | 149.00 | 1 |

**Paper A2: Metric Spaces and Complex Analysis**

| Question | Mean Mark | | Std Dev | Number of attempts | |
|---|---|---|---|---|---|
| | All | Used | | Used | Unused |
| Q1 | 15.65 | 15.89 | 6.76 | 105.00 | 2 |
| Q2 | 16.24 | 16.32 | 5.90 | 136.00 | 1 |
| Q3 | 11.16 | 11.47 | 5.70 | 66.00 | 4 |
| Q4 | 13.21 | 13.21 | 4.80 | 125.00 | 0 |
| Q5 | 15.71 | 16.00 | 5.46 | 140.00 | 4 |
| Q6 | 10.75 | 11.39 | 4.76 | 59.00 | 4 |

**Paper A3: Rings and Modules**

| Question | Mean Mark | | Std Dev | Number of attempts | |
|---|---|---|---|---|---|
| | All | Used | | Used | Unused |
| Q1 | 14.94 | 14.94 | 4.81 | 88.00 | 0 |
| Q2 | 11.54 | 11.72 | 4.81 | 74.00 | 2 |
| Q3 | 16.18 | 17.19 | 6.20 | 16.00 | 1 |

**Paper A4: Integration**

| Question | Mean Mark | | Std Dev | Number of attempts | |
|---|---|---|---|---|---|
| | All | Used | | Used | Unused |
| Q1 | 16.74 | 16.84 | 5.69 | 87.00 | 1 |
| Q2 | 15.71 | 16.10 | 4.88 | 107.00 | 3 |
| Q3 | 15.44 | 15.67 | 4.80 | 78.00 | 2 |

**Paper A5: Topology**

| Question | Mean Mark | | Std Dev | Number of attempts | |
|---|---|---|---|---|---|
| | All | Used | | Used | Unused |
| Q1 | 14.27 | 14.27 | 5.07 | 37.00 | 0 |
| Q2 | 15.32 | 15.32 | 5.45 | 100.00 | 0 |
| Q3 | 12.09 | 12.20 | 5.54 | 75.00 | 1 |

**Paper A6: Differential Equations 2**

| Question | Mean Mark | | Std Dev | Number of attempts | |
|---|---|---|---|---|---|
| | All | Used | | Used | Unused |
| Q1 | 17.89 | 17.89 | 3.98 | 70.00 | 0 |
| Q2 | 14.77 | 15.00 | 5.04 | 52.00 | 1 |
| Q3 | 18.04 | 18.20 | 3.86 | 54.00 | 1 |

**Paper A7: Numerical Analysis**

| Question | Mean Mark | | Std Dev | Number of attempts | |
|---|---|---|---|---|---|
| | All | Used | | Used | Unused |
| Q1 | 15.35 | 15.35 | 5.90 | 51.00 | 0 |
| Q2 | 19.45 | 19.45 | 4.86 | 56.00 | 0 |
| Q3 | 19.33 | 19.33 | 4.58 | 15.00 | 0 |

**Paper A8: Probability**

| Question | Mean Mark | | Std Dev | Number of attempts | |
|---|---|---|---|---|---|
| | All | Used | | Used | Unused |
| Q1 | 15.71 | 15.71 | 5.03 | 111.00 | 0 |
| Q2 | 17.52 | 17.52 | 3.89 | 117.00 | 0 |
| Q3 | 13.82 | 14.13 | 3.60 | 62.00 | 3 |

**Paper A9: Statistics**

| Question | Mean Mark | | Std Dev | Number of attempts | |
|---|---|---|---|---|---|
| | All | Used | | Used | Unused |
| Q1 | 9.44 | 10.56 | 4.77 | 34.00 | 5 |
| Q2 | 14.44 | 14.66 | 4.88 | 44.00 | 1 |
| Q3 | 14.92 | 15.14 | 5.13 | 58.00 | 1 |

**Paper A10: Fluids and Waves**

| Question | Mean Mark | | Std Dev | Number of attempts | |
|---|---|---|---|---|---|
| | All | Used | | Used | Unused |
| Q1 | 18.77 | 19.00 | 3.66 | 30.00 | 1 |
| Q2 | 17.35 | 17.60 | 5.85 | 30.00 | 1 |
| Q3 | 21.75 | 21.75 | 4.05 | 12.00 | 0 |

**Paper A11: Quantum Theory**

| Question | Mean Mark | | Std Dev | Number of attempts | |
|---|---|---|---|---|---|
| | All | Used | | Used | Unused |
| Q1 | 16.19 | 16.19 | 4.85 | 72.00 | 0 |
| Q2 | 17.81 | 17.81 | 4.68 | 70.00 | 0 |
| Q3 | 11.86 | 12.83 | 5.24 | 6.00 | 1 |

**Paper ASO: Short Options**

| Question | Mean Mark | | Std Dev | Number of attempts | |
|---|---|---|---|---|---|
| | All | Used | | Used | Unused |
| Q1 | 11.86 | 11.86 | 5.11 | 71.00 | 0 |
| Q2 | 18.32 | 18.32 | 3.10 | 44.00 | 0 |
| Q3 | 16.56 | 16.56 | 6.67 | 9.00 | 0 |
| Q4 | 16.50 | 16.50 | 4.43 | 4.00 | 0 |
| Q5 | 19.11 | 19.11 | 5.03 | 74.00 | 0 |
| Q6 | 16.07 | 16.07 | 5.82 | 54.00 | 0 |
| Q7 | 16.15 | 16.15 | 4.33 | 27.00 | 0 |
| Q8 | 11.33 | 11.33 | 7.09 | 3.00 | 0 |
| Q9 | 15.53 | 15.53 | 5.42 | 36.00 | 0 |

## D. Comments on papers and on individual questions

The following comments were submitted by the assessors.

### Core Papers

### A0: Algebra 1

**Question 1**. Common mistakes: (a)(i): Show spanning or linearly-independent, but not both. (a)(ii): Prove the existence of $\overline{T}$, but not its linearity. (a)(iii): Fail to prove that the bottom right block coincides with $\overline{T}$ in basis $q(\mathcal{B}_2)$, or do so using incorrect reasoning. (a)(iv): Take for granted that every matrix is upper triangularizable in a basis. (b)(i): Argue that $au + bv + ciu + div = 0 \Rightarrow au + bv = ciu + div = 0$ by taking "real" and "imaginary" parts. This is not a well-defined operation on $V$! Prove spanning or linearly-independent, but not both. Argue that $u, v, iu, iv$ is spanning (over $\mathbb{R}$) since a subset (namely, $\{u, v\}$) is spanning (over $\mathbb{C}$). (b)(ii): Incorrectly calculate action of $T$ on basis. Reverse rows and columns. (b)(iii): Incorrectly calculate characteristic polynomial (if errors were related to errors in 1bii, partial credit was given). Correctly calculate characteristic polynomial, but do not explain deduction of minimal polynomial. (b)(iv): Incorrectly assert that there is no JNF (since minimal polynomial has no real roots). Correctly determine diagonal entries, but miscalculate block sizes from minimal polynomial. Correctly deduce JNF, but do not provide justification.

**Question 2** was the least popular question on this paper. Part (a) was done very well by most people who attempted it; however, many people lost marks in (a)(ii) if they failed to give a counterexample. Part (b) was quite difficult, although (ii), (iii) and (iv) were amenable to a direct calculation with no theory being involved. It was pleasing to see several students to nevertheless obtain full marks for Question 2.

**Question 3** was very popular, and done relatively well by most students. Part (a) was straightforward, although part (a)(iii) required careful thought in order to arrive at the correct answer (strictly *lower* triangular matrices, not *upper*). Parts (b)(i) and (b)(ii) were done very well indeed, but part (b)(iii) was more difficult. It was nice to see that several students found the following alternative elegant solution: by rank-nullity, it is enough to prove that $\gamma$ is injective; now if $\gamma(v) = 0$ then $0 = \langle \gamma(v), v\rangle = \langle \alpha^* v, \alpha^* v\rangle + \langle \beta v, \beta v\rangle = 0$, so because $\langle -, -\rangle$ is positive definite this gives $\alpha^* v = \beta v = 0$. But then $v \in \ker \beta = \mathrm{im}\alpha$, and $v \in \ker \alpha^*$, so $v = 0$ by part (ii).

### A1: Differential Equations 1

**Question 1** was the least popular among the questions and was found to be reasonably challenging. Most of the candidates got the bookwork with no problems. Some forgot to give any justification for their answers, e.g., in (a)(iii) when deducing the series representation for the solution. Part (a)(iv) was done poorly with only few students deriving the IVP for $z$ (and even then often forgetting about the boundary condition, or to argue why this IVP has a unique solution). Roughly a third of the students failed to produce any attempt at the counterexample in (b)(i) and a third gave an example which did not satisfy the suitable Lipschitz continuity assumptions, with the remaining third usually giving one among many simple examples (e.g., $jjy(x) + xjj$). Many candidates were confused with (b)(ii) and failed to use triangular inequality and exploit the Lipschitz property in two variables. Part (b)(iii) proved challenging with most candidates either not having time to attempt it, or stopping halfway through.

**Question 2** was attempted by a large number of candidates. Some marks were lost in the bookwork part (a) by simply stating the characteristic equations without giving any sort of derivation. Part (b) was generally done well. The quality of sketches of the helical strip surface in (b)(iii) varied widely, but the marks were given as long as the idea was right. Part (c) produced a wide variety of approaches; the lack of a defined $P$ and $Q$ clearly caused conceptual issues, so that many candidates did not seem to know how to begin. The problem is best solved by drawing the characteristics and data curve, and then recognising a right-angled triangle that defines the Cauchy data boundary. Candidates who saw this geometric connection were able to quickly obtain the desired solution.

**Question 3** was attempted by most candidates. Part (a)(iii) tended to be hit or miss, with many candidates leaving it blank; full marks required a valid argument about uniqueness of the equations away from critical points, which was best done by appropriately invoking Picard $\tilde{O}$'s theorem for the ODE $dy/dx = Y/X$ in a neighbourhood of a non-critical point. Part (b) was generally well done, with just a few small errors appearing, mostly due to small algebraic mistakes. Part (c) was again hit or miss. As the question hinted at making an argument about symmetry, vague statements about symmetry were not awarded marks; rather, clear arguments demonstrating how the $(u, v)$ system is symmetric, and how the symmetry implies closed trajectories, were required to earn the marks.

### A2: Metric Spaces and Complex Analysis

**Question 1**. This was a popular question and generally done very well, with many candidates scoring high marks. Almost everyone had little trouble with the bookwork in (a), and most recognised that (c) was a variation of a question on the problem sheets in which the real interval $[0; 1]$ is replaced by the finite set $\{0, 1\}$. Some had trouble correctly interpreting the statement in (b)(i), but most candidates were able to find a counterexample for (b)(ii) (related to material in Analysis II)

**Question 2** seems to have gone very well in general. Part (a) was mainly bookwork and most candidates received full marks; the only exception was the last part where some did not an example of a contraction without fixed points. Part (b) (i) was the most difficult part of this question. Nevertheless, a majority of candidates was able to see that the sequence $(x_n)_{n\in N}$ of fixed points of $(1-\frac{1}{n})T$ had to be considered (following the hint). The existence of a fixed point of $T$ now follows from a convergence argument which not everyone found. Part (b)(ii) was answered correctly by the great majority of candidates; part (b)(iii) caused more difficulty than what was perhaps expected with many simply asserting an affirmative answer, even though counterexamples are fairly easy (though not trivial) to spot. Part (b)(iv) again went well; the main reason for loss of marks was candidates simply asserting an answer. Part (c) seems to have been found fairly easy by most candidates. Almost all proved accurately that the topology on $\mathbb{N}$ induced by $d$ is discrete and that $(\mathbb{N}, d)$ is complete. A majority of candidates had the right idea regarding the proof that $(\mathbb{N}, \delta)$ is discrete but this caused some technical difficulty. The great majority gave correct examples of non-convergent Cauchy sequences in $(\mathbb{N}, \delta)$ but not all of them were able to justify this adequately. Surprisingly few spotted that $(\mathbb{N}, \delta)$ is isomorphic to $(\frac{1}{n} : n \in \mathbb{N}, d)$, where $d$ is the Euclidean metric.

**Question 3**. Part (a) was bookwork and was answered well by most people. I deducted marks for answers that were not as detailed as those in the solutions, and of course for the minority of students that wrote down the formula for the coefficients incorrectly. In (b)(ii) the vast majority of students correctly understood that the sequence $p_n$ was the truncated Taylor expansion, for which I already awarded 1 mark even if presented without justification. The remaining marks were to be gained by justifying why convergence is uniform on $D(0, r)$, $r < 1$. I gave a lot of partial credit to students who made estimation arguments that did not reach the level of detail in the proposed solution, e.g. many said that convergence is uniform "by the Weierstrass $M$-test" but without writing down the estimate for the coefficients: most of these students got 3/4. I was a bit stricter with students who did not write estimates but just argued based on the radius of convergence (though a few got full marks, provided the result used was stated clearly and correctly). In (b)(ii), and in further questions that required a counterexample — namely (c)(i) and (c)(iii) — my rule was: 0 marks for no or incorrect counterexample (including answers such as "No.", or "No, because the same argument of the previous point does not work", even if correct), 1 mark for a correct counterexample with no or incorrect justification, and further marks for attempts at justifying that the proposed function violates the required property. In (b)(ii) and in (c)(iii) a common misconception was that the sequence of polynomials/rational functions that approximate $f$ had to be the truncated Taylor/Laurent expansion, which is not stated in the question; in fact, one has to exclude that more general such sequences (with varying coefficients of low order) must always exist. In (b)(ii) most students guessed a correct counterexample and many intuited that the principle at play is that a sequence of bounded (even if not uniformly bounded) functions may not uniformly approximate an unbounded one, though few obtained the full 3 marks for a correct justification. In (c)(i), most students correctly guessed $f(x) = 1/x$ and most of these got the full marks by deriving the contradiction $0 = 2\pi i$, although a few provided an example that failed to be holomorphic on the punctured plane (writing, e.g. "$f(x) = 1/(1-x)$ and argue as in the previous point"); I did not award any marks to such answers, as they completely missed the point of the question. In (c)(ii) a majority of students correctly wrote down the correct sequence of approximants, although a few wrote things such as $q_n(x) = \sum_{m=-\infty}^{n} c_m z^m$ which is not a rational function, for which I did not award any marks. Here I was a little more demanding for precise justifications compared to (b)(i): this is because I consider uniform convergence of the Laurent series within the two radii to be a less standard result than that of the Taylor series (I noticed that uniformity is briefly mentioned in a proof in the lecture notes, but not in the statement of the theorem). Not many candidates understood that they had to separately estimate the principal part to use the $M$-test. (c)(iii) was the most difficult question, and I awarded partial credit to many arguments that were not very convincing but caught the spirit of the problem.

**Question 4**. Most candidates correctly answered the question (a). Question (a) involves a fundamental question regarding the holomorphic branch of the logarithmic function. However, it is crucially used to answer the question (b)(i). It turns out that question (b)(ii) was the most challenging problem. Only a few candidates provided the correct answer. For questions (b)(iii) and (c)(i), some candidates made (small) computation mistakes in the application of Taylor/ Laurent expansion. Mostly this mistake was made for the regular part of the Laurent expansion. Therefore many candidates got the marks for question (c)(ii). In summary, I think the subproblems of Question 4 are connected very nicely. The candidates who noticed this connection provided more smooth answers.

**Question 5**. Part (a): This question was attempted by a large number of students, with many excellent answers.

Many otherwise good proofs had issues with computing the residue. In particular finding $(-1)^{-2/3}$ caused difficulties, even for candidates that had given a correct definition of a branch of $z^{1/3}$ earlier in their answer.

Other common errors included mistaking $1/(z+1)^2$ for $1/(z^2+1)$, confusion about the cube root symbol $\sqrt[3]{z}$ (which means $z^{1/3}$ rather than $z^{3/2}$) and mistakenly assuming that the integrals along the contours either side of the branch cut took equal values.

Part (b): This question was again attempted by a large number of students, with many good answers and different approaches. A key point was that to gain full marks, all steps in the calculation needed to be carefully justified.

To evaluate the first integral, many candidates successfully set up the problem as a contour integral. Some otherwise good answers lost marks here by not taking care of constant factors; in particular the prefactor of $z^2$ in the denominator was often forgotten later in the calculation (i.e., $\frac{1}{(az^2+bz+c)} = \frac{1}{a(z-z_0)(z-z_1)} \neq \frac{1}{(z-z_0)(z-z_1)}$, where the roots of the denominator are given by $z_0, z_1$).

For the second part, many good attempts were made following the line of reasoning that uses the previous integral, i.e., expanding both sides in powers of $\alpha$ (with correct justification). Two common alternative approaches were to calculate the integral on the left-hand-side directly using residue calculus, or to prove the result by induction.

**Question 6**. A sizeable proportion of candidates attempted this question, but it was not done that well and few scored high marks. The vast majority did not get beyond stating the theorem in (c). Most did well on the early parts of (a), but few gave convincing arguments for (a)(iii). Part (b) was on the whole done quite well.

## Long Options

### A3: Rings and Modules

**Question 1** was attempted by almost all candidates. In (a)(ii) the most common mistake was to say $(3 + 2\sqrt{2}) = (1 + \sqrt{2})^2$ and so it factors and is not irreducible, rather than noting that it is a unit and so not irreducible. (a)(iii) could be done by directly applying Krull's Theorem, or a longer answer could be given by repeating the lecture proof that a PID has the ACCP and then applying (a)(i). (a)(iv) was quite well done with the most common mistake to confuse maximal proper principal ideal and maximal proper ideal. (b)(i) was fairly well done. The idea in (b)(ii) was that the previous part could be used. The most common error was to claim that $\mathbb{Z}[\alpha + \beta] = \mathbb{Z}[\alpha] + \mathbb{Z}[\beta]$. For (b)(iii) the idea was to note that if $\alpha \in \overline{\mathbb{Z}}$ then $\sqrt{\alpha} \in \overline{\mathbb{Z}}$, so if $\alpha$ is irreducible then $\alpha \approx \sqrt{\alpha}$. This means that either $\alpha$ is a unit or $\alpha = 0$. This last case was missed by almost everyone, but allowed for the use of (a)(iv) coupled with the observation that $\overline{\mathbb{Z}}$ is not a field.

**Question 2** was attempted by the vast majority of candidates. It was found harder than intended, but it did give a very good spread to marks which made it easier to distinguish candidates. A common mistake in parts (b) and (c) was to show the isomorphisms were linear isomorphisms rather than ring isomorphisms. While (c)(i) was natural for the structure of the question, solutions to this seemed to be uncorrelated with solutions to other parts of the question. Part (d)(i) as worded admitted taking $p = 0$ though that was not what was intended. Full marks were given for a careful proof of this, though that made the result less useful for (d)(ii). (d)(ii) was found to be very difficult.

**Question 3** was attempted by very few candidates, but was certainly the easiest question on the paper so those who were prepared to work with modules were rewarded for their efforts. The most common mistake was to claim for (a)(iii) that the Uniqueness Theorem applied, but the point of the question was that it does not for non-commutative rings such as $R = M_2(\mathbb{F})$. Here it was best to note that $R^n$ is $\mathbb{F}$-linearly isomorphic to $\mathbb{F}^{4n}$ and so if $R^n$ is $R$-linearly isomorphic to $R^m$ then $4n = 4m$ by the Dimension Theorem for vector spaces.

### A4: Integration

**Question 1**. This was a popular question, and part (a) was generally well done. As a note for future students, in part (a)(iv) most candidates considered $F_n = \cup_{r \geq n} E_r$ and applied (iii) as intended - however marks where often lost for failing to explicitly check that the hypotheses of (iii) apply (particularly $m(F_1) < \infty$). In (a)(iii) many students chose to prove the lemma that if $E_1 \subset E_2 \subset \dots$ are measurable then $m(\bigcup_n E_n) = \sum_n m(E_n)$ en-route to the result; this is perfectly valid (though longer than finding a way of using countable additivity directly). Regardless of the approach, it would have been advisable to clearly indicate in answers where the important hypothesis $m(F_1) < \infty$ is used, though this was not penalised this year unless it actually led to an error.

Part (b) discriminated well between candidates: (i) was well answered by a reasonable number of candidates demonstrating strong command of the material, but it also saw quite a few attempts in the style of a plausibility argument and also featured a number of false claims such as measurable functions on closed and bounded intervals being bounded. In (ii) a considerable number of candidates had the idea to look at $E = \cap_n \cup_{r \geq n} \{x : |f_r(x)|/a_r > 1/r\}$ which has measure 0 by (a)(iv), but this was often not correctly related to $\{x : f_n(x) \not\to 0\}$

(often these sets where claimed to be equal, whereas if $x \notin E$, then $f_n(x) \to 0$). A common error was to attempt to take limits in $m(\{x : |f_n(x)|/a_n > 1/n\}) < 2^{-n}$ either inside the set, or in the $2^{-n}$, without any form of justification (these arguments where invariably invalid).

**Question 2**. The bookwork in (a) was very well done. In (b), (i) was well done, though quite a few candidates where a bit too caviller in computing $\int_0^\infty e^{-xy} dx$ without any comment about how to handle the $\infty$: expressions like $[\frac{e^{-xy}}{y}]_0^\infty$ are inadvisable in this sort of course, and certainly the first time something like this appears it's a chance to show knowledge of how these integrals can be computed using the baby monotone convergence theorem. Most candidates had looked ahead at (ii) and realised they were aiming for $\log a$ — though a few ended up bounding the integral in a somewhat complicated fashion. (ii) was invariably very well done, but (iii) caused more difficulties. Many candidates found the argument published in the solutions (or the a slightly simpler version added to the solutions); another successful though involved method was to explicitly calculate $\int_0^1 \int_1^\infty |g| dx dy$ by working out when $g$ is positive and negative. However, quite a number of candidates where hampered by calculation slips (sign errors, or inaccurate integrals) which had the effect of destroying the problem (and some candidates failed to recall that for $g$ to be integrable, the integral of $|g|$ must be finite).

Part (c) was very mixed; a few candidates found the very short solution using Fubini's theorem. A number of other candidates had this idea, but wrote it down in a sloppy fashion, which didn't convey understanding (often expressions like $\int_0^1 f(x)dx - \int_0^1 f(y)dx < \infty$ — it is a not a good idea to be writing $\int < \infty$ to mean integrable unless one is working with a non-negative function). There was also a large variety of incorrect counter examples on offer.

**Question 3**. Question (a) was typically well done, though some candidates in (iv) didn't argue from the definitions introduced in (ii) and (iii) and relied on linearity of the integral; a much harder result. (b) was also broadly well done. I'd advise future candidates to be very explicit in how they use the substitution theorem in questions like (b)(ii) to show their knowledge of the material of the course, and avoid answering this in a calculus style fashion. Part (c) proved challenging, and featured various incorrect applications of the DCT. Not many candidates realised they could take limits (a.e.) to obtain $|f| \le g$ and hence deduce integrability of $f$ from that of $g$. Those candidates that spotted that Fatou's lemma could be used (the statement is a kind of variation of the DCT so it's natural to consider how the proof of the DCT goes) typically solved part (i). Very few candidates spotted that one can apply the result of (i) together with integrability to obtain (ii) (considering $|f_n| + |f| \le g_n + |f|$).

### A5: Topology

**Question 1** was chosen by the least number of students, possibly because they saw the drawings of Möbius bands and Klein bottles. Indeed, the question asking to show that the connected sum of two Möbius bands is homeomorphic to a twice punctured Klein bottle was skipped by many. The last question, about recognising open subsets of Hausdorff spaces was difficult, and only a very small percentage of students got it right.

**Question 2** was chosen by most. In (a)(ii)($\alpha$), some reproved the Heine-Borel theorem, which was not the intention of the question. In (b)(ii), almost nobody knew how to properly prove that a convex polygon is homeomorphic to a disc. I ended up giving full points for less then perfect answers.

**Question 3** was done mostly fine, with the notable exception of (b)(iii), which most students found very hard. Indeed, arguing that the map $[0, b) \to S^1 : x \mapsto e^{2\pi i}$ is a quotient map was beyond the ability of most students. The last question, about identifying two models of $\mathbb{R}P^2$ was difficult, but of the expected level of difficulty.

### A6: Differential Equations 2

**Question 1**. 1(a) was overall very well done, with some difficulties to derive expressions for the adjoint boundary conditions. For 1(b), most students did well, and most errors came from calculations errors. In 1(c), arguments were not always sufficiently solid. 1(d) was the most challenging question, especially the first part of the question, but also in finding the eigenvalues for the second part of the problem.

**Question 2**. Question 2 involves the longest calculations, and students sometimes failed to reach the last parts of the question. 2(a) went relatively well, even if some students failed to give satisfactory explanations. 2(b) went very well, except for some minor calculation mistakes. A same observation can be made for 2(c). 2(d) was the most challenging part, and several students got lost in calculations, or did not consider it.

**Question 3**. 3(a) was bookwork and went well, even if explanations were not always clearly exposed. 3(b) went relatively well, but some students struggled in finding the regime where the solution breaks done. Question 3(c) went relatively well. The students clearly had a good understanding overall, but the succession of steps to arrive at the solution, and the explanation between steps were sometimes not satisfactory.

### A7: Numerical Analysis

This seems to have been a reasonably successful exam with a spread of marks including some very high ones. On all three questions there was at least one candidate that achieved full marks, but none that did overall.

**Question 1** on LU and QR factorisation was attampted by many candidates. Several were unable to carry out a simple example of Gauss Elimination with partial pivoting in practice even though they were able to explain correctly what partial pivoting was. It was dissappointing to see several candidates plough on with simple GE without pivoting even though it was explicitly stated in the question that such an approach would secure no credit! QR factorisation of a square matrix using Given rotations also challenged some candidates with many not being careful enough to note if there was any overwriting of created zero entries in their suggested procedure. In the final part, there were many different long-winded proofs of the requested result, but few simply noted that the result dropped out because of orthogonality of $PQ$.

**Question 2** on best approximation in inner product spaces and orthogonal polynomials was attempted by the vast majority of candidates with most being successful in the calculation of a specific best approximation and several producing reasonable arguments in the last parts on orthogonal polynomials.

**Question 3** on ODE IVP's had the fewest attempts but attracted the highest marks. Some failed to notice that the resulting recurrence relation in part (d) could be simply solved by elementary methods seen in the first year.

**A8: Probability**

*See Mathematics and Statistics report.*

**A9: Statistics**

*See Mathematics and Statistics report.*

**A10: Fluids and Waves**

**Question 1**. Part (a) of this question was well done in general. A few candidates derived the Conservation of Mass equation using the Lagrangian approach, but did not state that (in the absence of sources and sinks) the time rate of change of the mass of a material volume of fluid that is advected with the flow is zero, instead making an incorrect argument involving fluxes across a boundary. In (a)(iii) many candidates did not correctly show that when the flow is incompressible, the Conservation of Mass equation gives that the density of fluid is preserved following the flow. Regrettably, part (b) of this question had a typo and the half domain $\mathrm{Re}(z) \geq 0$ should have read $\mathrm{Im}(z) \geq 0$. The question did self correct in (b)(ii) where the position of the wall was stated to be at $y = 0$. Adjustments were made to ensure all candidates were treated fairly. Part (b)(i) was well done with the majority of candidates correctly considering the image vortex. A very small number of candidates got confused with the required boundary condition stating that the velocity should be zero on the boundary, instead of the condition of zero normal velocity on the boundary. In (b)(ii) some candidates did not employ Bernoulli's theorem for steady irrotational flow, instead using the unsteady version. A proportion of candidates incorrectly plotted the pressure profile on the wall, instead plotting the negative of the profile. In (b)(iii) only a few candidates determined the correct value of $x*$.

**Question 2**. Part (a) was well done in general. Part (b)(i) was well done. In b(ii) a small number of candidates did not include the necessary image potentials, and hence were unable to verify the appropriate boundary condition on the do- main was satisfied. In c(i) a significant proportion of candidates struggled to determine the complex potential in the required limit, incorrectly computing limiting expressions as $a \to 0$.

**Question 3**. This was very well done in general. Some candidates did not pose the correct separable solution for $\phi(x, y, t)$. Not all candidates gave a correct physical interpretation of the condition in part (c).

**A11: Quantum Theory**

**Question 1** is on a quantum particle in a box. Part (a) is all bookwork, and was generally answered very well. In part (b) many candidates managed to correctly compute the formula for the coefficients $c_n$ (with occasional computational slips), although a common error was getting the range of integration wrong, and many candidates didn't notice that $n = 3$ needs to be treated separately. Very few candidates noticed in part (b)(ii) that the expected value of the energy is the same as it was in the original ground state, although many explained why it is independent of time $t$. Candidates who got close to the correct formula for $c_n$ in part (b) also typically got close to the final result in part (c).

**Question 2** is on a three-dimensional quantum harmonic oscillator, investigating the Fradkin tensor $F_{ij}$ using raising and lowering operators. Although strictly speaking none of this question is bookwork, it is both similar to the one-dimensional harmonic oscillator in lectures, and similar to a question in the 2021 exam. Parts (a) and (b) were generally very well answered, with most errors made in computing the commutator in (b)(ii) (which is zero): a common problem was using the free indices $i$ and $j$ also as summation indices, and then mixing up which is which, or getting signs of the commutators wrong. Part (c) regrettably has a typo (with many apologies from the assessor): $a_i\psi = 0$ should have read $a_i^-\psi = 0$. However, almost all candidates either corrected this in their answer, or read it as the correct latter statement. Many candidates made good progress through part (c), and despite the typo this question had the highest average mark.

**Question 3** is on the first excited state wave functions for the hydrogen atom, and was attempted by only a handful of candidates. Although the wave functions should be familiar from lectures, the way they are approached in part (b) perhaps looked too unfamiliar. One can answer part (c) independently of the rest of the question (apart from knowing the value of $\kappa = 1/2a$), although hardly any candidates attempted it.

### Short Options

### ASO: Q1. Number Theory

Part (a) was well-answered by many candidates. Part (b) was also well-answered by a good many candidates, although many candidates made the mistake of assuming that if $p-1$ does not divide $k$ then $g^k, g^{2k}, \dots, g^{(p-1)k}$ is a permutation of $1, 2, \dots, p-1$.

One or two candidates found the following nice variant solution: $g, 2g, \cdots, (p-1)g$ is a permutation of $1, 2, \dots, p-1$ modulo $p$, so

$$g^k(1^k + 2^k + \cdots + (p-1)^k) \equiv 1^k + 2^k + \cdots + (p-1)^k.$$

Since $g$ is a primitive root, if $p-1 \nmid k$ then $g^k \neq 1 \mod p$ and so $1^k + 2^k + \cdots + (p-1)^k \equiv 0$.

Almost no candidates made significant progress with (c) and (d), which is slightly surprising since these do not appear, on the face of it, to be particularly tricky and are certainly well within the syllabus for the course. Indeed for (d) there is a shorter solution than the official one which is just to count solutions to $x_1^2 + 1 = x_2^2$ ($p-1$ solutions) and then adjust to remove $x_1 = 0$ and $x_1^2 \equiv -1$ (if there is a solution to this), then observe that this counts each $y$ for which both $y$ and $y+1$ are quadratic residues four times.

### ASO: Q2. Group Theory

This question was about solvability and related conditions (in particular nilpotence) and various kinds of series (derived, lower central) associated to a group. The question works through these concepts for the dihedral group.

The material on the structure of the dihedral group in Part (a) was generally done well, but many candidates were too sketchy in their discussion of what was meant by defining a group via generators and relations. Several candidates omitted the special case $n = 2$ in their discussion of the centre.

Part (b) was generally also well done, with most candidates managing the commutator calculations and finding the derived subgroup and hence the derived series, which reaches $\{1\}$

after two steps.

Part (c) introduced the new concepts of lower central series, and (implicitly) the nilpotence condition. Many candidates got confused with the calculation at this point, but quite a few got the idea that repeated commutation with $G = D_{2n}$ kept on squaring the generator, so that the series only reached $\{1\}$ when the order of the group was a power of 2.

A reasonable number of candidates got the basic idea for the last part (choosing a nonabelian simple group) though some could have given a bit more detail.

**ASO: Q3. Projective Geometry**

The first part of the question was generally well done, either using coordinates or using duality which provides an elegant solution. The second part of the question definitely requires a coordinate-based approach; this part proved more problematic, with students making calculational errors which made interpretation of results difficult.

**ASO: Q4. Multidimensional Analysis and Geometry**

Relatively few candidates attempted this question, but most of those who did scored well on it. In part (a) candidates usually saw how property C implies differentiability, but the converse proved more challenging (likely for a similar reason the same statement in an arbitrary normed vector space is harder the condition on Mx does not determine it uniquely, so one has to out how to produce a choice of Mx which varies continuously). In part (b) many candidates gave a proof which did not really use part (a). In part (c) most attempts managed to show that $S$ is a submanifold of $\mathbb{R}^3$, but fewer managed to show that $T$ is a submanifold of $\mathbb{R}^6$.

**ASO: Q5. Integral Transforms**

This was quite a straightforward question which had many good answers. The primary source of lost marks was calculation errors, especially in calculating partial fractions (i.e. residues in this instance): the 'cover-up rule' would have saved a lot of trouble!

**ASO: Q6. Calculus of Variations**

Overall the question seemed to work well. There was a good spread of marks and so it seemed to distinguish the stronger candidates from the weaker ones. It was very pleasing that almost all candidates seemed to have appreicated the fundamental idea of the course, even if they sometimes struggled with the execution. Some candidates struggled to solve the differential equation, despite essentially the same one appearing in lectures, and only a small minority correctly answered the more conceptual final part of the question.

**ASO: Q7. Graph Theory**

Part (a) was on minimum cost spanning trees and Kruskal's algorithm. It was standard material that was done very well. Part (b) was on matchings that are maximal, in the sense that they are not a subset of a larger matching. Generally, this was done well, although few students could give the correct answer to (b)(iv). Part (c) was concerned with matchings and spanning trees. Most students were able to prove that a tree has at most one perfect matching. Many also spotted that any matching in a connected graph can be extended to a spanning tree. Quite a few also realised that Kruskal's algorithm could be used to prove this, although there are of course proofs that do not use this algorithm. The final subpart (c)(iv) was challenging, although there were a few correct solutions to this.

**ASO: Q8. Special Relativity**

The problem examines the basics of special relativity: notion of the proper time, basic Minkowski geometry and the basic dynamics in SR. The first and the second parts are either quite standard or straightforward bookwork. The result is not satisfactory. The dynamics problem is not meant to be challenging, part of which could be solved without resorting to SR. However, what surprised me is that students are not familiar with the concepts of 4-acceleration and 4-velocity, all of which are the key ingredients of the course.

**ASO: Q9. Modelling in Mathematical Biology**

(a)(i) A surprising number of candidates did not state the Law of Mass Action correctly, yet virtually all candidates used it correctly to derive the system of ordinary differential equations. (ii) To show that the total number of enzyme was conserved, candidates need to first exaplin why the total amount of enzyme was $e(t) + s(t)$ and then show from the system of ODEs derived in (i) that $\frac{d(e+s)}{dt} = 0$, hence $e(t) + s(t) = constant$. However, most candidates simply wrote down $\frac{d(e+s)}{dt} = 0$ without explaining why or how it relates to total enzyme concentration.

(b)(i) A number of candidates did not realise that a combination of the equations for the QSSA and conservation of enzyme needed to be used to eliminate $e(t)$ so that $c(t)$ can be calculated explicitly.

(ii), (iii) were bookwork and done reasonably well, although a number of candidates quoted the velocity for the product, whilst the question asked for the velocity of the reaction $(*)$. I accepted either.

Part (iv) was done very well.

(c) This was a challenging question with some subtle aspects to it, but some candidates made good attempts. In (i), some candidates got derailed by confusing the chain rule for differentiation with the product rule.

For (ii), many candidates saw that $W(0)$ satisfies $W(0)\exp(W(0)) = 0$ and, from this, concluded that $W(0) = 0$. However, there is another solution to this equation, namely $W(0) = -\infty$, and only one or two candidates picked this up. The assumption given (that $s(t)$ is non-negative) means that we can ignore this solution.

## E. Names of members of the Board of Examiners

- **Examiners:**
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  Prof Jan Obloj
  Prof. Konstantin Ardakov
  Prof. Derek Moulton
  Prof Neil Strickland (External Examiner)
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- **Assessors:**
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  Prof. Andrew Dancer
  Prof. Ben Green
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