Part B Mathematics 2024-25

November 18, 2024

Contents

1 Foreword

The confirmed synopses for Part B 2024-25 will be available on the course management portal `https://courses.maths.ox.ac.uk/` before the start of Michaelmas Term 2024.

Honour School of Mathematics - Units

See the current edition of the *Examination Regulations* (`https://examregs.admin.ox.ac.uk/` for the full regulations governing these examinations. Examination Conventions can be found at: `https://www.maths.ox.ac.uk/members/students/undergraduate-courses/examinations-assessments/examination-conventions`.

In Part B each candidate shall offer a total of eight units from the schedule of units. Each unit is the equivalent of a sixteen-hour lecture course.

(a) A total of at least four units offered should be from the schedule of 'Mathematics Department units'.

(b) Candidates may offer up to four units from the schedule of 'Other Units' but with no more than two from each category (Statistics options, Computer Science options, Other options).

(c) Candidates may offer at most one double unit which is designated as an extended essay or a structured project, with the exception of BO1.1 History of Mathematics, which additionally includes an exam. (Units which may be offered under this heading are indicated in the synopses.)

All Mathematics Department lecture courses are independently available as units.

Mathematics Department Units

- B1.1 Logic (MT)
- B1.2 Set Theory (HT)
- B2.1 Introduction to Representation Theory (MT)
- B2.2 Commutative Algebra (HT)
- B2.3 Lie Algebras (HT)
- B3.1 Galois Theory (HT)
- B3.2 Geometry of Surfaces (MT)
- B3.3 Algebraic Curves (HT)
- B3.4 Algebraic Number Theory (HT)
- B3.5 Topology and Groups (MT)
- B4.1 Functional Analysis I (MT)
- B4.2 Functional Analysis II (HT)
- B4.3 Distribution Theory (HT)

- B4.4 Fourier Analysis (HT)
- B5.1 Stochastic Modelling of Biological Processes (HT)
- B5.2 Applied Partial Differential Equations (MT)
- B5.3 Viscous Flow (MT)
- B5.4 Waves and Compressible Flow (HT)
- B5.5 Further Mathematical Biology (MT)
- B5.6 Nonlinear Dynamics, Bifurcations and Chaos (HT)
- B6.1 Numerical Solution of Partial Differential Equations (MT)
- B6.2 Optimisation for Data Science (HT)
- B6.3 Integer Programming (MT)
- B7.1 Classical Mechanics (MT)
- B7.2 Electromagnetism (HT)
- B7.3 Further Quantum Theory (HT)
- B8.1 Probability, Measure and Martingales (MT)
- B8.2 Continuous Martingales and Stochastic Calculus (HT)
- B8.3 Mathematical Models of Financial Derivatives (HT)
- B8.4 Information Theory (MT)
- B8.5 Graph Theory (MT)
- B8.6 High Dimensional Probability
- SB3.1 Applied Probability (HT)
- BSP Structured Projects (MT/HT) [double unit]

Other Units

1. **Statistics Options**

   Students in Part B may take the units listed below, which are drawn from Part B of the Honour School of Mathematics and Statistics.

   The Statistics units available are as follows:

   - SB1 Applied and Computational Statistics [double unit] (MT/HT)
   - SB2.1 Foundations of Statistical Inference [unit] (MT)
   - SB2.2 Statistical Machine Learning [unit] (HT)

For full details of these units see the syllabus and synopses for Part B of the Honour School Mathematics and Statistics, which are available on the web at `https://www.stats.ox.ac.uk/bammath-mathematics-and-statistics-student-resources`.

2. **Computer Science Options**

   Students in Part B may take the units listed below, which are drawn from Part B of the Honour School of Mathematics and Computing.

   The Computer Science units available are as follows:

   - Lambda Calculus and Types [unit] (HT)
   - Computational Complexity [unit] (HT)
   - Combinatorial Optimisation (MT)

   For full details of these units see the Department of Computer Science's website `http://www.cs.ox.ac.uk/teaching/courses/`.

3. **Other Options**

   - BO1.1 History of Mathematics [double unit] (MT/HT)
   - BOE: "Other Mathematical" Extended Essay [double unit] (MT/HT)
   - Philosophy options - see below for details (double units)

   Students in Part B may take the units listed below, which are drawn from Part B of the Honour School of Mathematics and Philosophy. Students interested in taking a Philosophy double unit are encouraged to contact their college tutors well in advance of term, to ensure that teaching arrangements can be made.

   The Philosophy units available are as follows:

   - 101 Early Modern Philosophy [double unit] (MT/HT)
   - 102 Knowledge and Reality [double unit] (MT/HT)
   - 122 Philosophy of Mathematics [double unit] (MT)
   - 127 Philosophical Logic [double unit] (HT)

   [Paper 101, 102, 122 and 127 in all Honour Schools including Philosophy. Teaching responsibility of the Faculty of Philosophy.]

   For full details of these units see the Philosophy Faculty website `http://www.philosophy.ox.ac.uk/`.

Registration for Part B courses 2024-25

Students will be asked to register for the options they intend to take by the end of week 11, Trinity Term 2024. It is helpful if their registration is as accurate as possible as the data is used to make teaching resource arrangements. Towards the start of the academic year students will be given the opportunity to make edits to their course registration. Students will then be asked to sign up for classes via the Course Materials site, otherwise known as Moodle (`https://courses.maths.ox.ac.uk/`) at the start of Michaelmas Term 2024. Further information about this will be sent via email before the start of term.

Students who register for a course or courses for which there is a quota should consider registering for an additional course (by way of a ”reserve choice”) in case they do not receive a place on the course with the quota. They may also have to give the reasons why they wish to take a course which has a quota and provide the name of a tutor who can provide a supporting statement for them should the quota be exceeded. Where this is necessary students will be contacted by email after they have registered. In the event that the quota for a course is exceeded, the Mathematics Teaching Committee will decide who may have a place on the course on the basis of the supporting statements from the student and tutor, and all relevant students will be notified of the decision by email.

Every effort will be made when timetabling lectures to ensure that mathematics lectures do not clash. However, because of the large number of options, this may sometimes be unavoidable. The timing of lectures for a course taught by another faculty will usually be set by that faculty and the Mathematical Institute has little control over the arrangements. In the event of clashes being necessary, then students will be notified of the clashes by email and in any case, options will only be allowed to clash when the take-up of both options is unlikely or inadvisable.

Pathways

By and large, Part B options lead naturally to one or more related Part C options in the fourth year. The most obvious exceptions to this are the options

- BSP Structured Projects
- BO1.1 History of Mathematics

Any student taking any of these options should realize that this may limit or make difficult a choice of options in the fourth year. The only current means of continuing with the history of mathematics in the fourth year is via a dissertation.

2 B1.1 Logic

2.1 General Prerequisites

There are no formal prerequisites, but familiarity with basic mathematical objects and notions will be helpful at some points in the course, in particular with some examples. In particular, such objects/notions as: the rational, real and complex number fields; sets; the idea of surjective, injective and bijective functions; order relations; the definitions of basic abstract mathematical structures such as groups and fields (all covered in Mathematics I and II in Prelims).

2.2 Overview

To give a rigorous mathematical treatment of the fundamental ideas and results of logic that is suitable for the non-specialist mathematicians and will provide a sound basis for more advanced study. Cohesion is achieved by focusing on the Completeness Theorems and the relationship between provability and truth. Consideration of some implications of the Compactness Theorem gives a flavour of the further development of model theory. To give a concrete deductive system for predicate calculus and prove the Completeness Theorem, including easy applications in basic model theory.

2.3 Learning Outcomes

Students will be able to use the formal language of propositional and predicate calculus and be familiar with their deductive systems and related theorems. For example, they will know and be able to use the soundness, completeness and compactness theorems for deductive systems for predicate calculus.

2.4 Synopsis

The notation, meaning and use of propositional and predicate calculus. The formal language of propositional calculus: truth functions; conjunctive and disjunctive normal form; tautologies and logical consequence. The formal language of predicate calculus: satisfaction, truth, validity, logical consequence.

Deductive system for propositional calculus: proofs and theorems, proofs from hypotheses, the Deduction Theorem; Soundness Theorem. Maximal consistent sets of formulae; completeness; proof of completeness.

Statement of Soundness and Completeness Theorems for a deductive system for predicate calculus; derivation of the Compactness Theorem; simple applications of the Compactness Theorem.

A deductive system for predicate calculus; proofs and theorems; prenex form. Proof of Completeness Theorem. Existence of countable models, the downward Löwenheim-Skolem Theorem.

2.5 Reading List

1. R. Cori and D. Lascar, *Mathematical Logic: A Course with Exercises (Part I)* (Oxford University Press, 2001), sections 1, 3, 4.
2. A. G. Hamilton, *Logic for Mathematicians* (2nd edition, Cambridge University Press, 1988), pp.1-69, pp.73-76 (for statement of Completeness (Adequacy) Theorem), pp.99-103 (for the Compactness Theorem).
3. W. B. Enderton, *A Mathematical Introduction to Logic* (Academic Press, 1972), pp.101-144.
4. D. Goldrei, *Propositional and Predicate Calculus: A model of argument* (Springer, 2005).
5. A. Prestel and C. N. Delzell, *Mathematical Logic and Model Theory* (Springer, 2010).

2.6 Further Reading

1. R. Cori and D. Lascar, *Mathematical Logic: A Course with Exercises (Part II)* (Oxford University Press, 2001), section 8.
2. M. Hils and F. Loeser, *A First Journey through Logic* (American Mathematical Society, 2019), Chapter 2.
3. D. Marker, *An Invitation to Mathematical Logic* (Springer, 2024), Chapters 1-5.

3 B1.2 Set Theory

3.1 General Prerequisites

There are no formal prerequisites, but familiarity with some basic mathematical objects and notions such as: the rational and real number fields; the idea of surjective, injective and bijective functions, inverse functions, order relations; the notion of a continuous function of a real variable, sequences, series, and convergence, and the definitions of basic abstract structures such as fields, vector spaces, and groups (all covered in Mathematics I and II in Prelims) will be helpful at points.

3.2 Overview

Introduce sets and their properties as a unified way of treating mathematical structures. Emphasise the difference between an intuitive collection and a formal set. Define (infinite) cardinal and ordinal numbers and investigate their properties. Frame the Axiom of Choice and its equivalent forms and study their implications.

3.3 Learning Outcomes

Students will have a sound knowledge of set theoretic language and be able to use it to codify mathematical objects. They will have an appreciation of the notion of infinity and arithmetic of the cardinals and ordinals. They will have developed a deep understanding of the Axiom of Choice, Zorn's Lemma and the Well-Ordering Principle.

3.4 Synopsis

What is a set? Introduction to the basic axioms of set theory. Ordered pairs, cartesian products, relations and functions. Axiom of Infinity and the construction of the natural numbers; induction and the Recursion Theorem.

Cardinality; the notions of finite and countable and uncountable sets; Cantor's Theorem on power sets. The Tarski Fixed Point Theorem. The Schröder-Bernstein Theorem. Basic cardinal arithmetic.

Well-orders. Comparability of well-orders. Ordinal numbers. Transfinite induction; transfinite recursion [informal treatment only]. Ordinal arithmetic.

The Axiom of Choice, Zorn's Lemma, the Well-ordering Principle; comparability of cardinals. Equivalence of WO, CC, AC and ZL. Cardinal numbers.

3.5 Reading List

1. D. Goldrei, *Classic Set Theory* (Chapman and Hall, 1996).
2. H. B. Enderton, *Elements of Set Theory* (Academic Press, 1978).

3.6 Further Reading

1. R. Cori and D. Lascar, *Mathematical Logic: A Course with Exercises (Part II)* (Oxford University Press, 2001), section 7.1-7.5.
2. M. Hils and F. Loeser, *A First Journey through Logic* (American Mathematical Society, 2019), Chapters 1 and 6.
3. R. Rucker, *Infinity and the Mind: The Science and Philosophy of the Infinite* (Princeton University Press, 1995). An accessible introduction to set theory.
4. J. W. Dauben, *Georg Cantor: His Mathematics and Philosophy of the Infinite* (Princeton University Press, 1990). For some background, you may find JW Dauben's biography of Cantor interesting.
5. M. D. Potter, *Set Theory and its Philosophy: A Critical Introduction* (Oxford University Press, 2004). An interestingly different way of establishing Set Theory, together with some discussion of the history and philosophy of the subject.
6. W. Sierpinski, *Cardinal and Ordinal Numbers* (Polish Scientific Publishers, 1965). More about the arithmetic of transfinite numbers.
7. J. Stillwell, *Roads to Infinity* (CRC Press, 2010).

4 B2.1 Introduction to Representation Theory

4.1 General Prerequisites

Rings and Modules is essential. Group Theory is recommended.

4.2 Overview

This course gives an introduction to the representation theory of finite groups. Representation theory is a fundamental tool for studying symmetry by means of linear algebra: it is studied in a way in which a given group may act on vector spaces, giving rise to the notion of a representation.

The first part of the course will deal with the structure theory of semisimple algebras and their modules (representations). We will prove that any finite-dimensional semisimple algebra is isomorphic to a product of matrix rings (Wedderburn's Theorem).

In later parts of the course we apply the developed material to group algebras, and classify when group algebras are semisimple (Maschke's Theorem). All of this material will be applied to the study of characters and representations of finite groups.

4.3 Learning Outcomes

They will know in particular simple modules and semisimple algebras and they will be familiar with examples. They will appreciate important results in the course such as Schur's Lemma, Wedderburn's Theorem, the Row Orthogonality Theorem and Burnside's $p^\alpha\ q^\beta$ Theorem. They will be familiar with the classification of semisimple algebras over $\mathbb{C}$ and be able to apply this to representations and characters of finite groups.

4.4 Synopsis

Representations of groups. Maschke's Theorem. The group ring.

Modules and their relationship with representations. Semisimple algebras, Schur's Lemma and the Wedderburn Theorem. Characters of complex representations. Orthogonality relations, finding character tables. Tensor product of representations. Induction and restriction of representations. Application: Burnside's $p^\alpha\ q^\beta$ Theorem.

4.5 Reading List

1. G. D. James and M. Liebeck, *Representations and Characters of Finite Groups* (2nd edition, Cambridge University Press, 2001).

2. J.-P. Serre, *Linear Representations of Finite Groups*, Graduate Texts in Mathematics 42 (Springer-Verlag, 1977).

4.6 Further Reading

1. J. L. Alperin and R. B. Bell, *Groups and Representations*, Graduate Texts in Mathematics 162 (Springer-Verlag, 1995).
2. P. M. Cohn, *Classic Algebra* (Wiley & Sons, 2000). (Several books by this author available.)
3. C. W. Curtis, and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras* (Wiley & Sons, 1962).
4. L. Dornhoff, *Group Representation Theory* (Marcel Dekker Inc., New York, 1972).
5. I. M. Isaacs, *Character Theory of Finite Groups* (AMS Chelsea Publishing, American Mathematical Society, Providence, Rhode Island, 2006).
6. P. Etingof, *Introduction to representation theory* (Online course notes, MIT 2011).
7. K. Erdmann, T. Holm *Algebras and Representation Theory*, Springer Undergraduate Mathematical Series (2018), ISSN 1615-2085

5 B2.2 Commutative Algebra

5.1 General Prerequisites

Rings and Modules is essential. Galois Theory is strongly recommended.

5.2 Overview

Amongst the most familiar objects in mathematics are the ring of integers and the polynomial rings over fields. These play a fundamental role in number theory and in algebraic geometry, respectively. The course explores the basic properties of such rings.

5.3 Synopsis

Modules, ideals, prime ideals, maximal ideals. Noetherian rings; Hilbert basis theorem. Minimal primes. Localization. Polynomial rings and algebraic sets. Weak Nullstellensatz. Nilradical and Jacobson radical; strong Nullstellensatz. Integral extensions. Prime ideals in integral extensions. Noether Normalization Lemma. Krull dimension; dimension of an affine algebra. Noetherian rings of small dimension, Dedekind domains.

5.4 Reading List

1. M. F. Atiyah and I. G. MacDonald: Introduction to Commutative Algebra, (Westview Press, 1994). 2. D. Eisenbud: Commutative Algebra with a view towards Algebraic Geometry, (Springer GTM, 2004).

6 B2.3 Lie Algebras

6.1 General Prerequisites

A thorough understanding of the material of Part A Linear Algebra, and of basic notions of abstract algebra such as group actions, rings, ideals, quotients *etc.*.

The Michaelmas term Part B course "Introduction to Representation Theory" is recommended. Although the results of that course are not largely not logically needed for the Lie algebras course, the representation theory of Lie algebras mirrors that of groups, and so some familiarity with the concepts of the Michaelmas course is very helpful. Some notions are also introduced in the Part A Group Theory course.

6.2 Overview

Lie Algebras are mathematical objects which, besides being of interest in their own right, elucidate problems in several areas in mathematics. The classification of the finite-dimensional complex Lie algebras is a beautiful piece of applied linear algebra. The aims of this course are to introduce Lie algebras, develop some of the techniques for studying them, and outline the techniques and structures that go into this classification theorem, which shows that semisimple Lie algebras are encoded in finite sets of highly symmetric vectors in a Euclidean vector space known as root systems, which in turn are classified by a kind of graph known as a Dynkin diagram.

6.3 Learning Outcomes

By the end of the course, students will be able to identify the basic classes of Lie algebras - nilpotent, solvable and semisimple, give examples of each, and appreciate the role they play in understanding the structure of Lie algebras. They should be able to use basic notions such as ideals and representations to analyse the structure of Lie algebras, and employ the Cartan criteria. They should also be able to analyse concrete examples of semisimple Lie algebras and identify the associated Dynkin diagram.

Although not all of the key theorems are proved in this course, should they need to, a student who has internalised the techniques used in these lectures should not have too much difficulty filling in these gaps using any of the standard textbooks on the subject.

6.4 Synopsis

- Definition of Lie algebras, small-dimensional examples, some classical groups and their Lie algebras (treated informally). Ideals, subalgebras, homomorphisms, isomorphism theorems.

- Basics of representation theory: $\mathfrak{g}$-modules (or equivalently $\mathfrak{g}$-representations). Irreducible and indecomposable representations, semisimplicity. Composition series and the Jordan-Hölder theorem for modules. Operations on representations: subrepresentations, quotients, Homs and tensor products.

- Composition series for Lie algebras. Short exact sequences and the notion of an extension. Split sequences and semi-direct products. Definition of solvable and nilpotent Lie algebras. A representation in which every element of the Lie algebra acts nilpotently has the trivial representation as its only composition factor. Engel's theorem.

- Representations of solvable Lie algebras over an algebraically closed field of characteristic zero including Lie's theorem. Decomposition of representations of nilpotent Lie algebras into generalised weight spaces.

- Cartan subalgebras and the Cartan decomposition. Trace forms and Cartan's criterion for solvability. The solvable radical and Cartan's criterion for semisimplicity. Semisimple and simple Lie algebras. The Jordan decomposition. The representations of $\mathfrak{sl}_2$ (*to be done through problem set questions – only the classification of irreducibles is needed for use elsewhere in the course*).

- The Cartan decomposition of a semisimple Lie algebra and the structure of the root system using the representation theory of $\mathfrak{sl}_2$. The abstract root system attached to a semisimple Lie algebra and the reduction of the classification of abstract root systems to the classification of Cartan matrices/Dynkin diagrams. Statement of the classification theorem and informal discussion of the proof of the classification of semisimple Lie algebras.

6.5 Reading List

- *Introduction to Lie algebras*, K. Erdmann, M. Wildon, Springer Undergraduate Mathematics Series. (Available online through the Bodleian at `https://link.springer.com/book/10.1007/1-84628-490-2`.)

- *Introduction to Lie Groups and Lie algebras*, A. Kirillov, Jr. Cambridge Studies in Advanced Mathematics, C.U.P. Available online at `https://www.cambridge.org/core/books/an-introduction-to-lie-groups-and-lie-algebras/98E68056F3EE57686421863E2B0B5DF4`

- *Lie algebras: Theory and algorithms*, Willem A. de Graff, North-Holland Mathematical Library. Available online at `https://www.sciencedirect.com/bookseries/north-holland-mathematical-library/vol/56/suppl/C`

- *Lie algebras of finite and affine type*, R. Carter, Cambridge Studies in Advanced Mathematics, C.U.P. Available online at `https://www.cambridge.org/core/books/lie-algebras-of-finite-and-affine-type/4E6820728C16DC1F812860C974FBB4F6`

- *Lie Groups, Lie Algebras, and Representations*, Brian C. Hall, Graduate Texts in Mathematics, Springer. Available online at `https://link.springer.com/book/10.1007/978-3-319-13467-3`

- *Representation theory: A First Course*, W. Fulton, J. Harris, Graduate Texts in Mathematics, Springer. Available online at `https://www.vlebooks.com/Account/Logon/?returnurl=https%3a%2f%2fwww.vlebooks.com%2fProduct%2fIndex%2f1922982%3fpage%3d0%26startBookmarkId%3d-1`

7 B3.1 Galois Theory

7.1 General Prerequisites

Rings and Modules is essential and Group Theory is recommended.

7.2 Overview

The course starts with a discussion of the classical problem of solving polynomial equations by radicals. This is followed by the classical theory of Galois field extensions, culminating in some of the classical theorems in the subject: the insolubility of the general quintic equation, the classification of finite fields and the irreducibility of the cyclotomic polynomials over the rational numbers.

7.3 Learning Outcomes

Understanding of the relation between symmetries of roots of a polynomial and its solubility in terms of simple algebraic formulae; working knowledge of interesting group actions in a nontrivial context; working knowledge, with applications, of a nontrivial notion of finite group theory (soluble groups); understanding of the relation between algebraic properties of field extensions.

7.4 Synopsis

Solvability of cubic and quartic equations by radicals. Review of algebraic extensions, the Tower Law, Gauss' Lemma and Eisenstein's criterion. Review of groups acting on sets; upper bounds on the size of the Galois group; the theorem of the Primitive Element. Splitting fields and separable extensions. Characterisation of Galois extensions. The fundamental theorem of Galois theory; explicit examples. Solvability by radicals. Normal extensions. Kummer extensions. Techniques for calculating Galois groups: insolubility of certain quintics. Finite fields, the Frobenius automorphism and classification of finite fields. Cyclotomic extensions and the irreducibility of cyclotomic polynomials over the rationals.

7.5 Reading List

1. J. Rotman, *Galois Theory* (Springer-Verlag, NY Inc, 2001/1990).

2. I. Stewart, *Galois Theory* (Chapman and Hall, 2003/1989).

3. D.J.H. Garling, *A Course in Galois Theory* (Cambridge University Press I.N., 1987).

4. Herstein, *Topics in Algebra* (Wiley, 1975).

8 B3.2 Geometry of Surfaces

8.1 General Prerequisites

Part A Topology is recommended. Multidimensional Analysis and Geometry would be useful but not essential. Also, B3.2 is helpful, but not essential, for B3.3 (Algebraic Curves).

8.2 Overview

Different ways of thinking about surfaces (also called two-dimensional manifolds) are introduced in this course: first topological surfaces and then surfaces with extra structures which allow us to make sense of differentiable functions ('smooth surfaces'), holomorphic functions ('Riemann surfaces') and the measurement of lengths and areas ('geometric surfaces' or 'Riemannian 2-manifolds').

These geometric structures interact in a fundamental way with the topology of the surfaces. A striking example of this is given by the Euler number, which is a manifestly topological quantity, but can be related to the total curvature, which at first glance depends on the geometry of the surface.

The course ends with an introduction to hyperbolic surfaces modelled on the hyperbolic plane, which gives us an example of a non-Euclidean geometry (that is, a geometry which meets all of Euclid's axioms except the axiom of parallels).

8.3 Learning Outcomes

Students will be able to implement the classification of surfaces for simple constructions of topological surfaces such as planar models and connected sums; be able to relate the Euler characteristic to branching data for simple maps of Riemann surfaces; be able to describe the definition and use of Gaussian curvature; know the geodesics and isometries of the hyperbolic plane and their use in geometrical constructions.

8.4 Synopsis

The concept of a topological surface (or 2-manifold); examples, including polygons with pairs of sides identified. Orientability and the Euler characteristic. Classification theorem for compact surfaces (the proof will not be examined).

Smooth surfaces in Euclidean three-space. Tangent space. Abstract topological and smooth surfaces. The fundamental forms. The concept of a Riemannian 2-manifold; isometries; Gaussian curvature and the Theorema Egregium.

Geodesics. The Gauss-Bonnet Theorem (statement of local version and deduction of global version). Critical points of real-valued functions on compact surfaces. The Poincaré-Hopf Theorem. Morse functions and the gradient field.

The hyperbolic plane, its isometries and geodesics. Compact hyperbolic surfaces.

Riemann surfaces; examples, including the Riemann sphere, the quotient of the complex numbers by a lattice, and double coverings of the Riemann sphere. Holomorphic maps of

Riemann surfaces and the Riemann-Hurwitz formula. Elliptic functions.

8.5 Reading List

1. A. Pressley, *Elementary Differential Geometry*, Springer Undergraduate Mathematics Series (Springer-Verlag, 2001). (Chapters 4-8 and 10-11.)
2. G. B. Segal, *Geometry of Surfaces*, Mathematical Institute Notes (1989).
3. M. P. do Carmo, *Differential Geometry of Curves and Surfaces* (Dover, 2016)
4. F. Kirwan, *Complex Algebraic Curves*, Student Texts 23 (London Mathematical Society, Cambridge, 1992) (Chapter 5.2 only).

8.6 Further Reading

1. P. A. Firby and C. E. Gardiner, *Surface Topology* (Ellis Horwood, 1991) (Chapters 1-4 and 7).
2. J. McCleary, *Geometry from a Differentiable Viewpoint*, (Cambridge, 1997).
3. B. O'Neill, *Elementary Differential Geometry* (Academic Press, 1997).

9 B3.3 Algebraic Curves

9.1 General Prerequisites

Part A Topology. Multidimensional Analysis and Geometry would be useful but not essential. Projective Geometry is recommended. Also, B3.2 (Geometry of Surfaces) is helpful, but not essential.

9.2 Overview

A real algebraic curve is a subset of the plane defined by a polynomial equation $p(x, y) = 0$. The intersection properties of a pair of curves are much better behaved if we extend this picture in two ways: the first is to use polynomials with complex coefficients, the second to extend the curve into the projective plane. In this course projective algebraic curves are studied, using ideas from algebra, from the geometry of surfaces and from complex analysis.

9.3 Learning Outcomes

Students will know the concepts of projective space and curves in the projective plane. They will appreciate the notion of nonsingularity and know some basic features of intersection theory. They will view nonsingular algebraic curves as examples of Riemann surfaces, and be familiar with divisors, meromorphic functions and differentials.

9.4 Synopsis

Projective spaces, homogeneous coordinates, projective transformations.

Algebraic curves in the complex projective plane. Irreducibility, singular and nonsingular points, tangent lines.

Bezout's Theorem (the proof will not be examined). Points of inflection, and normal form of a nonsingular cubic.

Nonsingular algebraic curves as Riemann surfaces. Meromorphic functions, divisors, linear equivalence. Differentials and canonical divisors. The group law on a nonsingular cubic.

The Riemann-Roch Theorem (the proof will not be examined). The geometric genus. Applications.

9.5 Reading List

1. F. Kirwan, *Complex Algebraic Curves*, Student Texts 23 (London Mathematical Society, Cambridge, 1992), Chapters 2-6.

2. W. Fulton, *Algebraic Curves*, 3rd ed., downloadable at `http://www.math.lsa.umich.edu/~wfulton`

10 B3.4 Algebraic Number Theory

10.1 General Prerequisites

Rings and Modules and Number Theory. B3.1 Galois Theory is an essential pre-requisite. All second-year algebra and arithmetic. Students who have not taken Part A Number Theory should read about quadratic residues in, for example, the appendix to Stewart and Tall. This will help with the examples.

10.2 Overview

An introduction to algebraic number theory. The aim is to describe the properties of number fields, but particular emphasis in examples will be placed on quadratic fields, where it is easy to calculate explicitly the properties of some of the objects being considered. In such fields the familiar unique factorisation enjoyed by the integers may fail, and a key objective of the course is to introduce the class group which measures the failure of this property.

10.3 Learning Outcomes

Students will learn about the arithmetic of algebraic number fields. They will learn to prove theorems about integral bases, and about unique factorisation into ideals. They will learn to calculate class numbers, and to use the theory to solve simple Diophantine equations.

10.4 Synopsis

Field extensions, minimum polynomial, algebraic numbers, conjugates, discriminants, Gaussian integers, algebraic integers, integral basis

Examples: quadratic fields

Norm of an algebraic number

Existence of factorisation

Factorisation in $\mathbb{Q}(\sqrt{d})$

Ideals, $\mathbb{Z}$-basis, maximal ideals, prime ideals

Unique factorisation theorem of ideals

Relationship between factorisation of number and of ideals

Norm of an ideal

Ideal classes

Statement of Minkowski convex body theorem

Finiteness of class number

Computations of class number to go on example sheets

10.5 Reading List

I. Stewart and D. Tall, *Algebraic Number Theory and Fermat's Last Theorem* (Third Edition, Peters, 2002).

10.6 Further Reading

D. Marcus, *Number Fields* (Springer-Verlag, New York-Heidelberg, 1977). ISBN 0-387-90279-1.

11 B3.5 Topology and Groups

11.1 General Prerequisites

Part A Topology is essential and Group Theory is recommended.

11.2 Overview

This course introduces the important link between topology and group theory. On the one hand, associated to each space, there is a group, known as its fundamental group. This can be used to solve topological problems using algebraic methods. On the other hand, many results about groups are best proved and understood using topology. For example, presentations of groups, where the group is defined using generators and relations, have a topological interpretation. One of the highlights of the course is the Nielsen-Schreier Theorem, an important, purely algebraic result, which is proved using topological techniques.

11.3 Learning Outcomes

Students will develop a sound understanding of simplicial complexes, cell complexes and their fundamental groups. They will be able to use algebraic methods to analyse topological spaces and compute the fundamental groups of many spaces, including compact surfaces. They will also be able to address questions about groups using topological techniques.

11.4 Synopsis

Homotopic mappings, homotopy equivalence. Simplicial complexes. Simplicial approximation theorem.

The fundamental group of a space. The fundamental group of a circle. Application: the fundamental theorem of algebra. The fundamental groups of spheres.

Free groups. Existence and uniqueness of reduced representatives of group elements. The fundamental group of a graph.

Groups defined by generators and relations (with examples). Tietze transformations.

The free product of two groups. Amalgamated free products.

The Seifert-van Kampen Theorem.

Cell complexes. The fundamental group of a cell complex (with examples). The realization of any finitely presented group as the fundamental group of a finite cell complex.

Covering spaces. Liftings of paths and homotopies. A covering map induces an injection between fundamental groups. The use of covering spaces to determine fundamental groups: the circle again, and real projective n-space. The correspondence between covering spaces and subgroups of the fundamental group. Regular covering spaces and normal subgroups.

Cayley graphs of a group. The relationship between the universal cover of a cell complex, and the Cayley graph of its fundamental group. The Cayley 2-complex of a group.

The Nielsen-Schreier Theorem (every subgroup of a finitely generated free group is free) proved using covering spaces.

11.5 Reading List

1. John Stillwell, *Classical Topology and Combinatorial Group Theory* (Springer-Verlag, 1993).

11.6 Further Reading

1. D. Cohen, *Combinatorial Group Theory: A Topological Approach*, Student Texts 14 (London Mathematical Society, 1989), Chapters 1-7.

2. A. Hatcher, *Algebraic Topology* (CUP, 2001), Chapter. 1.

3. M. Hall, Jr, *The Theory of Groups* (Macmillan, 1959), Chapters. 1-7, 12, 17 .

4. D. L. Johnson, *Presentations of Groups*, Student Texts 15 (Second Edition, London Mathematical Society, Cambridge University Press, 1997). Chapters. 1-5, 10,13.

5. W. Magnus, A. Karrass, and D. Solitar, *Combinatorial Group Theory* (Dover Publications, 1976). Chapters. 1-4.

12 B4.1 Functional Analysis I

12.1 General Prerequisites

Part A Integration is essential; the only concepts which will be used are the convergence theorems and the theorems of Fubini and Tonelli, and the notions of measurable functions, integrable functions, null sets and L^p spaces. No knowledge is needed of outer measure, or of any particular construction of the integral, or of any proofs. A good working knowledge of Part A Core Analysis (both metric spaces and complex analysis) is expected.

12.2 Overview

The course provides an introduction to the methods of functional analysis.

It builds on core material in analysis and linear algebra studied in Part A. The focus is on normed spaces and Banach spaces; a brief introduction to Hilbert spaces is included, but a systematic study of such spaces and their special features is deferred to B4.2 Functional Analysis II. The techniques and examples studied in the Part B courses Functional Analysis I and II support, in a variety of ways, many advanced courses, in particular in analysis and partial differential equations, as well as having applications in mathematical physics and other areas.

12.3 Learning Outcomes

Students will have a firm knowledge of real and complex normed vector spaces, with their geometric and topological properties. They will be familiar with the notions of completeness, separability and density, will know the properties of a Banach space and important examples, and will be able to prove results relating to the Hahn-Banach Theorem. They will have developed an understanding of the theory of bounded linear operators on a Banach space.

12.4 Synopsis

Brief recall of material from Part A Metric Spaces and Part A Linear Algebra on real and complex normed vector spaces, their geometry and topology and simple examples of completeness. The norm associated with an inner product and its properties. Banach spaces, exemplified by ℓ^p, L^p, $C(K)$, spaces of differentiable functions. Finite-dimensional normed spaces, including equivalence of norms and completeness. Hilbert spaces as a class of Banach spaces having special properties; examples (Euclidean spaces, ℓ^2, L^2), projection theorem, Riesz Representation Theorem.

Density. Approximation of functions, Stone-Weierstrass Theorem. Separable spaces; separability of subspaces.

Bounded linear operators, examples (including integral operators). Continuous linear functionals. Dual spaces. Statement of the Hahn-Banach Theorem; applications, including density of subspaces and embedding of a normed space into its second dual. Adjoint operators.

12.5 Reading List

1. B.P. Rynne and M.A. Youngson, *Linear Functional Analysis* (Springer SUMS, 2nd edition, 2008), Chapters 2, 4, 5.
2. E. Kreyszig, *Introductory Functional Analysis with Applications* (Wiley, revised edition, 1989), Chapters 2, 4.2-4.3, 4.5, 7.1-7.4.

13 B4.2 Functional Analysis II

13.1 General Prerequisites

B4.1 Functional Analysis I is an essential pre-requisite. A4 Integration is also essential; the only concepts which will be used are the convergence theorems and the theorems of Fubini and Tonelli, and the notions of measurable functions, integrable functions, null sets and L^p spaces. No knowledge is needed of outer measure, or of any particular construction of the integral, or of any proofs. A good working knowledge of Part A Core Analysis (both metric spaces and complex analysis) is expected.

13.2 Overview

The course provides further introduction to the methods of functional analysis. It builds on core material in Part A analysis and linear algebra and in Part B B4.1 Functional Analysis I. On one hand, it delves deeper into operator theory on Banach spaces, and on the other, it gives a systematic study of Hilbert spaces, operators on Hilbert spaces and their special features. The techniques and examples studied in the course, together with that in B4.1, support, in a variety of ways, many advanced courses, in particular in analysis and partial differential equations, as well as having applications in mathematical physics and other areas.

13.3 Learning Outcomes

Students will appreciate the role of completeness through the Baire category theorem and its consequences for operators on Banach spaces. They will have a demonstrable knowledge of the properties of a Hilbert space, including orthogonal complements, orthonormal sets, complete orthonormal sets together with related identities and inequalities. They will be familiar with the theory of linear operators on a Banach or Hilbert space, including adjoint operators, compact, self-adjoint and unitary operators with their spectra. They will know the L^2-theory of Fourier series and be aware of the classical theory of Fourier series.

13.4 Synopsis

Orthogonality, orthogonal complement, closed subspaces.

Linear operators on Hilbert space, adjoint operators. Self-adjoint operators, orthogonal projections, unitary operators. Compact operators on Banach spaces.

Baire Category Theorem and its consequences for operators on Banach spaces (Uniform Boundedness, Open Mapping, Inverse Mapping and Closed Graph Theorems). Strong convergence of sequences of operators.

Weak convergence. Weak precompactness of the unit ball (proof for Hilbert spaces).

Orthonormal sets, Pythagoras, Bessel's inequality. Complete orthonormal sets, Parseval. L^2-theory of Fourier series, including completeness of the trigonometric system.

Spectral theory in Banach and Hilbert spaces, in particular spectra of self-adjoint and unitary operators. Spectral theorem for compact self-adjoint operators.

Brief contextual comments on the classical theory of Fourier series and modes of convergence; exposition of failure of pointwise convergence of Fourier series of some continuous functions.

13.5 Reading List

1. B.P. Rynne and M.A. Youngson, *Linear Functional Analysis* (Springer SUMS, 2nd edition, 2008), Chapters 3, 4.4, 6.

2. E. Kreyszig, *Introductory Functional Analysis with Applications* (Wiley, revised edition, 1989), Chapters 3, 4.7-4.9, 4.12-4.13, 9.1-9.2.

3. N. Young, *An Introduction to Hilbert Space* (Cambridge University Press, 1988), Chs 1-7.

13.6 Further Reading

1. E.M. Stein and R. Shakarchi, *Real Analysis: Measure Theory, Integration & Hilbert Spaces* (Princeton Lectures in Analysis III, 2005), Chapter 4.

2. M. Reed and B. Simon, *Methods of Modern Mathematical Physics Vol. I. Functional Analysis* (Academic Press, 1980).

3. H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations* (Universitext. Springer, 2011).

4. P. Lax, *Functional Analysis* (Wiley, 2012).

14 B4.3 Distribution Theory

14.1 General Prerequisites

Part A Integration is essential. A good working knowledge of Part A core Analysis is expected. Part A Integral Transforms is desirable but not essential.

14.2 Overview

Distribution theory can be thought of as the completion of differential calculus, just as Lebesgue integration theory can be thought of as the completion of integral calculus. It was created by Laurent Schwartz in the 20th century, as was Lebesgue's integration theory.

In this course we give an introduction to distributions. It builds on core material in analysis and integration studied in Part A. One of the main areas of applications of distributions is to the theory of partial differential equations, and a brief treatment, mainly through examples, is included.

14.3 Learning Outcomes

Students will become acquainted with the basic techniques that in many situations form the starting point for the modern treatment of PDEs.

14.4 Synopsis

Test functions and distributions on Rn: definitions and examples, Dirac δ- function, approximate identities and constructions using convolution of functions. Density of test functions in Lebesgue spaces. Smooth partitions of unity. [4 lectures]

The calculus of distributions on Rn : functions as distributions, operations on distributions, adjoint identities, consistency of derivatives, convolution of test functions and distributions. The Fundamental Theorem of Calculus for distributions. Support and singular support of a distribution. Convolution with a compactly supported distribution.

Examples of distributions defined by principal value integrals and finite parts. Examples of distributional boundary values of holomorphic functions defined in a half plane. [8 lectures]

Distributional and weak solutions of PDEs, absolutely continuous functions, Sobolev functions. Examples of fundamental solutions. Weyl's Lemma for distributions. Convolution rules for support and singular support. [4 lectures]

14.5 Reading List

The main recommended book is 1) R.S. Strichartz, A Guide to Distribution Theory and Fourier Transforms (World Scientific, 1994. Reprinted: 2008, 2015) In particular, Chapters 1, 2 and 6.

14.6 Further Reading

2) L.C. Evans, Partial Differential Equations (Amer. Math. Soc. 1998) 3) E.H. Lieb and M. Loss, Analysis (Amer. Math. Soc. 1997) 4) E.M. Stein and R. Shakarchi, Fourier analysis. An introduction (Princeton Univ. Press 2003)

15 B5.1 Stochastic Modelling of Biological Processes

15.1 General Prerequisites

Part A Probability and Part A Integral Transforms

15.2 Overview

"Stochastic Modelling of Biological Processes" provides an introduction to stochastic methods for modelling biological systems, covering a number of applications, ranging in size from simulations of small biomolecules to stochastic modelling of groups of animals. The focus is on the underlying mathematics, i.e. it is not assumed that students have taken any advanced courses in biology or chemistry.

The course discusses the essence of mathematical methods which appear (under different names) in a number of interdisciplinary scientific fields (including mathematical biology, non-equilibrium statistical physics, computational chemistry, and biophysics). New mathematical approaches and their analysis are explained using simple examples of biological models.

The course starts with stochastic (non-spatial) modelling of chemical reactions, introducing stochastic simulation algorithms and mathematical methods which can be used for analysis of stochastic models. Different stochastic spatio-temporal models are then studied, including diffusion, advection-diffusion, and reaction-diffusion models. The methods covered include Brownian dynamics, velocity jump processes and compartment-based (lattice-based) models.

15.3 Learning Outcomes

The student will learn:

(i) mathematical techniques for the analysis of stochastic models of the kind arising in biology and chemistry;

(ii) how stochastic models can be efficiently simulated using a computer;

(iii) connections and differences between different stochastic methods, and between stochastic and deterministic modelling.

15.4 Synopsis

Stochastic simulation of chemical reactions in well-stirred systems: Gillespie algorithm, chemical master equation, analysis of simple systems, deterministic vs. stochastic modelling.

Stochastic differential equations: numerical methods, Fokker-Planck equation, first exit time, backward Kolmogorov equation, chemical Langevin equation, chemical Fokker-Planck equation.

Stochastic reaction-diffusion modelling: compartment-based (lattice-based) models, reaction-diffusion master equation, Brownian dynamics, diffusion-limited reactions.

Stochastic models of dispersal in biological systems: velocity-jump processes, chemotaxis, collective animal behaviour.

15.5 Reading List

R. Erban and S. J. Chapman, *Stochastic Modelling of Reaction-Diffusion Processes* (Cambridge University Press, 2020). Available at `https://www.cambridge.org/core/books/stochastic-modelling-of-reactiondiffusion-processes/9BB8B46DE0B898FC019AFBEA95608FAE`

R. Erban, S. J. Chapman and P. K. Maini, *A practical guide to stochastic simulation of reaction-diffusion processes* (2007). Available at `http://arxiv.org/abs/0704.1908`

15.6 Further Reading

Students are by no means expected to read all these sources. There are suggestions intended to be helpful to students interested exploring the subjects covered in further detail.

1. L. J. S. Allen, *An Introduction to Stochastic Processes with Applications to Biology* (CRC, 2010).
2. H. Berg, *Random Walks in Biology* (Princeton University Press, 1993).
3. D. T. Gillespie, *Markov Processes, an Introduction for Physical Scientists* (Gulf Professional Publishing, 1992).
4. P. Attard, *Non-Equilibrium Thermodynamics and Statistical Mechanics* (Oxford University Press, 2012).
5. A. Nitzan, *Chemical Dynamics in Condensed Phases* (Oxford University Press, 2006).
6. P. Krapivsky, S. Redner and E. Ben-Naim, *A Kinetic View of Statistical Physics* (Cambridge University Press, 2010).
7. D. Anderson and T. Kurtz, *Stochastic Analysis of Biochemical Systems* (Springer, 2015).

16 B5.2 Applied Partial Differential Equations

16.1 General Prerequisites

Differential Equations 1 and Differential Equations 2 from Part A are prerequisites, and the material in these courses will be assumed to be known. Calculus of Variations and Fluids and Waves from Part A are desirable but not essential. Integral Transforms from Part A is strongly desirable.

16.2 Overview

This course continues the Part A Differential Equations courses. In particular, first-order conservation laws are solved and the idea of a shock is introduced; general nonlinear and quasi-linear first-order partial differential equations are solved, the classification of second-order partial differential equations is extended to systems, with hyperbolic systems being solved by characteristic variables. Then Riemann's function, Green's function and similarity variable methods are demonstrated.

16.3 Learning Outcomes

Students will know a range of techniques to characterise and solve PDEs including non-linear first-order systems, and second-order. They will be able to demonstrate various principles for solving PDEs including the method of characteristics, Green's functions, similarity solutions and Riemann functions.

16.4 Synopsis

First-order equations; applications. Characteristics, domain of definition. [2 lectures]

Weak solutions, conservation laws, shocks. [2 lectures]

Non-linear equations; Charpit's equations; eikonal equation. [3 lectures]

Systems of partial differential equations, characteristics. Shocks; weak solutions. [3 lectures]

2nd order semilinear equations. Hyperbolic equations, Riemann functions. [2 lectures]

Elliptic equations, parabolic equations. Well-posed problems, Green's function, similarity solutions. [4 lectures]

16.5 Reading List

1. J. R. Ockendon, S. D. Howison, A. A. Lacey and A. B. Movchan, *Applied Partial Differential Equations* (revised edition, Oxford University Press, Oxford, 2003).

2. M. Renardy and R.C. Rogers, *An Introduction to Partial Differential Equations* (Springer-Verlag, New York, 2004).

3. J. P. Keener, *Principles of Applied Mathematics: Transformation and Approximation* (revised edition, Perseus Books, Cambridge, Mass., 2000).

17 B5.3 Viscous Flow

17.1 General Prerequisites

The Part A (second-year) courses 'Waves and Fluids' and 'DEs 2' would be desirable. This course combines well with B5.2 Applied Partial Differential Equations. Though the two units are intended to be self-contained, they will complement each other.

17.2 Overview

Viscous fluids are important in so many facets of everyday life that everyone has some intuition about the diverse flow phenomena that occur in practice. This course is distinctive in that it shows how quite advanced mathematical ideas such as asymptotic expansions and partial differential equation theory can be used to analyse the underlying differential equations and hence give scientific understanding about flows of practical importance, such as air flow round wings, oil flow in a journal bearing and the flow of a raindrop on a windscreen.

17.3 Learning Outcomes

Students will have developed an appreciation of diverse viscous flow phenomena and they will have a demonstrable knowledge of the mathematical theory necessary to analyse such phenomena.

17.4 Synopsis

Euler's identity and Reynolds' transport theorem. The continuity equation and incompressibility condition. Cauchy's stress theorem and properties of the stress tensor. Cauchy's momentum equation. The incompressible Navier-Stokes equations. Vorticity. Energy. Exact solutions for unidirectional flows; Couette flow, Poiseuille flow, Rayleigh layer, Stokes layer. Dimensional analysis, Reynolds number. Derivation of equations for high and low Reynolds number flows.

Thermal boundary layer on a semi-infinite flat plate. Derivation of Prandtl's boundary-layer equations and similarity solutions for flow past a semi-infinite flat plate. Discussion of separation and application to the theory of flight.

Slow flow past a circular cylinder and a sphere. Non-uniformity of the two dimensional approximation; Oseen's equation. Lubrication theory: bearings, squeeze films, thin films; Hele-Shaw cell and the Saffman-Taylor instability.

17.5 Reading List

1. D. J. Acheson, *Elementary Fluid Dynamics* (Oxford University Press, 1990), Chapters 2, 6, 7, 8.
2. H. Ockendon and J. R. Ockendon, *Viscous Flow* (Cambridge Texts in Applied Mathematics, 1995).

3. R. P. Feynman, R.B. Leighton, M. Sands, *The Feynman Lectures on Physics, volume II*, (Addison Wesley, 1964) Chapter 41 "The Flow of Wet Water" (`http://www.feynmanlectures.caltech.edu/II_41.html`)

17.6 Further Reading

1. M. Van Dyke, *An Album of Fluid Motion* (Parabolic Press, 1982). ISBN 0915760029.
2. G.K. Batchelor, *An Introduction to Fluid Dynamics* (CUP, 2000). ISBN 0521663962.
3. C.C. Lin & L.A. Segel, *Mathematics Applied to Deterministic Problems in Natural Sciences* (Society of Industrial and Applied Mathematics, 1998). ISBN 0898712297.
4. L.A Segel, *Mathematics Applied to Continuum Mechancis* (Society for Industrial and Applied Mathematics, 2007). ISBN 0898716209.

18 B5.4 Waves and Compressible Flow

18.1 General Prerequisites

Familiarity is assumed with A10 Fluids and Waves and ASO Integral Transforms. This course combines well with B5.2 Applied Partial Differential Equations and B5.3 Viscous Flow.

18.2 Overview

In this course, we derive, analyse and solve models for linear and nonlinear wave propagation, focusing on examples in fluid dynamics. The governing equations derived in Part A Waves and Fluids are extended to incorporate energy conservation and further physical effects, including stratification and rotation. Linearised models for wave propagation are analysed using normal modes and transform methods, to explain physical phenomena such as dispersion, group velocity, and the transition from subsonic to supersonic flow. Nonlinear wave problems in gas dynamics and shallow water theory are solved using the theory of characteristics. Models for the propagation of shock waves and bores are derived and solved using conservation principles.

18.3 Learning Outcomes

By the end of this course, students will be able:

- to derive the governing equations for inviscid compressible flow from first principles;
- to derive linearised models for small-amplitude perturbations, and to solve the resulting problems using Fourier analysis and the method of characteristics;
- to analyse Fourier integrals using the method of stationary phase;

- to solve nonlinear hyperbolic models for one-dimensional gas dynamics and shallow water theory using the method of characteristics;
- to derive and solve Rankine-Hugoniot relations and entropy conditions governing the propagation of shock waves and hydraulic jumps.

18.4 Synopsis

Equations of motion: Conservation of mass, momentum and energy for an inviscid compressible fluid. Entropy and the Second Law of Thermodynamics. Flow relative to a rotating frame.

Models for linear wave propagation: Acoustic waves, Stokes waves, internal gravity waves, and inertial waves in a rotating fluid.

Theories for Linear Waves: Normal modes, travelling waves, Fourier integrals, method of stationary phase, dispersion and group velocity. Sub- and supersonic flow past a thin wing.

Nonlinear Waves: method of characteristics for one-dimensional unsteady gas dynamics and shallow water theory.

Shock Waves: Rankine-Hugoniot relations and entropy conditions for one-dimensional unsteady shock waves, bores and hydraulic jumps, and steady oblique shocks.

18.5 Reading List

1. Ockendon, H. and Ockendon, J.R. (2015) *Waves and Compressible Flow.* Second edition. New York: Springer.
2. Acheson, D.J. (1990) *Elementary Fluid Dynamics.* Oxford: Clarendon.
3. Billingham, J. and King, A.C. (2000) *Wave Motion* Cambridge: Cambridge University Press.
4. Lighthill, M.J. (2001) *Waves in Fluids.* Cambridge: Cambridge University Press.
5. Whitham, G.B. (1999) *Linear and Nonlinear Waves.* New York: Wiley.

19 B5.5 Further Mathematical Biology

19.1 General Prerequisites

Part A Differential Equations I and Modelling in Mathematical Biology. Part A Differential Equations II is also preferable but not necessary.

19.2 Overview

Further Mathematical Biology provides an introduction to more complicated models of biological phenomena, including spatial models of pattern formation and free boundary problems modelling invasion. The course focuses on applications where continuum, deterministic models formulated using ordinary and/or partial differential equations are appropriate, but also includes an introduction to discrete, stochastic models and how to relate them to continuum models. By using particular modelling examples in ecology, chemistry, biology and physiology, the course demonstrates how applied mathematical techniques, such as linear stability, phase planes and travelling waves, can yield important information about the behaviour of complicated models.

19.3 Learning Outcomes

Students will have developed a sound knowledge and appreciation of the ideas and concepts related to modelling biological and ecological systems using ordinary and partial differential equations and been introduced to discrete stochastic models.

19.4 Synopsis

- Non-spatial models with delays, including biological examples from population ecology and physiology.
- Age-structured models with biological and physiological examples.
- Introduction to spatial models, including morphogen gradients, chemotaxis and patterning.
- Travelling wave propagation with biological examples, including Fisher's equation and epidemics.
- Biological pattern formation, including Turing's model for animal coat markings and chemotaxis models.
- Moving boundary problems with biological examples, including colonisation and wound healing.
- Discrete-to-continuum models, including space- and velocity-jump models for diffusion and chemotaxis.

19.5 Reading List

1. J. D. Murray, Mathematical Biology, Volume I: An Introduction. 3rd Edition (Springer, 2002).
2. J. D. Murray, Mathematical Biology, Volume II: Spatial Models and Biomedical Applications. 3rd edition (Springer, 2003).
3. L. Edelstein-Keshet, Mathematical Models in Biology. (SIAM, 2005).

19.6 Further Reading

1. J. Keener and J. Sneyd, Mathematical Physiology. First Edition (Springer, 1998).

2. N. F. Britton, Essential Mathematical Biology. (Springer, 2003).

3. H. G. Othmer, S. R. Dunbar and W. Alt, Models of dispersal in biological systems. Journal of Mathematical Biology, 26:263-298 (1988).

20 B5.6 Nonlinear Dynamics, Bifurcations and Chaos

20.1 General Prerequisites

There is no optional Part B course as a formal prerequisite. This course builds on the material, which appears in several mandatory Prelims and Part A courses, including courses on differential equations, calculus, probability, linear algebra, constructive mathematics, computational mathematics, dynamics, metric spaces and analysis. Problem Sheet 0 includes questions covering some relevant background material.

20.2 Overview

This course aims to provide an introduction to the tools and concepts of dynamical systems theory which have become a central tool of both applied and pure mathematics with applications including chemical reaction networks, celestial mechanics, mathematical biology, fluid dynamics and social sciences.

The course will focus on both ordinary differential equations and maps. It will draw examples from appropriate model systems and various application areas. The problem sheets will require basic skills in numerical computation (numerical integration and visualisation of solutions of differential equations).

20.3 Learning Outcomes

Students will have developed a sound knowledge and appreciation of some of the tools, concepts, and computations used in the study of dynamical systems. They will also get some exposure to some modern research topics in the field.

20.4 Synopsis

The first 8 *lectures of this course is part of the core syllabus for the MSc in Mathematical Modelling and Scientific Computing* A2 *Mathematical Methods.*

Lectures 1–8: Discrete-time (maps) and continuous-time (differential equations) dynamical systems. Notion of flows, stability of fixed points, Lyapunov function, invariant manifolds, stable manifold theorem, notion of hyperbolicity, center manifold. Chemical reaction networks. Stable, unstable and center subspaces. Poincaré-Bendixson theorem. Periodic solutions, stable and unstable limit cycles. Introduction to bifurcation theory, covering saddle-node, transcritical, supercritical pitchfork and subcritical pitchfork bifurcations. Extended center manifold. Logistic map. Periodic points of maps. Stability of N-cycles. Period-doubling bifurcation. Sharkovsky's theorem. Invariant distribution.

Lectures 9–16: Bifurcations of limit cycles, covering supercritical and subcritical Hopf bifurcations, saddle-node bifurcation of cycles, infinite-period (SNIC) bifurcation and homoclinic (saddle-loop) bifurcation. Oscillations in chemical reaction networks. Weekly nonlinear oscillators. Poincaré-Lindstedt method. Conservative and non-conservative systems. Liénard systems, van der Pol oscillator. Hilbert's 16th problem. Lorenz equations. Lorenz map. Poincaré section. Poincaré map. Converse of Sharkovsky's theorem. Bernoulli shift map, symbolic dynamics. Tent map. Dynamics on metric spaces, sensitive dependence on initial conditions, transitivity, conjugate maps, chaotic dynamics.

20.5 Reading List

1. G. Goodson, Chaotic Dynamics (Cambridge University Press, 2016)
2. S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos (Springer, 2010)
3. L. Perko, Differential Equations and Dynamical Systems Third edition, Springer, 2008).
4. Y. A. Kuznetsov, Elements of Applied Bifurcation Theory (Third edition, Springer, 2004).
5. S. H. Strogatz, Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry and Engineering (Westview Press, 2000).
6. P. G. Drazin, Nonlinear Systems (Cambridge University Press, Cambridge, 1992).

21 B6.1 Numerical Solution of Partial Differential Equations

21.1 General Prerequisites

Part A Differential Equations 1; A7 Numerical Analysis is desirable but not essential.

21.2 Overview

To introduce and give an understanding of numerical methods for the solution of elliptic, parabolic and hyperbolic partial differential equations, including their derivation, analysis and applicability.

21.3 Learning Outcomes

At the end of the course the student will be able to: Construct practical methods for the numerical solution of boundary-value problems arising from ordinary differential equations and elliptic partial differential equations; analyse the stability, accuracy, and uniqueness properties of these methods; construct methods for the numerical solution of initial-boundary-value problems for second-order parabolic partial differential equations, and first- and second-order hyperbolic partial differential equations, and analyse their stability and accuracy properties.

21.4 Synopsis

The course is devoted to the development and analysis of numerical approximations to boundary- value problems for second-order ordinary differential equations, boundary-value problems for second-order elliptic partial differential equations, initial-boundary-value problems for second-order parabolic equations, and first- and second-order hyperbolic partial differential equations. The course begins by considering classical techniques for the numerical solution of boundary-value problems for second-order ordinary differential equations and elliptic boundary-value problems, in particular the Poisson equation in two dimensions. Topics include: discretisation, stability and convergence analysis, and the use of the discrete maximum principle. The remaining lectures focus on the numerical solution of initial-boundary-value problems for second-order parabolic and first- and second-order hyperbolic partial differential equations with topics such as: approximation by finite difference methods, accuracy, stability (including the Courant-Friedrichs-Lewy (CFL) condition) and convergence.

21.5 Reading List

The course will be based on the following textbooks:

1. A. Iserles, A First Course in the Numerical Analysis of Differential Equations (Cambridge University Press, second edition, 2009). ISBN 978-0-521-73490-5. [Chapters 8, 16, 17].

2. B.S. Jovanović and E. Süli, Analysis of Finite Difference Schemes for Linear Partial Differential Equations with Generalized Solutions (Springer, 2014). ISBN 978-1-447-15461-7. [Sections 2.1, 2.2, 2.3, 3.1, 3.2, 4.1, 4.2].

3. R. LeVeque, Finite Difference Methods for Ordinary and Partial Differential Equations (SIAM, 2007). ISBN 978-0-898716-29-0. [Chapters 2, 3, 4, 5, 9, 10].

4. K.W. Morton and D.F. Mayers, Numerical Solution of Partial Differential Equations: An Introduction (Cambridge University Press, second edition, 2012). ISBN 978-0-521-60793-3. [Chapters 2-7].

22 B6.2 Optimisation for Data Science

22.1 General Prerequisites

Basic linear algebra (such as eigenvalues and eigenvectors of real matrices), multivariate real analysis (such as norms, inner products, continuity, differentiability), multivariable calculus (such as Taylor expansions, multivariate differentiation, gradients), basic probability and statistics. All these notions are covered in Prelims Linear Algebra, Analysis, Calculus (Intro and Multivariate), Probability and Statistics. Part A Numerical Analysis is recommended but not essential.

22.2 Overview

Optimisation problems occur naturally in statistics, data analysis, machine learning and in signal and information processing. Their efficient solution is for example, the cornerstone of successfully training machine learning models, principal component analysis, matrix completion, and many more. This course covers a few of the typical problem formulations in that appear in such large scaled modern models and then focuses specifically on gradient-based algorithms (namely, deterministic and stochastic first-order methods) and their convergence and complexity analyses for (mainly) convex problems.

22.3 Learning Outcomes

Course participants learn how to model applied problems as optimisation problems and gain an understanding of the underpinnings of gradient based algorithms for the solution of such problems. While many variants of such algorithms have been published in the modern literature, this course will cover the key formulations and analyses on which state of the art methods are based. The algorithms we study introduce the basic tools that will enable participants to understand the strengths and limitations of these and other algorithms more generally.

22.4 Synopsis

Mathematical preliminaries: Global and local optimisers, convexity, gradients and subgradients, optimality conditions, convergence rates.

Steepest descent method and its convergence analysis in the general case, the convex case and the strongly convex case.

Modelling: least squares, matrix completion, sparse inverse covariance estimation, sparse principal components, sparse plus low rank matrix decomposition, support vector machines, logistic regression, deep learning.

Proximal operators and prox-gradient methods.

Accelerating gradient methods: heavy ball method and Nesterov acceleration.

Oracle complexity and the stochastic gradient descent algorithm.

The variance reduced stochastic gradient descent algorithm.

Dimensionality reduction techniques for large scale optimisation (Johnson-Lindenstrauss Lemma)

Data sketching: linear least squares and sums of functions (batch stochastic gradient)

Parameter sketching: Randomised coordinate descent first order methods and random subspace methods.

22.5 Reading List

- Amir Beck. "First Order Methods in Optimization", MOS-SIAM Series on Optimization, 2017.

- S.J.Wright. “Optimization Algorithms for Data Analysis”, 2016.
- http://www.optimization-online.org/DBFILE/2016/12/5748
- S.J.Wright. “Coordinate descent algorithms” Mathematical Programming, 151:3–34, 2015. https://arxiv.org/abs/1502.04759
- D.P. Woodruff. ”Sketching as a Tool for Numerical Linear Algebra” https://arxiv.org/abs/1411.4357
- (further reading) L. Bottou, F.E. Curtis, and J. Nocedal. “Optimization methods for large-scale machine learning. SIAM Review, 59(1): 65-98, 2017.
- (further reading) Z. Allen-Zhu. Katyusha: The first direct acceleration of stochastic gradient methods. The Journal of Machine Learning Research, 18(1): 8194-8244, 2017.
- (further reading) J. Wright and Y. Ma. ”High-dimensional data analysis with low-dimensional models”, CUP, 2021.

23 B6.3 Integer Programming

23.1 Overview

In many practical problems that can be approached via linear optimisation problems, some or all of the variables are constrained to take binary or integer values. For example, in optimal crew scheduling a pilot cannot be fractionally assigned to two different flights at the same time. Likewise, in combinatorial optimisation an element of a given set either belongs to a chosen subset or it does not. Integer programming is the mathematical theory of such problems and of algorithms for their solution. The aim of this course is to provide an introduction to some of the general ideas on which attacks to integer programming problems are based: generating bounds through relaxations by problems that are easier to solve, and branch-and-bound.

23.2 Learning Outcomes

Students will understand some of the theoretical underpinnings that render certain classes of integer programming problems tractable (”easy” to solve), and they will learn how to solve them algorithmically. Furthermore, they will understand some general mechanisms by which intractable problems can be broken down into tractable subproblems, and how these mechanisms are used to design good heuristics for solving the intractable problems. Understanding these general principles will enable the students to guide the modelling phase of a real-world problem towards a mathematical formulation that has a reasonable chance of being solved in practice.

23.3 Synopsis

Week 1:

- Classical examples of Integer Programming problems (IP), modelling and basic terminology.
- Linear Programming I: The simplex method.

Week 2:

- Linear Programming II: Duality Theory.
- Total Unimodularity I: Ideal formulations of IPs and totally unimodular matrices.

Week 3:

- Total Unimodularity II: Exact theoretical characterisation, practical sufficient criteria, bipartite matching, the shortest path problem.
- Submodularity I: Submodular functions and submodular optimisation problems.

Week 4:

- Submodularity II: Submodular rank functions, matroids, the greedy algorithm and the maximum weight independent set problem.
- Branch-and-Bound I: LP based branch-and-bound for general integer programming problems.

Week 5:

- Branch-and-Bound II: General B&B, pre-processing, warm starting of LPs, dual simplex method.
- Dantzig-Wolfe decomposition, delayed column generation.

Week 6:

- Branch-and-Price, application to the cutting stock problem.
- Preprocessing of LPs and IPs, generating valid cuts, cutting plane algorithm.

Week 7:

- Chvátal cuts, Gomory cuts, branch-and-cut algorithm.
- The Generalised Assignment Problem.

Week 8:

- Lagrangian relaxation and Lagrangian duality.
- The subgradient algorithm.

23.4 Reading List

1. M. Conforti, G. Cornuéjols, G. Zambelli, *Integer Programming* (Springer 2014), ISBN 978-3-319-11007-3.

2. L. A. Wolsey, *Integer Programming* (John Wiley & Sons, 2021), parts of chapters 1-5 and 7, ISBN 978-1-119-60653-6.

24 B7.1 Classical Mechanics

24.1 General Prerequisites

Calculus of Variations recommended, but not essential.

24.2 Overview

This course builds on the Prelims Dynamics course, recasting Newtonian mechanics in the Lagrangian and Hamiltonian formalisms. As well as being elegant and computationally useful, these formulations of classical mechanics give important insights into symmetries and conservation laws, and are the language used to describe all modern theories of physics and mechanical systems.

24.3 Learning Outcomes

Students will be able to demonstrate knowledge and understanding of the Lagrangian and Hamiltonian formalisms. They will understand how symmetries and conserved quantities are described in this language, and be able to apply the ideas developed to small oscillations around equilibria, rigid body motion and other elementary systems.

24.4 Synopsis

Review of Newtonian mechanics. Generalized coordinates. The principle of least action. Constraints. Symmetries and Noether's Theorem. Examples of simple and compound constrained systems.

Equilibria. Small oscillations about a stable equilibrium and normal modes, with examples.

Rigid bodies. Angular velocity, angular momentum and the inertia tensor. Euler's equations and tops. Euler angles and $SO(3)$.

Legendre transformations and the Hamiltonian. Phase space and its geometry. Poisson brackets. Canonical transformations. Liouville's theorem. The Hamilton-Jacobi equation.

24.5 Reading List

1. H. Goldstein, C, Poole, J. Safko, *Classical Mechanics* Third Edition (Addison-Wesley, 2002).

2. L. D. Landau, E. M. Lifshitz, *Mechanics (Course of Theoretical Physics, Vol. 1)*, Third Edition, (Butterworth-Heinemann, 1976).

3. N. M. J. Woodhouse, *Introduction to Analytical Mechanics* (OUP, 1987)

25 B7.2 Electromagnetism

25.1 General Prerequisites

Prelims. (Essential: vector calculus, multivariable differentiation and integration, Stokes' theorem)

25.2 Overview

This is a classical course on Electromagnetism, similar to the one in a theoretical Physics degree. We will follow closely the book by Jackson, first 7 chapters.

25.3 Learning Outcomes

Students will have a clear understanding of classical electromagnetism and its mathematical description by Maxwell's equations. They will dominate several techniques and will be able to solve most classic problems of electromagnetism. This course should also enable them to continue learning by themselves, or take more advanced courses.

25.4 Synopsis

Basics of electrostatics;

Boundary value problems in electrostatics;

Multipoles, electrostatics of macroscopic media, dielectrics;

Magnetostatics; Time-varying fields, Maxwell equations, conservation laws;

Plane electromagnetic waves;

Aspects of Electromagnetism and Special Relativity

25.5 Reading List

The lectures will follow:

1. J.D. Jackson, *Classical Electrodynamics* (Wiley, 1998), chapters 1 to 7.

25.6 Further Reading

1. R. Feynman, *Lectures in Physics*, Vol.2. Electromagnetism, Addison Wesley.

2. D. J. Griffiths, Introduction to Electrodynamics, Pearson.

3. N. M. J. Woodhouse, Special Relativity, Springer Undergraduate Mathematics.

26 B7.3 Further Quantum Theory

26.1 General Prerequisites

Part A Quantum Theory.

26.2 Overview

This course builds on Part A Quantum Theory. The mathematical foundations of quantum theory are developed more deeply than in the Part A course, and general principles regarding the realisation of symmetries in quantum mechanical systems are developed. Systems of several particles are studied, including a consideration of identical particles and particle statistics. Along the way, a number of simple-but-relevant concepts from the theories of Lie groups, representation theory, and functional analysis are introduced in a hands-on fashion.

In semi-realistic systems, an exact solution to the quantum dynamics is rarely forthcoming. The second part of the course is largely devoted to developing several approximation techniques that can be applied in these more general settings. These are employed to study problems such as the determination of energy levels of the Helium atom.

The course concludes with a short introduction to scattering theory in one spatial dimension.

26.3 Learning Outcomes

Student will be able to state the postulates of nonrelativistic quantum theory and explain how they are realized in key examples. They will be able to analyse simple quantum systems that admit exact solutions by exploiting symmetries and algebraic techniques. They will be able to calculate approximations to energy levels, scattering states, and other properties of more complicated systems using perturbation theory, semi-classical techniques, and variational methods.

26.4 Synopsis

Abstract formulation of quantum mechanics in terms of linear operators on Hilbert spaces; Dirac notation; discrete and continuum states; time evolution and the propagator.

Systems of several particles and Hilbert space tensor products; distinguishable and indistinguishable particles; Fermi–Dirac and Bose–Einstein statistics; the Pauli exclusion principle; elementary aspects of quantum entanglement.

Symmetries in quantum mechanics as unitary and anti-unitary operators; rotations, angular momentum, and spin; spin-1/2 and projective representations of SO(3); addition of angular momentum; spin-statistics theorem; tensor operators and the Wigner–Eckart theorem.

Approximation methods: Rayleigh–Schrödinger perturbation theory; variational methods; WKB approximation and Bohr–Sommerfeld quantisation.

Elementary scattering theory in one dimension; relation between bound states and of poles/zeroes of the S-matrix.

26.5 Reading List

1. J. J. Sakurai and Jim Napolitano, *Modern Quantum Mechanics* (CUP, 2017).
2. Steven Weinberg, *Lectures on Quantum Mechanics* (CUP, 2015).
3. James Binney and David Skinner, *The Physics of Quantum Mechancis* (Oxford, 2014)

26.6 Further Reading

1. David Griffiths and Darrell Schroeter, *Introduction to Quantum Mechanics* (CUP, 2017).
2. Andrew Larkoski, *Quantum Mechanics: A Mathematical Introduction* (CUP, 2023)
3. Brian Hall, *Quantum Theory for Mathematicians* (Springer, 2013).
4. Eugene Merzbacher, *Quantum Mechanics* (Wiley, 1970).
5. Keith Hannabuss, *An Introduction to Quantum Theory* (OUP, 1997).

27 B8.1 Probability, Measure and Martingales

27.1 General Prerequisites

The course is self-contained but subsumes both Part A Probability and Part A Integration. It relies strongly on the intuition and knowledge built up in those two courses and both are strongly recommended.

27.2 Overview

Probability is both a fundamental way of viewing the world and a core mathematical discipline. In recent years there has been an explosive growth in the importance of probability in scientific research. Applications range from physics to neuroscience, from genetics to communication networks and, of course, finance.

This course develops the mathematical foundations essential for more advanced courses in probability theory. The first part of the course develops a more sophisticated understanding of measure theory and integration, first seen in Part A Integration. The second part focuses on key probabilistic concepts: independence and conditional expectation. We then introduce discrete time martingales and establish results needed to study their behaviour. This prepares the ground for continuous martingales, studied in B8.2, which are the cornerstone of stochastic calculus.

27.3 Learning Outcomes

The students will learn about measure theory, random variables, independence, expectation and conditional expectation, product measures, filtrations and stopping times, discrete-parameter martingales and their properties and applications.

27.4 Synopsis

Measurable sets, σ-algebras, π-λ systems lemma. Random variables, generated σ-algebras, monotone class theorem. Measures: properties, uniqueness of extension, Caratheodory's Extension Theorem; measure spaces, pushforward measure, product measure.

Independence of events, random variables and σ-algebras, relation to product measures.

The tail σ-algebra, Kolomogorov's 0-1 Law, lim sup and lim inf of a sequence of events, Fatou and reverse Fatou Lemma for sets, Borel-Cantelli Lemmas.

Integration and expectation, review and extension of elementary properties of the integral and convergence theorems [from Part A Integration for the Lebesgue measure on R]. Radon-Nikodym Theorem [without proof], Scheffé's Lemma. Integration on product space, Fubini/Tonelli Theorem. Different modes of convergence and their relations. Markov's and Jensen's inequalities. Lp spaces, Holder's and Minkowski's inequalities, completeness. Uniform integrability, Vitali's convergence theorem.

Conditional expectation: definition, properties, uniquness. Conditional convergence theorems and inequalities, link with uniform integrability. Orthogonal projection in L2, existence of conditional expectation.

Filtrations and stopping times. Examples and properties. σ-algebra associated to a stopping time.

Martingales in discrete time: definition, examples, properties, discrete stochastic integrals. Doob's decomposition theorem. Stopped martingales and Doob's Optional Sampling Theorem. Maximal and Lp Inequalities, Doob's Upcrossing Lemma and Martingale Convergence Theorem. Uniformly integrable martingales, convergence in L1. Backwards martingales and Kolmogorov's Strong Law of Large Numbers.

27.5 Reading List

1. Lecture Notes for the course.

2. D. Williams, *Probability with Martingales*, Cambridge University Press, 1995.

27.6 Further Reading

1. Z. Brzezniak and T. Zastawniak, Basic stochastic processes. A course through exercises. Springer Undergraduate Mathematics Series. (Springer-Verlag London, Ltd., 1999) [more elementary than D. Williams' book, but can provide with a complementary first reading].

2. M. Capinski and E. Kopp, Measure, integral and probability, Springer Undergraduate Mathematics Series. (Springer-Verlag London, Ltd., second edition, 2004).

3. R. Durrett, Probability: Theory and Examples (Second Edition Duxbury Press, Wadsworth Publishing Company, 1996).

4. J. Neveu, Discrete-parameter Martingales (North-Holland, Amsterdam, 1975).

28 B8.2 Continuous Martingales and Stochastic Calculus

28.1 General Prerequisites

B8.1 Probability, Measure and Martingales is a prerequisite. Consequently, Part A Integration and Part A Probability are also prerequisites.

28.2 Overview

Stochastic processes - random phenomena evolving in time - are encountered in many disciplines from biology, physics, geology through to economics and finance. This course focuses on developing the mathematics needed to describe stochastic processes evolving continuously in time and introduces the basic tools of stochastic calculus which are a cornerstone of modern probability theory. The canonical example of such a stochastic process is Brownian motion, also called the Wiener process. This mathematical object was initially proposed by Bachelier, as a model for asset prices, and by Einstein to describe the displacement of a pollen particle in a fluid. The paths of Brownian motion, or of any continuous martingale, are of infinite variation (they are in fact nowhere differentiable and have non-zero quadratic variation) and one of the aims of the course is to define a theory of integration along such paths as well as a suitable version of the chain rule, given by Itô's formula.

28.3 Learning Outcomes

The students will develop an understanding of Brownian motion and continuous martingales in continuous time. They will became familiar with stochastic calculus and in particular be able to use Itô's formula.

28.4 Synopsis

An introduction to stochastic processes in continuous time.

Brownian motion - definition, construction and basic properties, regularity of paths.

Filtrations and stopping times, first hitting times.

Brownian motion - martingale and strong Markov properties, reflection principle.

Martingales - definitions, regularisation and convergence theorems, optional sampling theorem, maximal and Doob's L^p inequalities.

Quadratic variation, local martingales, semimartingales.

Discussion of the Stieltjes integral.

Stochastic integration and Itô's formula with applications.

28.5 Reading List

Lecture notes will be provided, but there are also many textbooks which cover the course material with a varying degrees of detail/rigour. These include:

1. D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, Springer (Revised 3^{rd} ed.), 2001, Chapters 0-4.
2. I. Karatzas and S. Shreve, *Brownian motion and stochastic calculus*, Springer (2^{nd} ed.), 1991, Chapters 1-3.
3. R. Durrett, *Stochastic Calculus: A practical introduction*, CRC Press, 1996. Sections 1.1 - 2.10.
4. F. Klebaner, *Introduction to Stochastic Calculus with Applications*, 3rd edition, Imperial College Press, 2012. Chapters 1, 2, 3.1–3.11, 4.1-4.5, 7.1-7.8, 8.1-8.7.
5. J. M. Steele, *Stochastic Calculus and Financial Applications*, Springer, 2010. Chapters 3 - 8.
6. B. Oksendal, *Stochastic Differential Equations: An introduction with applications*, 6th edition, Springer (Universitext), 2007. Chapters 1 - 3.
7. S. Shreve, *Stochastic calculus for finance*, Vol 2: Continuous-time models, Springer Finance, Springer-Verlag, New York, 2004. Chapters 3 - 4.

29 B8.3 Mathematical Models of Financial Derivatives

29.1 General Prerequisites

B8.1 Probability, Measure and Martingales would be good background. Part A Probability is a prerequisite. Part A Integration is also good background, though not a prerequisite.

29.2 Overview

The course aims to introduce students to derivative security valuation in financial markets. At the end of the course the student should be able to formulate a model for an asset price and then determine the prices of a range of derivatives based on the underlying asset using arbitrage free pricing ideas.

29.3 Learning Outcomes

Students will have a familiarity with the mathematics behind the models and analytical tools used in Mathematical Finance. This includes being able to formulate a model for an asset price and then determining the prices of a range of derivatives based on the underlying asset using arbitrage free pricing ideas. Furthermore, the students will be familiar with electronic trading and the demand and supply of liquidity in electronic financial markets.

29.4 Synopsis

- Discrete-time models (binomial trees) and arbitrage in finance.
- Hedging in continuous time and the Black-Scholes model
- European-style options and
- Introduction to Brownian motion and Ito's Lemma
- American-style options, PDEs and the Feynman-Kac formula
- Perpetual and other exotic options
- Implied volatility: smiles and smirks
- The Merton Jump-Diffusion model and option prices
- Consumption-based pricing
- Limit order books and optimal execution models

29.5 Reading List

1. S.E Shreve, *Stochastic Calculus for Finance*, vols I and II, (Springer 2004).
2. T. Bjork, *Arbitrage Theory in Continuous Time* (Oxford University Press, 1998).
3. P. Wilmott, S. D. Howison and J. Dewynne, *Mathematics of Financial Derivatives* (Cambridge university Press, 1995).
4. A. Etheridge, *A Course in Financial Calculus* (Cambridge University Press, 2002).

29.6 Further Reading

Background on Financial Derivatives

1. J. Hull, *Options Futures and Other Financial Derivative Products*, 4th edition (Prentice Hall, 2001).

30 B8.4 Information Theory

30.1 General Prerequisites

Part A Probability would be very helpful, but not essential.

30.2 Overview

Information theory is a relatively young subject. It played an important role in the rise of the current information/digital/computer age and still motivates much research in diverse fields such as statistics and machine learning, physics, computer science and engineering. Every time you make a phone call, store a file on your computer, query an internet search engine, watch a DVD, stream a movie, listen to a CD or mp3 file, etc., algorithms run that are based on topics we discuss in this course. However, independent of such applications, the underlying mathematical objects arise naturally as soon as one starts to think about "information" in a mathematically rigorous way. In fact, a large part of the course deals with two fundamental questions:

1. How much information is contained in a signal/data/message, and how do we compress it? (source coding)
2. What are the limits to information transfer over a channel that is subject to noisy perturbations, and how to encode and decode in this setting? (channel coding)

30.3 Learning Outcomes

The student will have learned about entropy, mutual information and divergence, their basic properties, how they relate to information transmission. Knowledge of fundamentals of block/symbol/channel coding. Understand the theoretical limits of transmitting information due to noise. They will have experience applying these methods to data.

30.4 Synopsis

(Conditional) entropy, mutual information, divergence and their basic properties and inequalities (Fano, Gibbs'). (Strong and weak) typical sequences: the asymptotic equipartition property, and applications to block coding. Symbol codes: Kraft–McMillan, optimality, various symbol codes (Huffman, Elias, Arithmetic codes) and their construction and complexity. Channel coding: discrete memoryless channels, channel codes/rates/errors, Shannon's noisy channel coding theorem, block linear codes. Methods for decoding noisy signals.

30.5 Reading List

1. *B8.4 Information theory* (online lecture notes)
2. T. Cover and J. Thomas, *Elements of Information Theory* (Wiley, 1991), Chapters 1-8, 11.
3. D. MacKay, *Information Theory, Inference, and Learning Algorithms* (Cambridge, 2003)

30.6 Further Reading

1. R. B. Ash, *Information Theory* (Dover, 1990).

2. D. J. A. Welsh, *Codes and Cryptography* (Oxford University Press, 1988), Chapters 1-3, 5.

3. G. Jones and J. M. Jones, *Information and Coding Theory* (Springer, 2000), Chapters 1-5.

4. Y. Suhov & M. Kelbert, *Information Theory and Coding by Example* (Cambridge University Press, 2013), Relevant examples.

31 B8.5 Graph Theory

31.1 General Prerequisites

Part A Graph Theory is recommended.

31.2 Overview

Graphs (abstract networks) are among the simplest mathematical structures, but nevertheless have a very rich and well-developed structural theory. Since graphs arise naturally in many contexts within and outside mathematics, Graph Theory is an important area of mathematics, and also has many applications in other fields such as computer science.

The main aim of the course is to introduce the fundamental ideas of Graph Theory, and some of the basic techniques of combinatorics.

31.3 Learning Outcomes

The student will have developed a basic understanding of the properties of graphs, and an appreciation of the combinatorial methods used to analyze discrete structures.

31.4 Synopsis

Introduction: basic definitions and examples.

Trees and their characterization.

Euler circuits; long paths and cycles.

Vertex colourings: Brooks' theorem, chromatic polynomial.

Edge colourings: Vizing's theorem.

Planar graphs, including Euler's formula, dual graphs.

Maximum flow - minimum cut theorem: applications including Menger's theorem and Hall's theorem.

Tutte's theorem on matchings.

Extremal Problems: Turan's theorem, Zarankiewicz problem, Erdős-Stone theorem.

31.5 Reading List

1. B. Bollobas, *Modern Graph Theory*, Graduate Texts in Mathematics 184 (Springer-Verlag, 1998)

31.6 Further Reading

1. J. A. Bondy and U. S. R. Murty, *Graph Theory: An Advanced Course*, Graduate Texts in Mathematics 244 (Springer-Verlag, 2007).
2. R. Diestel, *Graph Theory*, Graduate Texts in Mathematics 173 (5th edition, Springer-Verlag, 2017).
3. D. West, *Introduction to Graph Theory*, Second edition, (Prentice-Hall, 2001).

32 B8.6 High Dimensional Probability

32.1 General Prerequisites

Part A Probability and Part A Integration are required.

32.2 Overview

High-dimensional probability and high-dimensional statistics have emerged in recent years as ever more important topics due to the need to analyse vast amounts of complex data. The ideas and methods developed for dealing with probability distributions on spaces of high-dimension (such as distributions of randomly sampled data with multiple attributes) have been used not only in pure mathematics but also in applications from stochastic simulation to statistics, data science to statistical mechanics.

This course will focus on the development of basic ideas and techniques such as elementary dimension-free tail estimates, concentration bounds, the metric entropy method, the Poincaré and logarithmic Sobolev inequalities, large deviation principles for rare events etc.

32.3 Learning Outcomes

The students will learn the fundamental ideas and modern tools for handling distributions on high-dimensional spaces, and understand some special features of probability distributions on such spaces, for example the concentration of probability laws on small regions of low dimensional subspaces.

32.4 Synopsis

1) (2 lectures) Derive a few elementary but important tail estimates for distributions in terms of moments, variances and other statistical characteristics.

2) (3 lectures) Strong law of large numbers, Cramér's large deviation principle.

3) (3 lectures) Elementary results on concentration of probabilities, concentration functions, Wasserstein distance and information inequality.

4) (4 lectures) Heat semi-group, Gaussian measures, Poincaré inequality, logarithmic Sobolev inequalities for Gaussian measures and some other distributions. Concentration for Gaussian measures.

5) (2 lectures) Lasso performance bound, and large eigenvalue of random matrices.

6) (3 lectures) Developing the entropy method by means of examples – bounded difference inequalities, concentration of convex Lipschitz functions and modified logarithmic Sobolev inequalities.

7) (3 lectures) Building the connection between concentration estimates and isoperimetric inequalities.

32.5 Reading List

Lecture notes will be available. We will follow no one of them closely though, but will do a selection from the following references.

1. M. J. Wainwright: *High-dimensional statistics – A non-asymptotic viewpoint.* (Cambridge Series in Statistical and Probability Mathematics). Cambridge University Press (2019). Chapters 1 - 5.

2. R. Vershynin: *High-dimensional probability – An introduction with applications in data science.* (Cambridge Series in Statistical and Probability Mathematics). Cambridge University Press (2018). Chapters 2, 3 and 5.

3. S. Boucheron, G. Lugosi and P. Massart: *Concentration inequalities – A nonasymptotic theory of independence.* Oxford University Press (2013). Chapters 1-7

32.6 Further Reading

1. M. Ledoux, *The Concentration of Measure Phenomenon* (Mathematical Surveys and Monographs 89). AMS 2001.

2. P. Massart, *Concentration inequalities and model selection* (Lecture Notes in Mathematics), Springer (2007).

33 SB3.1 Applied Probability

33.1 General Prerequisites

Part A Probability.

33.2 Overview

This course is intended to show the power and range of probability by considering real examples in which probabilistic modelling is inescapable and useful. Theory will be developed as required to deal with the examples.

33.3 Synopsis

Poisson processes and birth processes. Continuous-time Markov chains. Transition rates, jump chains and holding times. Forward and backward equations. Class structure, hitting times and absorption probabilities. Recurrence and transience. Invariant distributions and limiting behaviour. Time reversal. Renewal theory. Limit theorems: strong law of large numbers, strong law and central limit theorem of renewal theory, elementary renewal theorem, renewal theorem, key renewal theorem. Excess life, inspection paradox.

Applications in areas such as: queues and queueing networks - M/M/s queue, Erlang's formula, queues in tandem and networks of queues, M/G/1 and G/M/1 queues; insurance ruin models; applications in applied sciences.

33.4 Reading List

1. J. R. Norris, *Markov Chains* (Cambridge University Press, 1997).
2. G. R. Grimmett and D. R. Stirzaker, *Probability and Random Processes* (3rd edition, Oxford University Press, 2001).
3. G. R. Grimmett and D. R. Stirzaker, *One Thousand Exercises in Probability* (Oxford University Press, 2001).
4. S. M. Ross, *Introduction to Probability Models* (4th edition, Academic Press, 1989).
5. D. R. Stirzaker, *Elementary Probability* (2nd edition, Cambridge University Press, 2003).

34 BSP Structured Projects

34.1 Overview

Quota Students will be able to choose a project from a menu of six to eight possibilities, or suggest a project of their own. Each project has a quota of two to three students. This project may be offered for examination at Part B as a double unit. It is equivalent to a 32-hour lecture course. Generally, students will have 6 hours of supervision distributed over Michaelmas and Hilary terms. In addition there are lectures on writing mathematics and using LaTeX in Michaelmas and Hilary terms. See the lecture list for details.

Students considering offering an essay should read the *Part B Projects Guidance.* available at:

https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/projects

Students who wish to take BSP **must** register their interest in BSP in the July course registration. After the course registration form has closed, students will be contacted with further details of the projects available for next year. Students will then be informed at a later date whether they have been allocated a place on a BSP project.

34.2 Learning Outcomes

This option is designed to help students understand applications of mathematics to live research problems and to learn some of the necessary techniques. For those who plan to stay on for the MMath or beyond, the course will provide invaluable preliminary training. For those who plan to leave after the BA, it will offer insights into what mathematical research can involve, and training in key skills that will be of long term benefit in any career.

Students will gain experience of:

1. Applications of numerical computation to current research problems.
2. Reading and understanding research papers.
3. Working with new people in new environments.
4. Meeting the expectations of different disciplines.
5. Presenting a well structured written report, using LaTeX.
6. Making an oral presentation to a non-specialist audience.
7. Reading and assessing the work of other students.
8. Independent study and time management.

Students will be expected to:

1. Learn about a current research problem by reading one or more relevant research papers together with appropriate material from textbooks.
2. Carry out the required calculations using Maple, MuPAD or Matlab. Students are not expected to engage in original research but there will be scope for able students to envisage new directions.
3. Write up the problem and their findings in a report that is properly supported with detail, discussion, and good referencing.
4. Undertake peer review.
5. Give an oral presentation to a non-specialist audience.

34.3 Synopsis

In past years projects have included applications to biology, finance, and earth sciences. It is expected that a similar menu of topics, from which students will select one, will be available for 2024-2025.

Teaching

At the beginning of the course students will be given written instructions for their chosen project.

Michaelmas Term

There will be a group meeting with the organiser (Cath Wilkins) at the beginning of MT to set out expectations and deal with queries. The organiser will meet again with students individually at the end of MT. Between these meetings students will read around their chosen topic and take preparatory courses in LaTeX and Matlab, both of which are available from the department and are well documented online. Individual contact with the organiser by email, or if necessary in person, will be encouraged.

Hilary Term

Week 1 Lecture on expectations for the term, and advice on writing up.

Weeks 2 to 8 Students will meet regularly with their specialist supervisor. In addition, each student will meet at least once with the organiser, who will maintain an overview of the student's progress.

Week 10 Submission of written paper.

Easter vacation

Peer review

Trinity Term

Week 1 Oral presentation

Assessment

Students (and tutors) have sometimes expressed doubts about the predictability or reliability of project assessment. We are therefore concerned:

[i.] to make the assessment scheme as transparent as possible both to students and to assessors;

[ii.] that students who produce good project work should be able to achieve equivalent grades to students who write good exam papers.

The mark breakdown will be as follows:

[a.] Written work 75%, of which:

50% of available marks will be for general explanation and discussion of the problem;

50% of available marks will be for mathematical calculations and commentary

[b.] Oral presentation 15%

[c.] Peer review 10%

Note on (c): This may be a new kind of assessment for you. As with journal peer review, the anonymity of both writer and reviewer will be strictly maintained. Each student will be expected to read one other project write-up (from this or previous years) and to make a careful and well explained judgement on it. Credit for this will go to the reviewer, not to the writer, whose work will already have been assessed by examiners in the usual way.

35 BO1.1 History of Mathematics

35.1 Overview

Quota The maximum number of students that can be accepted is 20.

35.2 Learning Outcomes

This course is designed to provide the historical background to some of the mathematics familiar to students from A-level and the first four terms of undergraduate study, and looks at a period from approximately the mid-sixteenth century to the end of the nineteenth century. The course will be delivered through 16 lectures in Michaelmas Term, and a reading course consisting of 8 seminars (equivalent to a further 16 lectures) in Hilary Term. Guidance will be given throughout on reading, note-taking, and essay-writing.

Students will gain:

1. an appreciation of university mathematics in its historical context;
2. an enriched understanding of the mathematical content of the topics covered by the course;
3. a broader, multicultural view of mathematics

together with skills in:

1. reading and analysing primary historical mathematical sources;
2. reading and analysing secondary sources;
3. efficient note-taking;
4. essay-writing (from 1000 to 3000 words);
5. construction of references and bibliographies;
6. oral discussion and presentation.

35.3 Synopsis

Lectures

The Michaelmas Term lectures will cover the following material:

1. Introduction: ancient mathematical knowledge and its transmission to early modern Europe; the development of symbolic notation up to the end of the sixteenth century.
2. Seventeenth century: analytic geometry; the development of calculus; Newton's *Principia*.
3. Eighteenth century: from calculus to analysis; functions, limits, continuity; equations and solvability.
4. Nineteenth century: group theory and abstract algebra; the beginnings of modern analysis; rigorous definitions of real numbers; integration; complex analysis; set theory; linear algebra.

Classes to accompany the lectures will be held in Weeks 3, 5, 6, 7. For each class students will be expected to prepare one piece of written work (1000 words) and one discussion topic. Students will also be expected to present the content of their essays to the whole class.

Reading course

The Hilary Term part of the course is run as a reading course during which we will study a selection of primary texts in some detail, using original sources and secondary literature. Details of the materials to be read in HT 2025 will be made available towards the end of MT 2024. Students will be expected to write three essays (2000 words each) during the first six weeks of term.

Assessment The Michaelmas Term material will be examined in a two-hour written paper during Trinity Term. Candidates will be expected to answer two half-hour questions (commenting on extracts) and one one-hour question (essay). The paper will account for 50% of the marks for the course. The Reading Course will be examined by a 3000-word essay at the end of Hilary Term. The title will be set at the beginning of Week 7 and submission of an electronic version of the essay to Inspera will also be required by midday on Monday of Week 10. The essay will account for 50% of the marks for the course.

35.4 Reading List

1. Jacqueline Stedall, *Mathematics emerging: a sourcebook 1540-1900* (Oxford University Press, 2008).
2. Victor Katz, *A history of mathematics* (brief edition) (Pearson Addison Wesley, 2004), or:
3. Victor Katz, *A history of mathematics: an introduction* (third edition) (Pearson Addison Wesley, 2009).
4. Benjamin Wardhaugh, *How to read historical mathematics* (Princeton, 2010).
5. Jacqueline Stedall, *The history of mathematics: a very short introduction* (Oxford University Press, 2012).

35.5 Further Reading

1. John Fauvel and Jeremy Gray (eds), *The history of mathematics: a reader*, (Macmillan, 1987).

2. June Barrow-Green, Jeremy Gray and Robin J. Wilson, *The history of mathematics: a source-based approach*, 2 vols. (Mathematical Association of America, 2019, 2022).

Further suggestions of additional reading on particular topics will be given throughout the lecture course. Moreover, the intercollegiate classes in MT and the seminars in HT will also serve as a forum in which students will be encouraged to share any interesting reading materials that they have discovered themselves.

36 BOE: Other Mathematical Extended Essay

36.1 Overview

An essay on a topic related to mathematics may be offered for examination at Part B as a double unit. It is equivalent to a 32-hour lecture course. Generally, students will have 6 hours of supervision distributed over Michaelmas and Hilary terms. In addition there are lectures on writing mathematics and using LaTeX in Michaelmas and Hilary terms. See the lecture list for details.

Students considering offering an essay should read the *Guidance Notes on Extended Essays and Dissertations in Mathematics* available at:

`https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/projects`

Application Students must apply to the Mathematics Projects Committee for approval of their proposed topic in advance of beginning work on their essay. Proposals should be addressed to the Chair of the Projects Committee, c/o Undergraduate Studies Administrator, Room S0.15, Mathematical Institute and are accepted from the end of Trinity Term. All proposals must be received before 12noon on Friday of Week 0 of Michaelmas Full Term. The application form is available at `https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/projects`.

Once a title has been approved, it may only be changed by approval of the Chair of the Projects Committee.

Assessment Each project is independently double-marked, normally by the project supervisor and one other assessor. The two marks are then reconciled to give the overall mark awarded. The reconciliation of marks is overseen by the examiners and follows the department's reconciliation procedure (see `https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/projects`).

Submission An electronic copy of your dissertation should be submitted via the Mathematical Institute website to arrive no later than **12 noon on Monday of week 1, Trinity Term 2024**. Further details may be found in the *Guidance Notes on Extended Essays and Dissertations in Mathematics*.

37 An Introduction to LaTeX

37.1 General Prerequisites

There are no prerequisites. The course is mainly intended for students writing a Part B Extended Essay or a Part C Dissertation but any students are welcome to attend the two lectures given in Michaelmas Term. Note that there is no assessment associated with this course, nor credit for attending the course.

37.2 Overview

This short lecture series provides an introduction to LaTeX.

LaTeX is a markup language, released by Donald Knuth in 1984 and freely sourced, for the professional typesetting of mathematics. (It is based on the earlier TeX released in 1978.) A markup language provides the means for rendering text in various ways - such as bold, italicized or Greek symbols - with the main focus of LaTeX being the rendering of mathematics so that even complicated expressions involving equations, integrals and matrices and images can be professionally typeset.

37.3 Learning Outcomes

Following these introductory lectures, a student should feel comfortable writing their own LaTeX documents, and producing professionally typset mathematics. The learning curve to producing a valid LaTeX document is shallow, and students will further become familiar will some of the principal features of LaTeX such as chapters, item lists, typesetting mathematics, including equations, tables, bibliographies and images. Then, with the aid of a good reference manual, a student should feel comfortable researching out for themselves further features and expanding their LaTeX vocabulary

37.4 Reading List

The Department has a page of LaTeX resources at `https://www.maths.ox.ac.uk/members/it/faqs/latex` which has various free introductory guides to LaTeX.