Part C Mathematics 2024-25

July 16, 2024

Contents

1 Foreword

The confirmed synopses for Part C 2024-25 will be available on the course management portal `https://courses.maths.ox.ac.uk/` before the start of Michaelmas Term 2024.

Honour School of Mathematics - Units

See the current edition of the *Examination Regulations* (`https://examregs.admin.ox.ac.uk/`) for the full regulations governing these examinations. Examination Conventions can be found at: `https://www.maths.ox.ac.uk/members/students/undergraduate-courses/examinations-assessments/examination-conventions`.

In the unlikely event that any course receives a very low registration we may offer this course as a reading course (this would include some lectures but fewer classes).

Students staying on to take Part C will take a minimum of eight units and a maximum of ten units. **These 8-10 units must include a CCD Dissertation on a Mathematical Topic (which will count as a double Maths Department Unit) or a COD Dissertation on the History of Mathematics (which will count as a double unit from the "Other Units" list). One unit is the equivalent of a 16 hour lecture course.** The equivalent of four units must be taken from the schedule of "Mathematics Department units" (which include the CCD dissertation and, if applicable, a dissertation offered by the Statistics department). Up to four units may be taken from the schedule of "Other Units" with not more than two from each category (Statistics options, Computer Science options, Other options).

Most Mathematics Department lecture courses are independently available as units, the exception being:

- C7.1 Theoretical Physics - this is available as a double-unit only.

All the units described are "M-Level".

Mathematics Department Units

- C1.1 Model Theory (MT)
- C1.2 Godel's Incompleteness Theorem (HT)
- C1.3 Analytic Topology (MT)
- C1.4 Axiomatic Set Theory (HT)
- C2.2 Homological Algebra (MT)
- C2.3 Representation Theory of Semisimple Lie Algebras (HT)
- C2.4 Infinite Groups (MT)
- C2.5 Non-Commutative Rings (HT)
- C2.6 Introduction to Schemes (HT)
- C2.7 Category Theory (MT)

- C3.1 Algebraic Topology (MT)
- C3.2 Geometric Group Theory (HT)
- C3.3 Differentiable Manifolds (MT)
- C3.4 Algebraic Geometry (MT)
- C3.5 Lie Groups (MT)
- C3.6 Modular Forms (MT)
- C3.7 Elliptic Curves (HT)
- C3.8 Analytic Number Theory (HT)
- C3.9 Computational Algebraic Topology (HT)
- C3.10 Additive Combinatorics (MT)
- C3.11 Riemannian Geometry (HT)
- C3.12 Low-dimensional Topology and Knot Theory (HT)
- C4.1 Further Functional Analysis (MT)
- C4.3 Functional Analytic Methods for PDEs (MT)
- C4.4 Hyperbolic Equations (HT)
- C4.6 Fixed Point Methods for Nonlinear PDEs (HT)
- C4.9 Optimal Transport & Partial Differential Equations (MT)
- C5.1 Solid Mechanics
- C5.2 Elasticity and Plasticity (MT)
- C5.4 Networks (MT)
- C5.5 Perturbation Methods (MT)
- C5.6 Applied Complex Variables (HT)
- C5.7 Topics in Fluid Mechanics (MT)
- C5.11 Mathematical Geoscience (MT)
- C5.12 Mathematical Physiology (MT)
- C6.1 Numerical Linear Algebra (MT)
- C6.2 Continuous Optimisation (HT)
- C6.5 Theories of Deep Learning (MT)
- C7.1 Theoretical Physics (MT/HT)

- C7.4 Introduction to Quantum Information (HT)
- C7.5 General Relativity I (MT)
- C7.6 General Relativity II (HT)
- C7.7 Random Matrix Theory (HT)
- C8.1 Stochastic Differential Equations (MT)
- C8.2 Stochastic Analysis and PDEs (HT)
- C8.3 Combinatorics (MT)
- C8.4 Probabilistic Combinatorics (HT)
- C8.6 Limit Theorems and Large Deviations in Probability (HT)
- C8.7 Optimal Control (HT)
- CCD Dissertations on a Mathematical Topic [double unit] (MT/HT)

The Guide to Mathematics Options at Part C can be found at `https://www.maths.ox.ac.uk/system/files/attachments/Guide%20to%20Options%20at%20Part%20C_1.pdf`.

Other Units

1. **Statistics Options**

   Students in Part C may take the units below, which are drawn from Part C of the Honour School of Mathematics and Statistics.

   The Statistics units available are as follows:

   - SC1 Stochastic Models in Mathematical Genetics (MT)
   - SC2 Probability and Statistics for Network Analysis (MT)
   - SC4 Advanced Topics in Statistical Machine Learning (HT)
   - SC5 Advanced Simulation Methods (HT)
   - SC6 Graphical Models (MT)
   - SC7 Bayes Methods (HT)
   - SC8 Topics in Computational Biology (HT)
   - SC9 Probability on Graphs and Lattices (MT)
   - SC10 Algorithmic Foundations of Learning (MT)
   - Climate Statistics (HT)

   For full details of these units see the syllabus and synopses for Part C of the Honour School Mathematics and Statistics, which are available on the web at `https://www.stats.ox.ac.uk/bammath-mathematics-and-statistics-student-resources`.

2. **Computer Science Options**

   Students in Part C may apply to take any unit from the Part C of the Honour School of Mathematics and Computing. The Department of Computer Science will provide further information regarding how to apply for these courses. Students can expect to hear the outcome of their application early in Michaelmas 2024.

   Please note that some courses will be examined by mini-project (as for MSc students). Mini-projects will be handed out to candidates on the last Monday or Friday of the term in which the subject is being taught, and you will have to hand it in to the Exam Schools by noon on Monday of Week 0 of the following term. The mini-project will be designed to be completed in about four to five days. It will include some questions that are more open-ended than those on a standard sit-down exam. The work you submit should be your own work, and include suitable references.

   Please note that the Computer Science courses in Part C are 50% bigger than those in earlier years, i.e. for each Computer Science course in the 3rd year undergraduates are expected to undertake about 10 hours of study per week, but 4th year courses will each require about 15 hours a week of study. Lecturers are providing this extra work in a variety of ways, e.g. some will give 16 lectures with extra reading, classes and/or practicals, whereas others will be giving 24 lectures, and others still will be doing something in between. Students will need to look at each synopsis for details on this.

   For full details of all units offered see the Department of Computer Science's website `http://www.cs.ox.ac.uk/teaching/courses/`.

3. **Other Options**

   - COD Dissertations on the History of Mathematics [double unit] (MT/HT)

Registration for Part C courses 2024-25

Students will be asked to register for the options they intend to take by the end of week 11, Trinity Term 2024. It is helpful if their registration is as accurate as possible as the data is used to make arrangements for teaching resources. Towards the start of the academic year students will be given the opportunity to make edits to their course registration. Students will then be asked to sign up for classes via the Course Materials site, otherwise known as Moodle (`https://courses.maths.ox.ac.uk/`) at the start of Michaelmas Term 2024. Further information about this will be sent via email before the start of term.

Every effort will be made when timetabling lectures to ensure that mathematics lectures do not clash. However, because of the large number of options in Part C this may sometimes be unavoidable. The timing of lectures for a course taught by another faculty will usually be set by that faculty and the Mathematical Institute has little control over the arrangements. In the event of clashes being necessary, then students will be notified of the clashes by email and in any case options will only be allowed to clash when the take-up of both options is unlikely or inadvisable.

2 C1.1 Model Theory

2.1 General Prerequisites

This course presupposes basic knowledge of First Order Predicate Calculus up to and including the Soundness and Completeness Theorems. A familiarity with (at least the statement of) the Compactness Theorem would also be desirable.

2.2 Overview

The course deepens a student's understanding of the notion of a mathematical structure and of the logical formalism that underlies every mathematical theory, taking B1 Logic as a starting point. Various examples emphasise the connection between logical notions and practical mathematics.

The concepts of completeness and categoricity will be studied and some more advanced technical notions, up to elements of modern stability theory, will be introduced.

2.3 Learning Outcomes

Students will have developed an in depth knowledge of the notion of an algebraic mathematical structure and of its logical theory, taking B1 Logic as a starting point. They will have an understanding of the concepts of completeness and categoricity and more advanced technical notions.

2.4 Synopsis

Structures. The first-order language for structures. The Compactness Theorem for first-order logic. Elementary embeddings. Loewenheim-Skolem theorems. Preservation theorems for substructures. Model Completeness. Quantifier elimination.

Categoricity for first-order theories. Types and saturation. Omitting types. The Ryll Nardzewski theorem characterizing aleph-zero categorical theories. Theories with few types. Ultraproducts.

2.5 Reading List

1. C.C. Chang and H. Jerome Keisler, *Model Theory* (Third Edition (Dover Books on Mathematics)Paperback)

2. Tent and Ziegler, *A Course in Model Theory*, Cambridge University Press, April 2012

3 C1.2 Gödel's Incompleteness Theorems

3.1 General Prerequisites

This course presupposes knowledge of first-order predicate logic up to and including soundness and completeness theorems for a formal system of first-order predicate logic (B1.1 Logic).

3.2 Overview

The starting point is Gödel's mathematical sharpening of Hilbert's insight that manipulating symbols and expressions of a formal language has the same formal character as arithmetical operations on natural numbers. This allows the construction for any consistent formal system containing basic arithmetic of a 'diagonal' sentence in the language of that system which is true but not provable in the system. By further study we are able to establish the intrinsic meaning of such a sentence. These techniques lead to a mathematical theory of formal provability which generalizes the earlier results. We end with results that further sharpen understanding of formal provability.

3.3 Learning Outcomes

Understanding of arithmetization of formal syntax and its use to establish incompleteness of formal systems; the meaning of undecidable diagonal sentences; a mathematical theory of formal provability; precise limits to formal provability and ways of knowing that an unprovable sentence is true.

3.4 Synopsis

Gödel numbering of a formal language; the diagonal lemma. Expressibility in a formal language. The arithmetical undefinability of truth in arithmetic. Formal systems of arithmetic; arithmetical proof predicates. Σ_0-completeness and Σ_1-completeness. The arithmetical hierarchy. ω-consistency and 1-consistency; the first Gödel incompleteness theorem. Separability; the Rosser incompleteness theorem. Adequacy conditions for a provability predicate. The second Gödel incompleteness theorem; Löb's theorem. Provable Σ_1-completeness. The ω-rule. The system GL for provability logic. The fixed point theorem for GL. The Bernays arithmetized completeness theorem; undecidable Δ_2-sentences of arithmetic.

3.5 Reading List

Lecture notes for the course.

3.6 Further Reading

1. Raymond M. Smullyan, *Gödel's Incompleteness Theorems* (Oxford University Press, 1992).

2. George S. Boolos and Richard C. Jeffrey, *Computability and Logic* (3rd edition, Cambridge University Press, 1989), Chs 15, 16, 27 (pp 170-190, 268-284).

3. George Boolos, *The Logic of Provability* (Cambridge University Press, 1993).

4 C1.3 Analytic Topology

4.1 General Prerequisites

Part A Topology, including the notions of a topological space, a continuous function and a basis for a topology. A basic knowledge of Set Theory, including cardinal arithmetic, ordinals and the Axiom of Choice, will also be useful.

4.2 Overview

The aim of the course is to present a range of major theory and theorems, both important and elegant in themselves and with important applications within topology and to mathematics as a whole. Central to the course are separation properties and metrization and the general theory of compactness, compactifications and Tychonoff's theorem, one of the most important in all mathematics (with applications across mathematics and in mathematical logic) and computer science.

4.3 Synopsis

Subbases. Products.

Separation axioms. Urysohn's metrization theorem. Paracompactness. Stone's Theorem; that metric spaces are paracompact. Bing's Metrization Theorem. Lindelöf spaces, seperable spaces, first countable spaces.

Filters and ultrafilters. Convergence in terms of filters.

Tychonoff's theorem. Compactifications, in particular the Alexandroff One-Point Compactification and the Stone-Čech Compactification.

Connectedness. Components and quasi-components. Totally disconnected compact spaces.

Stone duality.

4.4 Reading List

1. 1. S. Willard, *General Topology* (Addison-Wesley, 1970), Chs. 1-8.

2. R. Engelking, *General Topology* (Sigma Series in Pure Mathematics, Vol 6, 1989)

5 C1.4 Axiomatic Set Theory

5.1 General Prerequisites

This course presupposes basic knowledge of First Order Predicate Calculus up to and including the Soundness and Completeness Theorems, together with a course on basic set theory, including cardinals and ordinals, the Axiom of Choice and the Well Ordering Principle.

5.2 Overview

Inner models and consistency proofs lie at the heart of modern Set Theory, historically as well as in terms of importance. In this course we shall introduce the first and most important of inner models, Gödel's constructible universe, and use it to derive some fundamental consistency results.

5.3 Synopsis

A review of the axioms of ZF set theory. Absoluteness, the recursion theorem. The Cumulative Hierarchy of sets and the consistency of the Axiom of Foundation as an example of the method of inner models. Levy's Reflection Principle. Gödel's inner model of constructible sets and the consistency of the Axiom of Constructibility ($V = L$). $V = L$ is absolute. The fact that $V = L$ implies the Axiom of Choice. Some advanced cardinal arithmetic. The fact that $V = L$ implies the Generalized Continuum Hypothesis.

5.4 Reading List

For the review of ZF set theory and the prerequisites from Logic:

1. D. Goldrei, *Classic Set Theory* (Chapman and Hall, 1996).
2. K. Kunen, *The Foundations of Mathematics* (College Publications, 2009).

For course topics (and much more):

1. K. Kunen, *Set Theory* (College Publications, 2011) Chapters (I and II).

5.5 Further Reading

1. K. Hrbacek and T. Jech, *Introduction to Set Theory* (3rd edition, M Dekker, 1999).

6 C2.2 Homological Algebra

6.1 General Prerequisites

A3 Rings and Modules is essential: good understanding of modules over fields (aka vector spaces), polynomial rings, the ring of integers, and the ring of integers modulo n; good familiarity with module homomorphisms, submodules, and quotient modules.

6.2 Overview

Homological algebra is one of the most important tools in mathematics with application ranging from number theory and geometry to quantum physics. This course will introduce the basic concepts and tools of homological algebra with examples in module theory and group theory.

6.3 Learning Outcomes

Students will learn about abelian categories and derived functors and will be able to apply these notions in different contexts. They will learn to compute Tor, Ext, and group cohomology and homology.

6.4 Synopsis

Overview of category theory: adjoint functors, limits and colimits, Abelian categories (2 hours)

Chain complexes: complexes of R-modules and in an abelian category, operations on chain complexes, long exact sequences, chain homotopies, mapping cones and cylinders (3 hours)

Derived functors: delta functors, projective and injective resolutions, left and right derived functors, adjoint functors and exactness, balancing Tor and Ext (5 hours)

Tor and Ext: Tor and flatness, Ext and extensions, universal coefficients theorems, Kunneth formula, Koszul resolutions (3 hours)

Group homology and cohomology: definition, basic properties, cyclic groups, interpretation of H^1 and H^2, the Bar resolution (3 hours).

6.5 Reading List

1. Weibel, Charles An introduction to Homological algebra (see Google Books)

7 C2.3 Representation Theory of Semisimple Lie Algebras

7.1 General Prerequisites

B2.3 Lie algebras is recommended, but not required. Results from that course will be used but stated. B2.1 Introduction to Representation Theory is recommended also, by way of a first introduction to ideas of representation theory.

7.2 Overview

The representation theory of semisimple Lie algebras plays a central role in modern mathematics with motivation coming from many areas of mathematics and physics, for example, the Langlands program. The methods involved in the theory are diverse and include remarkable interactions with algebraic geometry, as in the proofs of the Kazhdan-Lusztig and Jantzen conjectures.

The course will cover the basics of finite dimensional representations of semisimple Lie algebras (e.g., the Cartan-Weyl highest weight classification) in the framework of the larger Bernstein-Gelfand-Gelfand category $\mathcal{O}$.

The course will run as a reading course with 8 one-hour meetings during the term. Each week, the students will be required to read a section of the material, starting with the course lecture notes and continuing with the more advanced material from Humphreys' *Representations of semisimple Lie algebras in the BGG category* $\mathcal{O}$ (AMS, 2008).

The course will be assessed by miniprojects.

7.3 Learning Outcomes

The students will have developed a comprehensive understanding of the basic concepts and modern methods in the representation theory of semisimple Lie algebras, including the classification of finite dimensional modules, the classification of objects in category ◯, character formulas, Lie algebra cohomology and resolutions of finite dimensional modules.

7.4 Synopsis

Universal enveloping algebra of a Lie algebra, Poincaré-Birkhoff-Witt theorem, basic definitions and properties of representations of Lie algebras, tensor products.

The example of $sl(2)$: finite dimensional modules, highest weights.

Category $\mathcal{O}$: Verma modules, highest weight modules, infinitesimal characters and Harish-Chandra's isomorphism, formal characters, contravariant (Shapovalov) forms.

Finite dimensional modules of a semisimple Lie algebra: the Cartan-Weyl classification, Weyl character formula, dimension formula, Kostant's multiplicity formula, examples.

Homological algebra: Lie algebra cohomology, Bernstein-Gelfand-Gelfand resolution of finite dimensional modules, Ext groups in category $\mathcal{O}$.

Topics: applications, Bott's dimension formula for Lie algebra cohomology groups, characters of the symmetric group (via Zelevinsky's application of the BGG resolution to Schur-Weyl duality).

7.5 Reading List

1. Course Lecture Notes.

2. J. Bernstein "Lectures on Lie algebras", in *Representation Theory, Complex Analysis, and Integral Geometry* (Springer 2012).

7.6 Further Reading

1. J. Humphreys, *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$* (AMS, 2008).
2. J. Humphreys, *Introduction to Lie algebras and representation theory* (Springer, 1997).
3. W. Fulton, J. Harris, *Representation Theory* (Springer 1991).

8 C2.4 Infinite Groups

8.1 General Prerequisites

Knowledge of the first and second-year algebra courses is helpful but not mandatory; in particular Prelims *M1: Groups and Group Actions*, *A0: Linear Algebra*, and *ASO: Group Theory*. Likewise, the course *B3.5 Topology and Groups* would bring more familiarity and a different viewpoint of the notions treated in this course.

8.2 Overview

The course studies families of infinite groups with a high degree of commutativity (with an emphasis on nilpotent, polycyclic and solvable groups), various natural questions that one can ask about them, and various methods used to answer these questions. These involve, among other things, questions of finite presentability, linearity, torsion and growth.

8.3 Synopsis

Free groups; ping-pong lemma. Finitely generated and finitely presented groups. Residual finiteness and linearity.

Nilpotency, lower and upper central series. Polycyclic groups, length, Hirsch length, Noetherian induction. Solvable groups, derived series. Structure of linear nilpotent and linear solvable groups.

Solvable versus polycyclic: characterization of polycyclic groups as solvable noetherian and as solvable Z-linear groups. Solvable versus nilpotent: the Milnor -Wolf theorem characterizing nilpotent groups as solvable groups with sub-exponential (and in fact polynomial) growth.

8.4 Reading List

1. C. Drutu, M. Kapovich, *Geometric Group Theory*, (AMS, 2018), Chapters 13 and 14.
2. D. Segal, *Polycyclic groups*, (CUP, 2005) Chapters 1 and 2.

3. D. J. S. Robinson, *A course in the theory of groups*, 2nd ed., Graduate texts in Mathematics, (Springer-Verlag, 1995). Chapters 2, 5, 6, 15.

9 C2.5 Non-Commutative Rings

9.1 General Prerequisites

General Prerequisites: All the material of A3 Rings and Modules is essential: Basic properties of rings and modules. Ideals, prime ideals. Principal ideal rings, unique factorization rings, Euclidean rings. Finite fields. Modules over Euclidean rings.

Recommended material: From B2.1 Introduction to Representation Theory: semisimple modules and algebras, the Artin - Wedderburn theorem. From B2.2 Commutative Algebra: Noetherian rings and modules. Hilbert's basis theorem. Krull dimension.

9.2 Overview

This course builds on Algebra 2 from the second year. We will look at several classes of non-commutative rings and try to explain the idea that they should be thought of as functions on "non-commutative spaces". Along the way, we will prove several beautiful structure theorems for Noetherian rings and their modules.

9.3 Learning Outcomes

Students will be able to appreciate powerful structure theorems, and be familiar with examples of non-commutative rings arising from various parts of mathematics.

9.4 Synopsis

Examples of non-commutative Noetherian rings: enveloping algebras, rings of differential operators, group rings of polycyclic groups. Filtered and graded rings. (3 hours)

Jacobson radical in general rings. Jacobson's density theorem. Artin-Wedderburn. (3 hours)

Ore localisation. Goldie's Theorem on Noetherian domains. (3 hours)

Minimal prime ideals and dimension functions. Rees rings and good filtrations. (3 hours)

Bernstein's Inequality and Gabber's Theorem on the integrability of the characteristic variety. (4 hours)

9.5 Reading List

1. K.R. Goodearl and R.B. Warfield, *An Introduction to Noncommutative Noetherian Rings* (CUP, 2004).

9.6 Further Reading

1. M. Atiyah and I. MacDonald, *Introduction to Commutative Algebra* (Westview Press, 1994).

2. S.C. Coutinho, *A Primer of Algebraic D-modules* (CUP, 1995).

3. J. Björk, *Analytic D-Modules and Applications* (Springer, 1993).

10 C2.6 Introduction to Schemes - Draft

10.1 General Prerequisites

B2.2 Commutative Algebra is essential. *C2.2 Homological Algebra* is highly recommended and *C2.7 Category Theory* is recommended but the necessary material from both courses can be learnt during the course. *C3.4 Algebraic Geometry* is strongly recommended but not technically necessary. *C3.1 Algebraic Topology* contains many homological techniques also used in this course.

10.2 Overview

Scheme theory is the foundation of modern algebraic geometry, whose origins date back to the work from the 1950s and 1960s by Jean-Pierre Serre and Alexander Grothendieck. It unifies algebraic geometry with algebraic number theory. This unification has led to proofs of important conjectures in number theory such as the Weil conjecture by Deligne and the Mordell conjecture by Faltings.

This course will cover the basics of the theory of schemes.

10.3 Learning Outcomes

Students will have developed a thorough understanding of the basic concepts and methods of scheme theory. They will be able to work with affine and projective schemes, as well as with (quasi-)coherent sheaves and their cohomology groups.

10.4 Synopsis

The Spec of a ring, Zariski topology, comparison with classical algebraic geometry.

Pre-sheaves and stalks, sheaves, sheafification. The abelian category of sheaves of abelian groups on a topological space. Direct and inverse images of sheaves. Sheaves defined on a topological basis.

Ringed spaces and morphisms of ringed spaces. Affine schemes, construction of the structure sheaf, the equivalence of categories defined by Spec.

Schemes, closed subschemes. Global sections. The functor of points.

Properties of schemes: (locally) Noetherian, reduced, irreducible, and integral schemes. Properties of morphisms of schemes: finite type, open/closed immersions, flatness. Simple examples of flat families of schemes arising from deformations.

Gluing sheaves. Gluing schemes. Affine and projective n-space viewed as schemes.

Products, coproducts and fiber products in category theory. Existence of products of schemes. Fibers and pre-images of morphisms of schemes. Base change.

Further properties of morphisms of schemes: separated, universally closed, and proper morphisms. Projective n-space and projective morphisms. Abstract varieties. Complete varieties. Scheme structure on a closed subset of a scheme.

Sheaves of modules. Vector bundles and coherent sheaves. The abelian category of sheaves of modules over a scheme. Pull-backs.

Quasi-coherent sheaves. Gluing sheaves of modules. Classification of (quasi-)coherent sheaves on Spec of a ring.

Čech cohomology. Vanishing of higher cohomology groups of quasi-coherent sheaves on affine schemes. Independence of Čech cohomology on the choice of open cover. Line bundles, examples on projective n-space.

Sheaf cohomology. Acyclic resolutions. Comparison of sheaf cohomology and Čech cohomology.

Brief discussion of (quasi-)coherent sheaves on projective n-space, graded modules, and Proj of a graded ring.

10.5 Reading List

1. Robin Hartshorne, *Algebraic Geometry.*
2. Ravi Vakil, *Foundations of Algebraic Geometry*, online notes on the website of Stanford University (open access)
3. Geir Ellingsrud and John Christian Ottem, *Introduction to schemes*, Available online (notes in progress): `https://www.uio.no/studier/emner/matnat/math/MAT4215/data/masteragbook-2023.pdf`

10.6 Further Reading

1. David Mumford, *The Red Book of Varieties and Schemes.*
2. David Eisenbud and Joe Harris, *The Geometry of Schemes.*
3. George R. Kempf, *Algebraic Varieties.*
4. Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford University Press, 2002.

11 C2.7 Category Theory

11.1 General Prerequisites

There are no essential prerequisites, but familiarity with the basic theory of groups, rings, vector spaces, modules, and topological spaces would be very useful. Other topics such as Algebraic Geometry, Algebraic Topology, Homological Algebra and Representation Theory are relevant. Category Theory also has links with Logic and Set Theory, but this course will not stress these.

11.2 Overview

Category theory brings together many areas of pure mathematics (and also has close links to logic and to computer science). It is based on the observation that many mathematical topics can be unified and simplified by using descriptions in terms of diagrams of arrows; the arrows represent functions of suitable types. Moreover, many constructions in pure mathematics can be described in terms of 'universal properties' of such diagrams.

The aim of this course is to provide an introduction to category theory using a host of familiar examples, to explain how these examples fit into a categorical framework, and to use categorical ideas to make new constructions.

11.3 Learning Outcomes

Students will have developed a thorough understanding of the basic concepts and methods of category theory. They will be able to work with commutative diagrams, naturality and universality properties and adjoint functors, and to apply categorical ideas and methods in a wide range of areas of mathematics.

11.4 Synopsis

Introduction: universal properties in linear and multilinear algebra.

Categories, functors, natural transformations. Examples including categories of sets, groups, rings, vector spaces and modules, topological spaces. Groups, monoids and partially ordered sets as categories. Opposite categories and the principle of duality. Covariant, contravariant, faithful and full functors. Equivalences of categories.

Adjoints: definition and examples including free and forgetful functors and abelianisations of groups. Adjunctions via units and counits, adjunctions via initial objects.

Representables: definitions and examples including tensor products. The Yoneda lemma and applications.

Limits and colimits, including products, equalizers, pullbacks and pushouts. Monics and epics. Interaction between functors and limits.

Monads and comonads, algebras over a monad, Barr-Beck monadicity theorem (proof not examinable). The category of affine schemes as the opposite of the category of commutative rings.

Rudiments of higher category theory (non-examinable).

11.5 Reading List

1. T. Leinster, *Basic category theory*, (CUP, 2014) Chapters 1-5, available online arXiv:1612.09375.
2. E. Riehl, *Category theory in context* (Dover, 2016) Chapters 1-5, available online www.math.jhu.edu/ eriehl/context.pdf.

11.6 Further Reading

1. S. Awodey, *Category theory*, Oxford Logic Guides (OUP, 2010).
2. D. Eisenbud, J. Harris, *The geometry of schemes.*
3. S. Lang, *Linear algebra* 2nd edition, (Addison Wesley, 1971), Chapter XIII, out of print but may be available in college libraries.
4. J. Lurie, *Higher Topos Theory*, Princeton University Press, 2009), Chapter 1.
5. S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed., (Springer, 1998).
6. D.G. Northcott, *Multilinear algebra* (CUP, reissued 2009).
7. E. Riehl, *Categorical Homotopy theory* (CUP, 2014), available online at www.math.jhu.edu/ eriehl/cathtpy.p

12 C3.1 Algebraic Topology

12.1 General Prerequisites

A3 Rings and Modules is essential, in particular a solid understanding of groups, rings, fields, modules, homomorphisms of modules, kernels and cokernels, and classification of finitely generated abelian groups. A5 Topology is essential, in particular a solid understanding of topological spaces, connectedness, compactness, and classification of compact surfaces. B3.5 Topology and Groups is helpful but not necessary, in particular the notion of homotopic maps, homotopy equivalences, and fundamental groups will be recalled during the course. There will be little mention of homotopy theory in this course as the focus will be instead on homology and cohomology. It is recommended, but not required, that students take C2.2 Homological Algebra concurrently.

12.2 Overview

Homology theory is a subject that pervades much of modern mathematics. Its basic ideas are used in nearly every branch, pure and applied. In this course, the homology groups of topological spaces are studied. These powerful invariants have many attractive applications. For example we will prove that the dimension of a vector space is a topological invariant and the fact that 'a hairy ball cannot be combed'.

12.3 Learning Outcomes

At the end of the course, students are expected to understand the basic algebraic and geometric ideas that underpin homology and cohomology theory. These include the cup product and Poincaré Duality for manifolds. They should be able to choose between the different homology theories and to use calculational tools such as the Mayer-Vietoris sequence to compute the homology and cohomology of simple examples, including projective spaces, surfaces, certain simplicial spaces and cell complexes. At the end of the course, students should also have developed a sense of how the ideas of homology and cohomology may be applied to problems from other branches of mathematics.

12.4 Synopsis

Brief introduction to categories and functors. Applications of homology theory: Invariance of dimension, Brower fixed point theorem.

Chain complexes of free Abelian groups and their homology. Short exact sequences. of chain complexes, the induced long exact sequence in homology, and naturality. The snake lemma, the five lemma, splitting properties for short exact sequences.

Simplicial homology via Delta complexes.

Singular homology of topological spaces, and functoriality. Relative homology. Chain homotopies, homotopy equivalences. Homotopy invariance and excision (details of proofs not examinable). Retractions, deformation retractions, quotients.

Mayer-Vietoris Sequence. Wedge sums, cones, suspensions, connected sums.

Degree of a self-map of a sphere. Application: the hairy ball theorem.

Cell complexes and cellular homology. Equivalence of simplicial, cellular and singular homology.

Cochains and cohomology of spaces. Cup products.

Künneth Theorem (without proof). Euler characteristic. Ext and Tor groups via free resolutions. (Co)homology with different coefficients. The Universal Coefficient Theorem (proof not examinable).

Topological manifolds and orientability. The fundamental class of an orientable, closed manifold and the degree of a map between manifolds of the same dimension. Poincaré duality (proof not examinable). Manifolds with boundary and Poincaré-Lefschetz duality (proof not examinable). Brief discussion of locally finite homology, and cohomology with compact supports. Cap product.

Alexander duality. Applications: knot complements, Jordan curve theorem.

12.5 Reading List

1. A. Hatcher, *Algebraic Topology* (Cambridge University Press, 2001). Chapters 2 and 3.

2. G. Bredon, *Topology and Geometry* (Springer, 1997). Chapters 4 and 5.

3. J. Vick, *Homology Theory*, Graduate Texts in Mathematics 145 (Springer, 1973).

12.6 Further Reading

1. W.S. Massey, *A Basic Course in Algebraic Topology*, (Springer, GTM 127, 1991).
2. P. May, *A Concise Course in Algebraic Topology* (University of Chicago Press, 1999).
3. J. Davis and P. Kirk, *Lecture Notes in Algebraic Topology* (AMS, 2001).

13 C3.2 Geometric Group Theory

13.1 General Prerequisites

Knowledge of the material in the courses *B3.5 Topology and Groups* and *C2.4 Infinite Groups* is helpful but not mandatory, as it would only bring more familiarity and a different viewpoint of some of the notions treated in this course.

13.2 Overview

The aim of this course is to introduce fundamental methods of geometric group theory, used to investigate infinite groups. It focuses on "large groups", that is, groups that are similar to free non-abelian groups. The two main methods used are the study of actions of such groups on metric spaces, and the investigation of some algebraic problems by designing appropriate algorithms.

The course begins with an introduction to presentations and the list of problems of Max Dehn. It continues with the study of actions of groups on trees and the decomposition of infinite groups into simpler building blocks that such an action can yield. It provides a structural study of fundamental groups of graphs of groups.

The second part of the course focuses on modern geometric techniques and introduces a few key results from the theory of Gromov hyperbolic groups.

13.3 Synopsis

Free groups. Group presentations. Dehn's problems. Residually finite groups.

Group actions on trees. Amalgams, HNN-extensions, graphs of groups, subgroup theorems for groups acting on trees.

Quasi-isometries. Hyperbolic groups. Solution of the word and conjugacy problem for hyperbolic groups.

If time allows: Small Cancellation Groups, Stallings Theorem, Boundaries.

13.4 Reading List

1. J.P. Serre, *Trees* (Springer Verlag, 2012 or any earlier edition).

2. C. Drutu, M. Kapovich, *Geometric Group Theory*, (AMS, 2018), Chapters 7,8,11.

3. M. Bridson, A. Haefliger, *Metric Spaces of Non-positive Curvature, Part III* (Springer, 1999), Chapters I.8, III.H.1, III. Γ5.

13.5 Further Reading

1. H. Short *et al.*, 'Notes on word hyperbolic groups', Group Theory from a Geometrical Viewpoint, Proc. ICTP Trieste (eds E. Ghys, A. Haefliger, A. Verjovsky, World Scientific 1990), available online at: `http://www.cmi.univ-mrs.fr/~hamish/`

2. C.F. Miller, *Combinatorial Group Theory*, notes: `http://www.ms.unimelb.edu.au/~cfm/notes/cgt-notes.pdf`

3. P. de la Harpe, *Topics in Geometric Group Theory*, (University of Chicago Press, 2000).

4. G. Baumslag, *Topics in Combinatorial Group Theory* (Birkhauser, 1993).

5. R. Lyndon, P. Schupp, *Combinatorial Group Theory* (Springer, 2001).

6. W. Magnus, A. Karass, D. Solitar,*Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations* (Dover Publications, 2004).

14 C3.3 Differentiable Manifolds

14.1 General Prerequisites

A5: Topology and *ASO: Multidimensional Analysis and Geometry* are strongly recommended. (Notions of Hausdorff, open covers, smooth functions on $\mathbf{R}^n$ will be used without further explanation.) Useful but not essential: *B3.2 Geometry of Surfaces.*

14.2 Overview

A manifold is a space such that small pieces of it look like small pieces of Euclidean space. Thus a smooth surface, the topic of the Geometry of Surfaces course, is an example of a (2-dimensional) manifold.

Manifolds are the natural setting for parts of classical applied mathematics such as mechanics, as well as general relativity. They are also central to areas of pure mathematics such as topology and certain aspects of analysis.

In this course we introduce the tools needed to do analysis on manifolds. We prove a very general form of Stokes' Theorem which includes as special cases the classical theorems of Gauss, Green and Stokes. We also introduce the theory of de Rham cohomology, which is central to many arguments in topology.

14.3 Learning Outcomes

The candidate will be able to manipulate with ease the basic operations on tangent vectors, differential forms and tensors both in a local coordinate description and a global coordinate-free one; have a knowledge of the basic theorems of de Rham cohomology and some simple examples of their use; know what a Riemannian manifold is and what geodesics are.

14.4 Synopsis

Smooth manifolds and smooth maps. Tangent vectors, the tangent bundle, induced maps. Vector fields and flows, the Lie bracket and Lie derivative.

Exterior algebra, differential forms, exterior derivative, Cartan formula in terms of Lie derivative. Orientability. Partitions of unity, integration on oriented manifolds.

Stokes' theorem. De Rham cohomology. Applications of de Rham theory including degree.

Riemannian metrics. Isometries. Geodesics.

14.5 Reading List

1. M. Spivak, *Calculus on Manifolds*, (W. A. Benjamin, 1965).

2. M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Vol. 1, (1970).

3. W. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*, 2nd edition, (Academic Press, 1986).

4. M. Berger and B. Gostiaux, *Differential Geometry: Manifolds, Curves and Surfaces.* Translated from the French by S. Levy, (Springer Graduate Texts in Mathematics, 115, Springer–Verlag (1988)) Chapters 0-3, 5-7.

5. F. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, (Springer Graduate Texts in Mathematics, 1994).

6. D. Barden and C. Thomas, *An Introduction to Differential Manifolds.* (Imperial College Press, London, 2003.)

15 C3.4 Algebraic Geometry

15.1 General Prerequisites

A3 Rings and Modules and B2.2 Commutative Algebra are essential. Noetherian rings, the Noether normalisation lemma, integrality, the Hilbert Nullstellensatz and dimension theory will play an important role in the course. B3.3 Algebraic Curves is useful but not essential. Projective spaces and homogeneous coordinates will be defined in C3.4, but a working knowledge of them would be useful. There is some overlap of topics, as B3.3 studies the algebraic geometry of one-dimensional varieties. Courses closely related to C3.4 include C2.2 Homological Algebra, C2.7 Category Theory, C3.7 Elliptic Curves, C2.6 Introduction to Schemes; and partly related to: C3.1 Algebraic Topology, C3.3 Differentiable Manifolds, C3.5 Lie Groups.

15.2 Overview

Algebraic geometry is the study of algebraic varieties: an algebraic variety is roughly speaking, a locus defined by polynomial equations. One of the advantages of algebraic geometry is that it is purely algebraically defined and applied to any field, including fields of finite characteristic. It is geometry based on algebra rather than calculus, but over the real or complex numbers it provides a rich source of examples and inspiration to other areas of geometry.

15.3 Synopsis

Affine algebraic varieties, the Zariski topology, morphisms of affine varieties. Irreducible varieties.

Projective space. Projective varieties, affine cones over projective varieties. The Zariski topology on projective varieties. The projective closure of affine variety. Morphisms of projective varieties. Projective equivalence.

Veronese morphism: definition, examples. Veronese morphisms are isomorphisms onto their image; statement, and proof in simple cases. Subvarieties of Veronese varieties. Segre maps and products of varieties.

Coordinate rings. The geometric form of Hilbert's Nullstellensatz. Correspondence between affine varieties (and morphisms between them) and finitely generated reduced K-algebras (and morphisms between them). Graded rings and homogeneous ideals. Homogeneous coordinate rings.

Discrete invariants of projective varieties: degree, dimension, Hilbert function. Statement of theorem defining Hilbert polynomial.

Quasi-projective varieties, and morphisms between them. The Zariski topology has a basis of affine open subsets. Rings of regular functions on open subsets and points of quasi-projective varieties. The ring of regular functions on an affine variety is the coordinate ring. Localisation and relationship with rings of regular functions.

Tangent space and smooth points. The singular locus is a closed subvariety. Algebraic re-formulation of the tangent space. Differentiable maps between tangent spaces.

Function fields of irreducible quasi-projective varieties. Rational maps between irreducible varieties, and composition of rational maps. Birational equivalence. Correspondence between dominant rational maps and homomorphisms of function fields. Blow-ups: of affine space at a point, of subvarieties of affine space, and of general quasi-projective varieties along general subvarieties. Statement of Hironaka's Desingularisation Theorem. Every irreducible variety is birational to a hypersurface. Re-formulation of dimension. Smooth points are a dense open subset.

15.4 Reading List

1. Karen E. Smith, *An invitation to algebraic geometry*

15.5 Further Reading

1. M Reid, *Undergraduate Algebraic Geometry*
2. K Hulek, *Elementary Algebraic Geometry*
3. A Gathmann, *Algebraic Geometry lecture notes*, online: `www.mathematik.uni-kl.de/en/agag/members/professors/gathmann/notes/alggeom`
4. Shafarevich, *Basic Algebraic Geometry 1*
5. D Mumford, *The Red Book of Varieties and Schemes*

16 C3.5 Lie Groups

16.1 General Prerequisites

ASO: Group Theory, *A5: Topology* and *ASO: Multidimensional Analysis and Geometry* are all useful but not essential. It would be desirable to have seen notions of derivative of maps from $\mathbf{R}^n$ to $\mathbf{R}^m$, inverse and implicit function theorems, and submanifolds of $\mathbf{R}^n$. Acquaintance with the notion of an abstract manifold would be helpful but not really necessary.

16.2 Overview

The theory of Lie Groups is one of the most beautiful developments of pure mathematics in the twentieth century, with many applications to geometry, theoretical physics and mechanics. The subject is an interplay between geometry, analysis and algebra. Lie groups are groups which are simultaneously manifolds, that is geometric objects where the notion of differentiability makes sense, and the group multiplication and inversion are differentiable maps. The majority of examples of Lie groups are the familiar groups of matrices. The course does not require knowledge of differential geometry: the basic tools needed will be covered within the course.

16.3 Learning Outcomes

Students will have learnt the fundamental relationship between a Lie group and its Lie algebra, and the basics of representation theory for compact Lie groups. This will include a firm understanding of maximal tori and the Weyl group, and their role for representations.

16.4 Synopsis

Brief introduction to manifolds. Classical Lie groups. Left-invariant vector fields, Lie algebra of a Lie group. One-parameter subgroups, exponential map. Homomorphisms of Lie groups and Lie algebras. Ad and ad. Compact connected abelian Lie groups are tori. The Campbell-Baker-Hausdorff series (statement only).

Lie subgroups. Definition of embedded submanifolds. A subgroup is an embedded Lie subgroup if and only if it is closed. Continuous homomorphisms of Lie groups are smooth. Correspondence between Lie subalgebras and Lie subgroups (proved assuming the Frobenius theorem). Correspondence between Lie group homomorphisms and Lie algebra homomorphisms. Ado's theorem (statement only), Lie's third theorem.

Basics of representation theory: sums and tensor products of representations, irreducibility, Schur's lemma. Compact Lie groups: left-invariant integration, complete reducibility. Representations of the circle and of tori. Characters, orthogonality relations. Peter-Weyl theorem (statement only).

Maximal tori. Roots. Conjugates of a maximal torus cover a compact connected Lie group. Weyl group. Reflections. Weyl group of $U(n)$. Representations of a compact connected Lie group are the Weyl-invariant representations of a maximal torus (proof of inclusion only). Representation ring of maximal tori and $U(n)$.

Killing form. Remarks about the classification of compact Lie groups.

16.5 Reading List

1. J. F. Adams, *Lectures on Lie Groups* (University of Chicago Press, 1982).
2. T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups* (Graduate Texts in Mathematics, Springer, 1985).

16.6 Further Reading

1. R. Carter, G. Segal and I. MacDonald, *Lectures on Lie Groups and Lie Algebras* (LMS Student Texts, Cambridge, 1995).
2. W. Fulton, J. Harris, *Representation Theory: A First Course* (Graduate Texts in Mathematics, Springer, 1991).
3. F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups* (Graduate Texts in Mathematics, 1983).

17 C3.6 Modular Forms

17.1 General Prerequisites

Part A Number Theory, Topology and Part B Geometry of Surfaces, Algebraic Curves (or courses covering similar material) are useful but not essential.

17.2 Overview

The course aims to introduce students to the beautiful theory of modular forms, one of the cornerstones of modern number theory. This theory is a rich and challenging blend of methods from complex analysis and linear algebra, and an explicit application of group actions.

17.3 Learning Outcomes

The student will learn about modular curves and spaces of modular forms, and understand in special cases how to compute their genus and dimension, respectively. They will see that modular forms can be described explicitly via their q-expansions, and they will be familiar with explicit examples of modular forms. They will learn about the rich algebraic structure on spaces of modular forms, given by Hecke operators and the Petersson inner product.

17.4 Synopsis

Overview and examples of modular forms. Definition and basic properties of modular forms. Topology of modular curves: a fundamental domain for the full modular group; fundamental domains for subgroups Γ of finite index in the modular group; the compact surfaces X_Γ; explicit triangulations of X_Γ and the computation of the genus using the Euler characteristic formula; the congruence subgroups $\Gamma(\mathrm{N})$,$\Gamma_0(N)$ and $\Gamma_1(N)$; examples of genus computations. Dimensions of spaces of modular forms: general dimension formula (proof non-examinable); the valence formula (proof non-examinable). Examples of modular forms: Eisenstein series in level 1; Ramanujan's Δ-function; some arithmetic applications. The Petersson inner product. Modular forms as functions on lattices: modular forms of level 1 as functions on lattices; Eisenstein series revisited. Hecke operators in level 1: Hecke operators on lattices; Hecke operators on modular forms and their q-expansions; Hecke operators are Hermitian; multiplicity one.

17.5 Reading List

1. F. Diamond and J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Mathematics 228, Springer-Verlag, 2005

2. R.C. Gunning, *Lectures on Modular Forms*, Annals of mathematical studies 48, Princeton University Press, 1962.

3. J.S. Milne, *Modular Functions and Modular Forms*, www.jmilne.org/math/CourseNotes/mf.html

4. J.-P. Serre, *A Course in Arithmetic*, Chapter VII, Graduate Texts in Mathematics 7, Springer-Verlag, 1973.

18 C3.7 Elliptic Curves

18.1 General Prerequisites

It is helpful, but not essential, if students have already taken a standard introduction to algebraic curves and algebraic number theory. For those students who may have gaps in their background, I have placed the file "Preliminary Reading" permanently on the Elliptic Curves webpage, which gives in detail (about 30 pages) the main prerequisite knowledge for the course.

18.2 Overview

Elliptic curves give the simplest examples of many of the most interesting phenomena which can occur in algebraic curves; they have an incredibly rich structure and have been the testing ground for many developments in algebraic geometry whilst the theory is still full of deep unsolved conjectures, some of which are amongst the oldest unsolved problems in mathematics. The course will concentrate on arithmetic aspects of elliptic curves defined over the rationals, with the study of the group of rational points, and explicit determination of the rank, being the primary focus. Using elliptic curves over the rationals as an example, we will be able to introduce many of the basic tools for studying arithmetic properties of algebraic varieties.

18.3 Learning Outcomes

On completing the course, students should be able to understand and use properties of elliptic curves, such as the group law, the torsion group of rational points, and 2-isogenies between elliptic curves. They should be able to understand and apply the theory of fields with valuations, emphasising the p-adic numbers, and be able to prove and apply Hensel's Lemma in problem solving. They should be able to understand the proof of the Mordell-Weil Theorem for the case when an elliptic curve has a rational point of order 2, and compute ranks in such cases, for examples where all homogeneous spaces for descent-via-2-isogeny satisfy the Hasse principle. They should also be able to apply the elliptic curve method for the factorisation of integers.

18.4 Synopsis

Non-singular cubics and the group law; Weierstrass equations.

Elliptic curves over finite fields; Hasse estimate (stated without proof).

p-adic fields (basic definitions and properties).

1-dimensional formal groups (basic definitions and properties).

Curves over p-adic fields and reduction mod p.

Computation of torsion groups over $\mathbb{Q}$; the Nagell-Lutz theorem.

2-isogenies on elliptic curves defined over $\mathbb{Q}$, with a $\mathbb{Q}$-rational point of order 2.

Weak Mordell-Weil Theorem for elliptic curves defined over $\mathbb{Q}$, with a $\mathbb{Q}$-rational point of order 2.

Height functions on Abelian groups and basic properties.

Heights of points on elliptic curves defined over $\mathbb{Q}$; statement (without proof) that this gives a height function on the Mordell-Weil group.

Mordell-Weil Theorem for elliptic curves defined over $\mathbb{Q}$, with a $\mathbb{Q}$-rational point of order 2.

Explicit computation of rank using descent via 2-isogeny.

Factorising integers: Pollard's $(p-1)$ method and the elliptic curve method.

18.5 Reading List

1. J.W.S. Cassels, *Lectures on Elliptic Curves*, LMS Student Texts 24 (Cambridge University Press, 1991).
2. N. Koblitz, *A Course in Number Theory and Cryptography*, Graduate Texts in Mathematics 114, 2nd Edition (Springer, 1994).
3. J.H. Silverman and J. Tate, *Rational Points on Elliptic Curves*, Undergraduate Texts in Mathematics, 2nd Edition (Springer, 2015).
4. J.H. Silverman, *The Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics 106, 2nd Edition (Springer, 2009).

18.6 Further Reading

1. A. Knapp, *Elliptic Curves, Mathematical Notes 40* (Princeton University Press, 1992).
2. G, Cornell, J.H. Silverman and G. Stevans (editors), *Modular Forms and Fermat's Last Theorem* (Springer, 1997).
3. J.H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics 151 (Springer, 1994).

19 C3.8 Analytic Number Theory

19.1 General Prerequisites

Basic ideas of complex analysis. Elementary number theory. Some familiarity with Fourier series will be helpful but not essential.

19.2 Overview

The aim of this course is to study the prime numbers using the famous Riemann ζ-function. In particular, we will study the connection between the primes and the zeros of the ζ-function. We will state the Riemann hypothesis, perhaps the most famous unsolved problem in mathematics, and examine its implication for the distribution of primes. We will prove the prime number theorem, which states that the number of primes less than X is asymptotic to $X/\log X$.

19.3 Learning Outcomes

In addition to the highlights mentioned above, students will gain experience with different types of Fourier transform and with the use of complex analysis.

19.4 Synopsis

Introductory material on primes. Arithmetic functions: Möbius function, Euler's ϕ-function, the divisor function, the σ-function. Multiplicativity. Dirichlet series and Euler products. The von Mangoldt function.

The Riemann ζ-function for $\Re(s) > 1$. Euler's proof of the infinitude of primes. ζ and the von Mongoldt function.

Schwarz functions on $\mathbf{R}$, $\mathbf{Z}$, $\mathbf{R}/\mathbf{Z}$ and their Fourier transforms. *Inversion formulas and uniqueness*. The Poisson summation formula. The meromorphic continuation and functional equation of the ζ-function. Poles and zeros of ζ and statement of the Riemann hypothesis. Basic estimates for ζ.

The classical zero-free region. Proof of the prime number theorem. Implications of the Riemann hypothesis for the distribution of primes.

19.5 Reading List

Full printed notes will be provided for the course, including the non-examinable topics (marked with asterisks above). The following books are relevant to the course.

1. G. H. Hardy and E. M. Wright, *An introduction to the Theory of Numbers* (Sixth edition, OUP 2008). Chapters 16, 17, 18.

2. H. Davenport, *Multiplicative number theory* (Third Edition, Springer Graduate texts 74), selected parts of the first half.

3. M. du Sautoy, *Music of the primes* (this is a popular book which could be useful background reading for the course).

20 C3.9 Computational Algebraic Topology

20.1 General Prerequisites

Some familiarity with the main concepts from algebraic topology, homological algebra and category theory will be helpful.

20.2 Overview

Ideas and tools from algebraic topology have become more and more important in computational and applied areas of mathematics. This course will provide at the masters level an introduction to the main concepts of (co)homology theory, and explore areas of applications in data analysis.

20.3 Learning Outcomes

Students should gain a working knowledge of homology and cohomology computation for simplicial complexes and sheaves, and improve their geometric intuition. Furthermore, they should gain an awareness of a variety of applications (with an emphasis on data analysis).

20.4 Synopsis

This course consists of two parts:

1. the first part, comprising five weeks, will cover the basics of algebraic topology. Explicitly, the material includes simplicial complexes, geometric realisations and simplicial maps; homotopy equivalence, carriers, nerves and fibres; homology and its computation; exact sequences and the snake lemma; cohomology, cup and cap products, poincare duality.

2. the second part consists of more advanced material. In the last three weeks, this course will examine persistent homology; cellular sheaves and their cohomology; discrete Morse theory.

20.5 Reading List

V. Nanda, Computational Algebraic Topology lecture notes `https://people.maths.ox.ac.uk/nanda/cat/TDANotes.pdf`

See also, U. Tillmann, Lecture notes for CAT 2012, in `http://people.maths.ox.ac.uk/tillmann/CAT.html`

1. G. Carlsson, *Topology and data*, Bulletin A.M.S.46 (2009), 255-308.

2. H. Edelsbrunner, J.L. Harer, *Persistent homology: A survey*, Contemporary Mathematics 452 A.M.S. (2008), 257-282.

3. S. Weinberger, *What is ... Persistent Homology?*, Notices A.M.S. 58 (2011), 36-39.

4. P. Bubenik, J. Scott, *Categorification of Persistent Homology*, Discrete Comput. Geom. (2014), 600-627.

21 C3.10 Additive Combinatorics

21.1 General Prerequisites

There are very few prerequisites for this course. At one point we will state and use Euler's formula for planar graphs. This is covered in B8.5 Graph Theory, but the statement may be understood independently of that course. We will be discussing the notion of entropy, which is introduced in B8.4 Information Theory. However, we will develop what we need from first principles.

21.2 Overview

Additive combinatorics is the study of additive questions about finite sets of integers.

We will begin by proving a famous theorem of Roth: every set of integers with positive density contains three distinct elements in arithmetic progression. This proof uses some basic ideas from Fourier analysis, which we will develop from scratch. Then, we will turn to the corresponding question in the group $(\mathbf{Z}/3\mathbf{Z})^n$, where much stronger bounds are known using algebraic methods.

Next we will look at the structure of finite sets A of integers which are almost closed under addition in the sense that their sumset $A + A := \{a_1 + a_2 : a_1, a_2 \in A\}$ is relatively small. The highlight here is Freiman's theorem, which states that any such set has a precise combinatorial structure known as a generalised progression. The proof once again uses some Fourier analysis as well as a host of other ingredients such as the geometry of numbers, which we will develop from first principles.

After that, we will turn to the corresponding question in vector spaces over finite fields. We will introduce entropy methods and describe how they may be used to prove a rather precise description of sets with small sumset in this setting.

Finally, we will look at instances of the sum-product phenomenon, which says that it is impossible for a finite set of integers to be simultaneously additively- and multiplicatively structured. This section draws from a particularly rich set of other mathematical areas, including graph theory, geometry and analysis, as well as previous sections of the course. Nonetheless, prerequisites will be minimal and we will develop what we need from scratch.

A particular aim of the course will be to give a taster of the very large number of different methods which have been brought to bear on these topics: Fourier analysis, algebraic methods, methods from information theory, graph theory and geometric combinatorics.

21.3 Synopsis

Arithmetic progressions. Basic properties of Fourier transforms. Roth's theorem that every subset of $\{1, \ldots, N\}$ of size at least δN contains three elements in arithmetic progression, provided N is sufficiently large in terms of δ. The Croot-Lev-Pach method and strong bounds for arithmetic progressions in $(\mathbf{Z}/3\mathbf{Z})^n$.

Sumsets and Freiman's theorem. Basic sumset estimates. Additive energy and its relation to sumsets: statement (but not proof) of the Balog-Szemerédi-Gowers theorem. Bohr sets and

Bogolyubov's theorem. Minkowski's second theorem (statement only). Freiman's theorem on sets with small doubling constant. Freiman's lemma on the dimension of sets with small doubling.

Entropy methods and polynomial Freiman-Ruzsa. Basic notions of entropy and entropy analogues of sumset inequalities. The fibring inequality for entropy doubling. Marton's conjecture in characteristic 2. Deduction of the weak polynomial Freiman-Ruzsa conjecture over $\mathbf{Z}$.

Sum-product theorems. The crossing number inequality for graphs. The Szemerédi-Trotter theorem on point-line incidences, and application to prove that either $|A+A|$ or $|A \cdot A|$ has size at least $c|A|^{5/4}$. Proof of Bourgain and Chang's result that either the m-fold sumset $A + A + \cdots + A$ or the m-fold product set $A \cdot A \cdots \cdot A$ has size at least $|A|^{f(m)}$, where $f(m) \to \infty$.

If time allows the course will conclude with a brief non-examinable discussion of Gowers's work on Szemerédi's theorem for progressions of length 4 and longer, which ties together several earlier strands in the course.

21.4 Reading List

Full printed notes will be produced for the course and these will be the primary resource. A number of the topics are very recent and not covered in any textbook. The book *Graph theory and Additive Combinatorics: exploring structure and randomness* by Yufei Zhao is a good resource for the course. The much older book T. Tao and V. Vu *Additive Combinatorics* is also useful.

22 C3.11 Riemannian Geometry

22.1 General Prerequisites

Differentiable Manifolds is required. An understanding of covering spaces will be strongly recommended.

22.2 Overview

Riemannian Geometry is the study of curved spaces and provides an important tool with diverse applications from group theory to general relativity. The surprising power of Riemannian Geometry is that we can use local information to derive global results.

This course will study the key notions in Riemannian Geometry: geodesics and curvature. Building on the theory of surfaces in R^3 in the Geometry of Surfaces course, we will describe the notion of Riemannian submanifolds, and study Jacobi fields, which exhibit the interaction between geodesics and curvature.

We will prove the Hopf–Rinow theorem, which shows that various notions of completeness are equivalent on Riemannian manifolds, and classify the spaces with constant curvature.

The highlight of the course will be to see how curvature influences topology. We will see this by proving the Cartan–Hadamard theorem, Bonnet–Myers theorem and Synge's theorem.

22.3 Learning Outcomes

The candidate will have great familiarity working with Riemannian metrics, the Levi-Civita connection, geodesics and curvature, both in a local coordinate description and using coordinate-free expressions. The candidate will gain understanding of Riemannian submanifolds, Jacobi fields, completeness, and be able to prove and apply fundamental results in the subject, including the theorems of Hopf–Rinow, Cartan–Hadamard, Bonnet–Myers and Synge.

22.4 Synopsis

Riemannian manifolds: basic examples of Riemannian metrics, Levi-Civita connection.

Geodesics: definition, first variation formula, exponential map, minimizing properties of geodesics.

Curvature: Riemann curvature tensor, sectional curvature, Ricci curvature, scalar curvature.

Riemannian submanifolds: examples, second fundamental form, Gauss–Codazzi equations.

Jacobi fields: Jacobi equation, conjugate points.

Completeness: Hopf–Rinow and Cartan–Hadamard theorems

Constant curvature: classification of complete manifolds with constant curvature.

Second variation and applications: second variation formula, Bonnet–Myers and Synge's theorems.

22.5 Reading List

1. M.P. do Carmo, Riemannian Geometry, (Birkhauser, 1992).
2. J.M. Lee, Riemannian Manifolds: An Introduction to Curvature, (Springer, 1997).
3. S. Gallot, D. Hulin and J. Lafontaine, Riemannian Geometry, (Springer, 1987).
4. W. Boothby, An Introduction to Differentiable Manifolds and Riemannian Geometry, 2nd edition, (Academic Press, 1986).

23 C3.12 Low-Dimensional Topology and Knot Theory

23.1 General Prerequisites

B3.5 Topology and Groups (MT) and C3.1 Algebraic Topology (MT) are essential. We will assume working knowledge of the fundamental group, covering spaces, homotopy, homology, and cohomology. B3.2 Geometry of Surfaces (MT) and C3.3 Differentiable Manifolds (MT) are useful but not essential, though some prior knowledge of smooth manifolds and bundles should make the material more accessible.

23.2 Overview

Low-dimensional topology is the study of 3- and 4-manifolds and knots. The classification of manifolds in higher dimensions can be reduced to algebraic topology. These methods fail in dimensions 3 and 4. Dimension 3 is geometric in nature, and techniques from group theory have also been very successful. In dimension 4, gauge-theoretic techniques dominate. This course provides an overview of the rich world of low-dimensional topology that draws on many areas of mathematics. We will explain why higher dimensions are in some sense easier to understand, and review some basic results in 3- and 4-manifold topology and knot theory.

23.3 Learning Outcomes

The students will become acquainted with topological and smooth manifolds. They will master important techniques from Morse theory and learn how to manipulate handle decompositions of manifolds. They will get an idea about the role of the h-cobordism theorem and the Whitney trick in higher-dimensional topology. They will learn a variety of techniques in knot theory, including how to manipulate diagrams using Reidemeister moves, how to derive knot invariants from Seifert surfaces, and how some of these are related to 4-dimensional quantities. They will be able to represent 3-manifolds using Heegaard decompositions, how to write them as sums of prime pieces using normal surface theory, and how to construct 3-manifolds via Dehn surgery and branched double covers along links. Finally, they will be able to represent 4-manifolds using Kirby diagrams and how to determine their homeomorphism type using the intersection form.

23.4 Synopsis

The definition of topological and smooth manifolds. Morse theory, handle decompositions, surgery. Every group can be the fundamental group of a manifold in dimension greater than three. The h-cobordism theorem (without proof) and the Whitney trick. Application: The generalized Poincaré conjecture. Knots and links: Reidemeister moves, Seifert surface and genus, Alexander polynomial, fibred knots, Jones polynomial, prime decomposition, 4-ball genus. 3-manifolds: Heegaard decompositions, unique prime decomposition, loop theorem (without proof), lens spaces, Dehn surgery, branched double cover. 4-manifolds: Kirby calculus, the intersection form, Freedman's and Donaldson's theorems (without proof).

23.5 Reading List

1. John Milnor, *Morse theory*, Ann. of Math. Stud., No. 51, Princeton University Press (1963).
2. John Milnor, *Lectures on the h-cobordism theorem*, Princeton University Press (1965).
3. W. B. Raymond Lickorish, *An introduction to knot theory*, Grad. Texts in Math. 175, Springer-Verlag, New York (1997).
4. Dale Rolfsen, *Knots and links*, Math. Lecture Ser. 7, Publish or Perish, Inc., Houston, TX (1990).
5. Alexandru Scorpan, *The wild world of 4-manifolds*, American Mathematical Society, Providence, RI (2005).

23.6 Further Reading

1. András Juhász, *Differential and low-dimensional topology*, London Math. Soc. Stud. Texts 104, Cambridge University Press (2023).

24 C4.1 Further Functional Analysis

24.1 General Prerequisites

Students wishing to take this course are expected to have a thorough understanding of the basic theory of normed vector spaces (including properties and standard examples of Banach and Hilbert spaces, dual spaces, and the Hahn-Banach theorem) and of bounded linear operators (ideally including the Open Mapping Theorem, the Inverse Mapping Theorem and the Closed Graph Theorem). Some fluency with topological notions such as (sequential) compactness and bases of topological spaces will also be assumed, as will be basic familiarity with the Lebesgue integral. A number of these prerequisites will be reviewed (briefly) during the course, and there will be a document available on the course webpage summarising most of the relevant background material.

24.2 Overview

This course builds on what is covered in introductory courses on Functional Analysis, by extending the theory of Banach spaces and operators. As well as developing general methods that are useful in operator theory, we shall look in more detail at the structure and special properties of "classical" sequence spaces and function spaces.

24.3 Learning Outcomes

By the end of this course, students will be able to:

1. Establish and use both extension and separation versions of the Hahn Banach Theorem, and geometric properties of the norm, to obtain dualities between embeddings and quotients and give characterisations of reflexivity.

2. Work with the weak and weak*-topologies on Banach spaces, establish and use the Banach-Alaoglu theorem, relating this to characterisations of reflexivity, and describe closures in both norm and weaker topologies using annihilators, and preannhilators.

3. Manipulate properties of compact and Fredholm operators on Banach and Hilbert spaces, to establish and use the Fredholm alternative, and obtain spectral theorems for compact operators both in abstract and concrete settings.

24.4 Course Synopsis

Normed vector spaces and Banach spaces. Dual spaces. Direct sums and complemented subspaces. Quotient spaces and quotient operators.

The Baire Category Theorem and its consequences (review).

Hahn-Banach extension and separation theorems. The bidual space. Reflexivity. Completion of a normed vector space.

Weak and weak* topologies. The Banach-Alaoglu theorem. Goldstine's theorem. Equivalence of reflexivity and weak compactness of the closed unit ball. The Schur property of ℓ^1 Weakly compact operators.

Compactness in normed vector spaces. Compact operators. Schauder's theorem on compactness of dual operators. Completely continuous operators.

The Closed Range Theorem. Fredholm theory: Fredholm operators; the Fredholm index; perturbation results; the Fredholm Alternative. Spectral theory of compact operators. The Spectral Theorem for compact self-adjoint operators.

Schauder bases; examples in classical Banach spaces.

24.5 Reading List

1. M. Fabian et al., *Functional Analysis and Infinite-Dimensional Geometry* (Canadian Math. Soc. Springer 2001)

2. N.L. Carothers, *A Short Course on Banach Space Theory* (LMS Student Text CUP 2004).

24.6 Further Reading

1. H. Brezis, *Functional Analysis, Sobolev Spaces and PDEs* (Springer 2011)

2. J. Conway, *A course in Functional Analysis* (Springer 2007)

3. B. Bollobas, *Linear Analysis: An Introductory Course* (CUP 1999)

4. R.E. Megginson *An Introduction to Banach Space Theory* (Springer 1998)

5. W. Rudin *Functional Analysis* (McGraw-Hill 1991)

25 C4.3 Functional Analytic Methods for PDEs

25.1 General Prerequisites

A4 Integration. There will be a 'Users' Guide to Integration' on the subject website and anyone who has not learned Lebesgue Integration can read it up over the summer vacation. In addition some knowledge of functional analysis, in particular Banach spaces and compactness, is useful.

25.2 Overview

The course will introduce some of the modern techniques in partial differential equations that are central to the theoretical and numerical treatments of linear and nonlinear partial differential equations arising in science, geometry and other fields.

It provides valuable background for the Part C courses on Calculus of Variations, Fixed Point Methods for Nonlinear PDEs, and Finite Element Methods.

25.3 Learning Outcomes

Students will learn techniques and results about Lebesgue L^p and Sobolev $W^{1,p}$ Spaces, distributions and weak derivatives, embedding theorems, traces, weak solution to elliptic PDE's, existence, uniqueness, and smoothness of weak solutions.

25.4 Synopsis

Why functional analysis methods are important for PDE's?

Revision of relevant definitions and statements from functional analysis: completeness, separability, compactness, and duality.

Revision of relevant definitions and statements from Lebesgue integration theory: sequences of measurable functions, Lebesgue and Riesz theorems.

Lebesgue spaces L^p: completeness, dense sets, linear functionals and weak convergence.

Distributions and distributional derivatives.

Sobolev spaces $W^{1,p}$: mollifications and weak derivatives, completeness, Friedrichs inequality, star-shaped domains and dense sets, extension of functions with weak derivatives.

Embedding of Sobolev spaces into Lebesgue spaces: Poincaré inequality, Reillich-Kondrachov-Sobolev theorems on compactness.

Traces of functions with weak derivatives.

Dirichlet boundary value problems for elliptic PDE's, Fredholm Alternative (uniqueness implies existence), variational method, spectrum of elliptic differential operators under Dirichlet boundary conditions.

Smoothness of weak solutions: embedding from Sobolev spaces into spaces of Hölder continuous functions, regularity of distributional solutions to elliptic equations with continuous coefficients. Strong solutions to the Dirichlet problem for elliptic differential operators.

25.5 Reading List

1. Lawrence C. Evans, *Partial differential equations*, (Graduate Studies in Mathematics 2004), American Mathematical Society
2. Elliott H. Lieb and Michael Loss, *Analysis*, 2nd Edition, (Graduate Studies in Mathematics 2001), American Mathematical Society

25.6 Further Reading

1. E. Kreyszig, *Introductory Functional Analysis with Applications*, Wiley (revised edition, 1989)
2. P.D. Lax *Functional analysis* (Wiley-Interscience, New York, 2002).
3. J. Rauch, *Partial differential equations*, (Springer-Verlag, New York, 1992).

26 C4.4 Hyperbolic Equations

26.1 General Prerequisites

A good background in Multivariate Calculus and Lebesgue Integration is expected (for instance as covered in the Oxford Prelims and Part A Integration). It would be useful to know some basic Functional Analysis and Distribution Theory; however, this is not strictly necessary as the presentation will be self-contained.

26.2 Overview

We introduce analytical and geometric approaches to hyperbolic equations, by discussing model problems from transport equations, wave equations, and conservation laws. These approaches have been applied and extended extensively in recent research and lie at the heart of the theory of hyperbolic PDEs.

26.3 Synopsis

1. Transport equations and nonlinear first order equations: Method of characteristics, formation of singularities
2. Introduction to nonlinear hyperbolic conservation laws: Discontinuous solutions, Rankine-Hugoniot relation, Lax entropy condition, shock waves, rarefaction waves, Riemann problem, entropy solutions, Lax-Oleinik formula, uniqueness.
3. Linear wave equations: The solution of Cauchy problem, energy estimates, finite speed of propagation, domain of determination, light cone and null frames, hyperbolic rotation and Lorentz vector fields, Sobolev inequalities, Klainerman inequality.

4. Nonlinear wave equations: local well-posedness, weak solutions

If time permits, we might also discuss parabolic approximation (viscosity method), compactness methods, Littlewood-Paley theory, and harmonic analysis techniques for hyperbolic equations/systems (off syllabus - not required for exam)

26.4 Reading List

1. Alinhac, S, *Hyperbolic Partial Differential Equations*, Springer-Verlag: New York, 2009.
2. Evans, L, *Partial Differential Equations*, Second edition. Graduate Studies in Mathematics, 19. American Mathematical Society, 2010
3. John, F, *Partial Differential Equations*, Fourth edition. Applied Mathematical Sciences, 1. Springer-Verlag: New York, 1982

27 C4.6 Fixed Point Methods for Nonlinear PDEs

27.1 General Prerequisites

Basic results on weak derivatives and Sobolev spaces either from B4.3 Distribution Theory or from C4.3 Functional Analytic Methods for PDEs. Some knowledge of functional analysis, mainly notions of Banach spaces, and weak convergence, is useful.

27.2 Overview

This course gives an introduction to the techniques of nonlinear functional analysis with emphasis on the major fixed point theorems and their applications to nonlinear differential equations and variational inequalities, which abound in applications such as fluid and solid mechanics, population dynamics and geometry.

27.3 Learning Outcomes

Besides becoming acquainted with the fixed point theorems of Banach, Brouwer and Schauder, students will see the abstract principles in a concrete context. Hereby they also reinforce techniques from elementary topology, functional analysis, Banach spaces, compactness methods, calculus of variations and Sobolev spaces.

27.4 Synopsis

Examples of nonlinear differential equations and variational inequalities. Contraction Mapping Theorem and applications. Brouwer's fixed point theorem, proof via Calculus of Variations and Null-Lagrangians. Compact operators and Schauder's fixed point theorem. Recap of basic results on Sobolev spaces. Applications of Schauder's fixed point theorem to nonlinear elliptic equations. Variational inequalities and monotone operators. Applications of monotone operator theory to nonlinear elliptic equations (p-Laplacian, stationary Navier-Stokes)

27.5 Reading List

1. Lawrence C. Evans, *Partial Differential Equations,* Graduate Studies in Mathematics (American Mathematical Society, 2004).
2. E. Zeidler, *Nonlinear Functional Analysis I & II* (Springer-Verlag, 1986/89).
3. M. S. Berger, *Nonlinearity and Functional Analysis* (Academic Press, 1977).
4. K. Deimling, *Nonlinear Functional Analysis* (Springer-Verlag, 1985).
5. L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Courant Institute Lecture Notes (American Mathematical Society, 2001).
6. R.E. Showalter, *Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations*, Mathematical Surveys and Monographs, vol.49 (American Mathematical Society, 1997).

28 C4.9 Optimal Transport & Partial Differential Equations

28.1 General Prerequisites

Good command of Part A Integration, Probability and Differential Equations 1 are essential; the main concepts which will be used are the convergence theorems and the theorems of Fubini and Tonelli, and the notions of measurable functions, integrable functions, null sets and L^p spaces. The Cauchy-Lipschitz theory and Picard's theorem proofs will be used. Basic knowledge of random variables, laws, expectations, and independence are needed. A good working knowledge of Part A Core Analysis (metric spaces) is expected. Knowledge of B8.1 Probability, Measure and Martingales will certainly help but it is not essential.

28.2 Overview

This course will serve as an introduction to optimal transportation theory, its application in the analysis of PDE, and its connections to the macroscopic description of interacting particle systems.

28.3 Learning Outcomes

Getting familar with the Monge-Kantorovich problem and transport distances. Derivation of macroscopic models via the mean-field limit and their analysis based on contractivity of transport distances. Dynamic Interpretation and Geodesic convexity. A brief introduction to gradient flows and examples.

28.4 Synopsis

1. Interacting Particle Systems & PDE
    - Granular Flow Models and McKean-Vlasov Equations.
    - Nonlinear Diffusion and Aggregation-Diffusion Equations.
2. Optimal Transportation: The metric side
    - Functional Analysis tools: weak convergence of measures. Prokhorov's Theorem. Direct Method of Calculus of Variations.
    - Monge Problem. Kantorovich Duality.
    - Transport distances between measures: properties. The real line. Probabilistic Interpretation: couplings.
3. Mean Field Limit & Couplings
    - Continuity Equation: measures sliding down a convex valley.
    - Dobrushin approach: derivation of the Aggregation Equation.
    - Boltzmann Equation for Maxwellian molecules: Tanaka Theorem.
4. An Introduction to Gradient Flows
    - Dynamic Interpretation of optimal tranport.
    - McCann's Displacement Convexity: Internal, Interaction and Confinement Energies.
    - Gradient Flows: Differential and metric viewpoints.

28.5 Reading List

1. F. Golse, *On the Dynamics of Large Particle Systems in the Mean Field Limit, Lecture Notes in Applied Mathematics and Mechanics 3.* Springer, 2016.
2. L. C. Evans, *Weak convergence methods for nonlinear partial differential equations. CBMS Regional Conference Series in Mathematics 74*, AMS, 1990.
3. F. Santambrogio, *Optimal Transport for Applied Mathematicians: Calculus of Variations, PDEs, and Modeling, Progress in Nonlinear Differential Equations and Their Applications*, Birkhauser 2015.
4. C. Villani, *Topics in Optimal Transportation*, AMS Graduate Studies in Mathematics, 2003

28.6 Further Reading

1. L. Ambrosio, G. Savare, *Handbook of Differential Equations: Evolutionary Equations*, Volume 3-1, 2007.
2. C. Villani, *Optimal Transport: Old and New*, Springer 2009

29 C5.1 Solid Mechanics

29.1 General Prerequisites

There are no formal prerequisites. In particular it is not necessary to have taken any courses in fluid mechanics or Elasticity and Plasticity, though having done so provides some background in the use of similar concepts. Use is made of (i) elementary linear algebra in (e.g., eigenvalues, eigenvectors and diagonalization of symmetric matrices, and revision of this material, for example from the Prelims Linear Algebra course, is useful preparation); and (ii) some 3D calculus (mainly differentiation of vector-valued functions of several variables).

29.2 Overview

Solid mechanics is a vital ingredient of materials science and engineering, and is playing an increasing role in biology. It has a rich mathematical structure that all physical field theories have adopted. The aim of the course is to derive the basic equations of continuum mechanics (which covers both fluids and solids) and in particular, elasticity theory, the central model of solid mechanics, and give some interesting applications to the behaviour of materials. It is useful to combine this course with C5.2 Elasticity and Plasticity. Taken together the two courses will provide a broad overview of modern solid mechanics, with a variety of approaches.

29.3 Learning Outcomes

Students will learn basic techniques of modern continuum mechanics, such as kinematics of deformation, stress, constitutive equations and the relation between nonlinear and linearized models. The emphasis on the course is on the structure of the models, but some applications are also discussed.

29.4 Synopsis

1. Introduction: one-dimensional elasticity Kinematics, dynamics, balance equations, applications

2. Kinematics Lagrangian and Eulerian descriptions of motion, deformations, vectors and tensors, derivatives of vector and tensor fields, deformation gradients, transformation of volume, surface, line elements, polar decomposition theorems, strain tensors. Examples of deformations

3. Dynamics Balance laws of continuum mechanics (conservation of mass, linear momentum, angular momentum). Stress tensors, Energy balance, Constitutive equations for fluids and solids.

4. Nonlinear elasticity Nonlinear elasticity (frame indifference, constitutive equations, material symmetries, isotropy). Exact solutions in elastostatics: Incompressibility and models of rubber. Universal deformations for compressible materials. Exact

solutions for incompressible materials, e.g. the Rivlin cube, simple shear, inflation of a balloon.

5. Linear Elasticity Linear elasticity as a linearization of nonlinear elasticity. Compatibility conditions. Plne-strain, plane stress solutions, planar waves in elasto-dynamics.

29.5 Reading List

1. Ray Ogden, Nonlinear Elastic Deformations, (Dover, 1997).
2. M. E. Gurtin, A introduction to continuum mechanics, (Academic Press, 1982).

29.6 Further Reading

1. A. Goriely, The Mathematics and Mechanics of Biological Growth, Chapters 4 and 11 (Springer, 2017)
2. S. S. Antman, Nonlinear Problems of Elasticity, vol 107 of Applied Mathematical Sciences (Springer, 2015).
3. P. G. Ciarlet, Mathematical Elasticity, Studies in Mathematics and its Applications; v. 20, 27, 29 (North–Holland, 1988).

30 C5.2 Elasticity and Plasticity

30.1 General Prerequisites

Familiarity will be assumed with the Part A course options: *A2: Metric Spaces and Complex Analysis*, *A1: Differential Equations 1*, *A6 Differential Equations 2* and *ASO: Calculus of Variations*. A basic understanding of stress tensors from either *B5.3 Viscous Flow* or *C5.1 Solid Mechanics* will also be required. The following courses are also helpful: *B5.1 Techniques of Applied Mathematics*, *B5.2 Applied Partial Differential Equations*, *C5.5 Perturbation Methods*, and *C5.6 Applied Complex Variables*.

30.2 Overview

The course starts with a rapid overview of mathematical models for basic solid mechanics. Benchmark solutions are derived for static problems and wave propagation in linear elastic materials. It is then shown how these results can be used as a basis for practically useful problems involving thin beams and plates. Simple geometrically nonlinear models are then introduced to explain buckling, fracture and contact. Models for yield and plasticity are then discussed, both microscopically and macroscopically.

30.3 Learning Outcomes

By the end of this course, students will be able:

1. to formulate and solve problems in linear elasticity, including models for elastic waves, mode conversion and mode III cracks;
2. to derive models for thin elastic beams and to describe buckling behaviour using weakly nonlinear asymptotic analysis;
3. to derive, analyse and solve linear complementarity problems describing smooth contact of elastic strings and membranes;
4. to formulate, analyse and solve free boundary problems for linear perfectly plastic yield of granular materials and metals in simple geometries.

30.4 Synopsis

Review of tensors, conservation laws, Navier equations. Antiplane strain, torsion, plane strain. Elastic wave propagation, Rayleigh waves. Ad hoc approximations for thin materials; simple bifurcation theory and buckling. Simple mixed boundary value problems, brittle fracture and smooth contact. Perfect plasticity theories for granular materials and metals.

30.5 Reading List

1. P. D. Howell, G. Kozyreff and J. R. Ockendon, *Applied Solid Mechanics* (Cambridge University Press, 2008)
2. S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity* (McGraw-Hill, 1970)
3. L.D. Landau and E.M. Lifshitz, *Theory of Elasticity* (Pergamon Press, 1986)

31 C5.4 Networks

31.1 General Prerequisites

Basic notions of linear algebra, probability theory, and some computational experience. Numerical codes may be illustrated in tutorials, but the student has the possibility to use the language of their choice. Relevant notions of graph theory will be reviewed.

31.2 Overview

Network Science provides generic tools to model and analyse systems in a broad range of disciplines, including biology, computer science and sociology. There are commercial applications within media marketing, defence and security, ranking and seeking in sports, and many sectors of the digital economy. This course aims at providing an introduction to this interdisciplinary field of research, by integrating tools from graph theory, statistics, linear algebra, and dynamical systems. Most of the topics to be considered are active modern research areas.

31.3 Learning Outcomes

Students will have developed a sound knowledge and appreciation of some of the tools, concepts, models, and computations used in the study of networks. The study of networks is predominantly a modern subject, so the students will also be expected to develop the ability to read and understand current (2015-2025) research papers in the field.

31.4 Synopsis

1. Introduction and short overview of useful mathematical concepts : Networks as abstractions; Renewal processes; Random walks and diffusion; Power-law distributions; Matrix algebra; Markov chains; Branching processes.

2. Basic structural properties of networks : Definition; Degree distribution; Measures derived from walks and paths; Clustering coefficient; Centrality Measures; The Laplacian; Spectral properties.

3. Models of networks : Erdos-Rényi random graph; Configuration model; Small World graphs, Network motifs; Core–periphery structure; Growing network with preferential attachment.

4. Community detection (2 lectures): Newman-Girvan Modularity; Spectral optimization of modularity; Greedy optimization of modularity.

5. Dynamics, Time-scales and Communities; Consensus dynamics; Timescale separation in dynamical systems and networks; Dynamically invariant subspaces and externally equitable partitions

6. Dynamics: Random walks Discrete-time random walks on networks; PageRank; Mean first-passage and recurrence times; Respondent-driven sampling; Continuous-Time Random Walks, Models of epidemic processes

7: Scaling Laws for functionals of growing networks

8. Dynamics:of networks Birth and dearth of edges: Stochastic models, Mean-Field Theories and Approximations

9. Applications to large scale simulations of the human cortex

31.5 Reading List

Self-contained lecture notes will be made available at the end of Michaelmas.

32 C5.5 Perturbation Methods

32.1 General Prerequisites

Knowledge of core complex analysis and of core differential equations will be assumed, respectively at the level of the complex analysis in the Part A (Second Year) course Metric Spaces and Complex Analysis and the phase plane section in Part A Differential Equations

I. The final section on approximation techniques in Part A Differential Equations II is highly recommended reading if it has not already been covered

32.2 Overview

Perturbation methods underlie numerous applications of physical applied mathematics: including boundary layers in viscous flow, celestial mechanics, optics, shock waves, reaction-diffusion equations, and nonlinear oscillations. The aims of the course are to give a clear and systematic account of modern perturbation theory and to show how it can be applied to differential equations.

32.3 Synopsis

Introduction to regular and singular perturbation theory: approximate roots of algebraic and transcendental equations. Asymptotic expansions and their properties. Asymptotic approximation of integrals, including Laplace's method, the method of stationary phase and the method of steepest descent. Matched asymptotic expansions and boundary layer theory. Multiple-scale perturbation theory. WKB theory and semiclassics.

32.4 Reading List

1. E.J. Hinch, *Perturbation Methods* (Cambridge University Press, 1991), Chs. 1-3, 5-7.
2. C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (Springer, 1999), Chs. 6, 7, 9-11.
3. J. Kevorkian and J.D. Cole, *Perturbation Methods in Applied Mathematics* (Springer-Verlag, 1981), Chs. 1, 2.1-2.5, 3.1, 3.2, 3.6, 4.1, 5.2.

33 C5.6 Applied Complex Variables - Draft

33.1 General Prerequisites

The course requires second year core analysis (A2 complex analysis). It continues the study of complex variables in the directions suggested by contour integration and conformal mapping. A knowledge of the basic properties of Fourier Transforms is assumed. Part A Waves and Fluids and Part C Perturbation Methods are helpful but not essential.

33.2 Overview

The course begins where core second-year complex analysis leaves off, and is devoted to extensions and applications of that material. The solution of Laplace's equation using conformal mapping techniques is extended to general polygonal domains and to free boundary problems. The properties of Cauchy integrals are analysed and applied to mixed boundary value problems and singular integral equations. The Fourier transform is generalised to complex values of the transform variable, and used to solve mixed boundary value problems and integral equations via the Wiener-Hopf method.

33.3 Learning Outcomes

Students will be able to:

1. Solve Laplace's equation on various two-dimensional domains using conformal mapping techniques
2. Use conformal mapping to solve certain free-boundary fluid flow problems
3. Use the Plemeli formulae for Cauchy integrals to solve mixed boundary value problems and singular integral equations
4. Use contour integrals and the Wiener-Hopf technique to solve a range of PDE problems and integral equations

33.4 Synopsis

Review of core complex analysis, analytic continuation, multifunctions, contour integration, conformal mapping and Fourier transforms.

Riemann mapping theorem (in statement only). Schwarz-Christoffel formula. Solution of Laplace's equation by conformal mapping onto a canonical domain; applications including inviscid hydrodynamics; Free streamline flows in the hodograph plane. Unsteady flow with free boundaries in porous media.

Application of Cauchy integrals and Plemelj formulae. Solution of mixed boundary value problems motivated by thin aerofoil theory and the theory of cracks in elastic solids. Reimann-Hilbert problems. Cauchy singular integral equations. Complex Fourier transform. Contour integral solutions of ODE's. Wiener-Hopf method.

33.5 Reading List

1. G. F. Carrier, M. Krook and C.E. Pearson, *Functions of a Complex Variable* (Society for Industrial and Applied Mathematics, 2005.) ISBN 0898715954.
2. M. J. Ablowitz and A. S. Fokas, *Complex Variables: Introduction and Applications* (2nd edition, Cambridge University Press, 2003). ISBN 0521534291.
3. J. R. Ockendon, S. D. Howison, A. A. Lacey and A. B. Movchan, *Applied Partial Differential Equations: Revised Edition* (Oxford University Press, 2003). ISBN 0198527713. Pages 195-212.

34 C5.7 Topics in Fluid Mechanics

34.1 General Prerequisites

B5.3 Viscous Flow, B5.4 Waves and Compressible Flow.

34.2 Overview

The course will expand and illuminate the 'classical' fluid mechanics taught in the third year B5.3 and B5.4 courses, and illustrate its modern application in a number of different applications.

34.3 Synopsis

The Navier Stokes equations and simplifications.
Surface tension and thin films, including lubrication theory, coatings, gravity flows, Marangoni effects, the steady state drag-out problem.
Droplet dynamics, contact lines, menisci. Drying and wetting.
Flow in porous media: Darcy's law; thermal and solutal convection; gravity-driven flow and carbon sequestration.
Aspects of low Reynolds flow in biological actuation and cell motility

34.4 Reading List

1. L.G Leal, *Advanced Transport Phenomena*,(Cambridge University Press, Cambridge, 2007).
2. J.S. Turner, *Buoyancy Effects in Fluids*, (Cambridge University Press, Cambridge, 1973).
3. A.C. Fowler, *Mathematical Models in the Applied Sciences*, (Cambridge University Press, Cambridge, 1997).
4. E. Lauga, *The Fluid Dynamics of Cell Motility* (Cambridge University Press, Cambridge, 2020).

   Please note that e-book versions of many books in the reading lists can be found on SOLO and ORLO.

34.5 Further Reading

1. G.K. Batchelor, H.K. Moffatt and M.G. Worster (eds.), *Perspectives in Fluid Dynamics* (Cambridge University Press, Cambridge, 2000).

35 C5.11 Mathematical Geoscience

35.1 General Prerequisites

B5.2 Applied Partial Differential Equations and *B5.3 Viscous Flow* recommended.

35.2 Overview

The aim of the course is to illustrate the techniques of mathematical modelling in their particular application to environmental problems. The mathematical techniques used are drawn from the theory of ordinary differential equations and partial differential equations. The course requires a willingness to become familiar with a range of different scientific disciplines. In particular, some familiarity with the concepts of fluid mechanics will be useful. The course will provide exposure to some current research topics.

35.3 Learning Outcomes

Students will be able to:

1. Derive and explain mathematical models for a range of geoscientific problems
2. Analyse such models to determine the relative importance of different physical and chemical effects, and to formulate appropriate reduced models
3. Use a range of mathematical techniques to solve the models, and be able to interpret the results in terms of applications to real-world phenomena

35.4 Synopsis

Applications of mathematics to environmental or geophysical problems involving the use of models with ordinary and partial differential equations. Examples to be considered are:

- Climate dynamics (radiative balance, greenhouse effect, ice-albedo feedback, carbon cycle)

- River flows (conservation laws, flood hydrographs, St Venant equations, sediment transport, bed instabilities)

- Ice dynamics (glaciers, ice sheets, sea ice)

35.5 Reading List

1. A. C. Fowler, *Mathematical Geoscience* (Springer, 2011).
2. J. T. Houghton, *The Physics of Atmospheres* (3rd ed., Cambridge University Press., Cambridge, 2002).
3. K. Richards, *Rivers* (Methuen, 1982).
4. K. M. Cuffey and W. S. B. Paterson, *The Physics of Glaciers* (4th edition, Butterworth-Heinemann, 2011).

36 C5.12 Mathematical Physiology

36.1 General Prerequisites

B5.5 Further Mathematical Biology is highly recommended.

36.2 Overview

The course aims to provide an introduction which can bring students within reach of current research topics in physiology, by synthesising a coherent description of the physiological background with realistic mathematical models and their analysis. The concepts and treatment of oscillations, waves and stability are central to the course, which develops ideas introduced in the more elementary B5.5 course. In addition, the lecture sequence aims to build understanding of the workings of the human body by treating in sequence problems at the intracellular, intercellular, whole organ and systemic levels.

36.3 Learning Outcomes

Students will have developed an understanding of mathematical modelling of physiological systems and will have demonstrable knowledge of the mathematical theory necessary to analyse such models.

36.4 Synopsis

Trans-membrane ion transport: Hodgkin-Huxley and Fitzhugh-Nagumo models.

Excitable media; wave propagation in neurons.

Calcium dynamics; calcium-induced calcium release. Intracellular oscillations and wave propagation.

The electrochemical action of the heart. Spiral waves, tachycardia and fibrillation.

Discrete delays in physiological systems. The Glass-Mackey model of respiration. Regulation of stem cell and blood cell production.

36.5 Reading List

The principal text is:

1. J. Keener and J. Sneyd, *Mathematical Physiology* (Springer-Verlag, 1998). First edition or Second edition Vol I: Chs. 2, 7. Vol II: Chs. 11, 13, 14. (Springer-Verlag, 2009)]

Subsidiary mathematical texts are:

1. J. D. Murray, *Mathematical Biology* (Springer-Verlag, 2nd ed., 1993). [Third edition, Vols I and II, (Springer-Verlag, 2003).]

2. L. Glass and M. C. Mackey, *From Clocks to Chaos* (Princeton University Press, 1988).

3. P. Grindrod, *Patterns and Waves* (OUP, 1991).

General physiology texts are:

1. R. M. Berne and M. N. Levy, *Principles of Physiology* (2nd ed., Mosby, St. Louis, 1996).

2. J. R. Levick, *An Introduction to Cardiovascular Physiology* (3rd ed. Butterworth-Heinemann, Oxford, 2000).

3. A. C. Guyton and J. E. Hall, *Textbook of Medical Physiology* (10th ed., W. B. Saunders Co., Philadelphia, 2000).

37 C6.1 Numerical Linear Algebra

37.1 General Prerequisites

Only elementary linear algebra is assumed in this course. The Part A Numerical Analysis course would be helpful, indeed some swift review and extensions of some of the material of that course is included here.

37.2 Overview

Linear Algebra is a central and widely applicable part of mathematics. It is estimated that many (if not most) computers in the world are computing with matrix algorithms at any moment in time whether these be embedded in visualization software in a computer game or calculating prices for some financial option. This course builds on elementary linear algebra and in it we derive, describe, and analyse a number of widely used constructive methods (algorithms) for various problems involving matrices.

Numerical Methods for solving linear systems of equations, computing eigenvalues and singular values and various related problems involving matrices are the main focus of this course.

37.3 Learning Outcomes

Students should understand the Singular Value Decomposition and its wide uses, state-of-the art algorithms for eigenvalue computation and core algorithms for solving linear systems, including in particular iterative solution methods of Krylov subspace type and randomised algorithms.

37.4 Synopsis

Common problems in linear algebra. Matrix structure, singular value decomposition. QR factorization, the QR algorithm for eigenvalues. Direct solution methods for linear systems, Gaussian elimination and its variants. Iterative solution methods for linear systems. Chebyshev polynomials conjugate gradients, convergence analysis, preconditioning. Randomised algorithms for least-squares problems and low-rank approximation.

37.5 Reading List

1. L. N. Trefethen and D. Bau III, *Numerical Linear Algebra* (SIAM, 1997).
2. J. W. Demmel, *Applied Numerical Linear Algebra* (SIAM, 1997).
3. A. Greenbaum, *Iterative Methods for Solving Linear Systems* (SIAM, 1997).
4. G. H. Golub and C. F. van Loan, *Matrix Computations* (John Hopkins University Press, 3rd edition, 1996).
5. H. C. Elman, D. J. Silvester and A. J. Wathen, *Finite Elements and Fast Iterative Solvers* (Oxford University Press, 1995), only chapter 2.

38 C6.2 Continuous Optimisation

38.1 General Prerequisites

Basic linear algebra (such as eigenvalues and eigenvectors of real matrices), multivariate real analysis (such as norms, inner products, multivariate linear and quadratic functions, basis) and multivariable calculus (such as Taylor expansions, multivariate differentiation, gradients).

38.2 Overview

The solution of optimal decision-making and engineering design problems in which the objective and constraints are nonlinear functions of potentially (very) many variables is required on an everyday basis in the commercial and academic worlds. A closely-related subject is the solution of nonlinear systems of equations, also referred to as least-squares or data fitting problems that occur in almost every instance where observations or measurements are available for modelling a continuous process or phenomenon, such as in weather forecasting. The mathematical analysis of such optimization problems and of classical and modern methods for their solution are fundamental for understanding existing software and for developing new techniques for practical optimization problems at hand.

38.3 Synopsis

Part 1: Unconstrained Optimization

Optimality conditions, steepest descent method, Newton and quasi-Newton methods, General line search methods, Trust region methods, Least squares problems and methods.

Part 2: Constrained Optimization

Optimality/KKT conditions, penalty and augmented Lagrangian for equality-constrained optimization, interior-point/ barrier methods for inequality constrained optimization. SQP methods.

38.4 Reading List

Lecture notes will be made available for downloading from the course webpage.

A useful textbook is J.Nocedal and S.J.Wright, *Numerical Optimisation*, (Springer, 1999 or 2006).

39 C6.5 Theories of Deep Learning

39.1 General Prerequisites

Only elementary linear algebra and probability are assumed in this course; with knowledge from the following prelims courses also helpful: linear algebra, probability, analysis, constructive mathematics, and statistics and data analysis. It is recommended that students have familiarity with some of: more advanced statistics, optimisation (B6.3, C6.2), networks (C5.4), and numerical linear algebra (C6.1), though none of these courses are required as the material is self contained.

39.2 Overview

A course on theories of deep learning.

39.3 Learning Outcomes

Students will become familiar with the variety of architectures for deep nets, including the scattering transform and ingredients such as types of nonlinear transforms, pooling, convolutional structure, and how nets are trained. Students will focus their attention on learning a variety of theoretical perspectives on why deep networks perform as observed, with examples such as: dictionary learning and transferability of early layers, energy decay with depth, Lipschitz continuity of the net, how depth overcomes the curse of dimensionality, constructing adversarial examples, geometry of nets viewed through random matrix theory, and learning of invariance.

39.4 Synopsis

Deep learning is the dominant method for machines to perform classification tasks at reliability rates exceeding that of humans, as well as outperforming world champions in games such as go. Alongside the proliferating application of these techniques, the practitioners have developed a good understanding of the properties that make these deep nets effective, such as initial layers learning weights similar to those in dictionary learning, while deeper layers instantiate invariance to transforms such as dilation, rotation, and modest diffeomorphisms. There are now a number of theories being developed to give a mathematical theory to accompany these observations; this course will explore these varying perspectives.

39.5 Reading List

1. I. Goodfellow, Y. Bengio, and A. Courville, *Deep Learning*, Adaptive computation and machine learning series.

Additional reading from recent publications will be provided as needed.

40 C7.1 Theoretical Physics (C6)

40.1 General Prerequisites

A11: Quantum Theory, *B7.1 Classical Mechanics*, *B7.2 Electromagnetism*.

40.2 Overview

This course is intended to give an introduction to some aspects of many-particle systems, field theory and related ideas. These form the basis of our current theoretical understanding of particle physics, condensed matter and statistical physics. An aim is to present some core ideas and important applications in a unified way. These applications include the classical mechanics of continuum systems, the quantum mechanics and statistical mechanics of many-particle systems, and some basic aspects of relativistic quantum field theory.

40.3 Synopsis

1. Path Integrals in Quantum Mechanics
2. Quantum Many-Particle Systems
3. Phase Transitions
4. Stochastic Processes
5. Classical Field Theory
6. Canonical Quantisation of Fields
7. Interacting Quantum Fields

40.4 Reading List

The lecturers are aware of no book that presents all parts of this course in a unified way and at an appropriate level. For this reason, detailed lecture notes will be made available.

41 C7.4 Introduction to Quantum Information

41.1 General Prerequisites

Quantum Theory.

The course material should be of interest to physicists, mathematicians, computer scientists, and engineers. The following will be assumed as prerequisites for this course: - elementary probability, complex numbers, vectors and matrices; - Dirac bra-ket notation; - a basic knowledge of quantum mechanics especially in the simple context of finite dimensional state spaces (state vectors, composite systems, unitary matrices, Born rule for quantum measurements); - basic ideas of classical theoretical computer science (complexity theory) would be helpful but are not essential.

Prerequisite notes will be provided giving an account of the necessary material. It would be desirable for you to look through these notes slightly before the start of the course.

41.2 Overview

The classical theory of computation usually does not refer to physics. Pioneers such as Turing, Church, Post and Goedel managed to capture the correct classical theory by intuition alone and, as a result, it is often falsely assumed that its foundations are self-evident and purely abstract. They are not! Computers are physical objects and computation is a physical process. Hence when we improve our knowledge about physical reality, we may also gain new means of improving our knowledge of computation. From this perspective it should not be very surprising that the discovery of quantum mechanics has changed our understanding of the nature of computation. In this series of lectures you will learn how inherently quantum phenomena, such as quantum interference and quantum entanglement, can make information processing more efficient and more secure, even in the presence of noise.

41.3 Learning Outcomes

By the end of the course, students will be able to:

- Use single qubit and two qubits quantum logic gates to construct quantum circuits of increasing complexity.
- Provide comparative analysis of quantum and classical algorithms.
- Solve problems involving quantum entanglement and its applications.
- Perform calculations using density matrices and completely positive maps.
- Use mathematical techniques that protect quantum evolutions, such as quantum error correction and fault tolerant computation.

41.4 Synopsis

Bits, gates, networks, Boolean functions, reversible and probabilistic computation

"Impossible" logic gates, amplitudes, quantum interference

One, two and many qubits

Entanglement and entangling gates

From interference to quantum algorithms

Algorithms, computational complexity and Quantum Fourier Transform

Phase estimation and quantum factoring

Non-local correlations and cryptography

Bell's inequalities

Density matrices and CP maps

Decoherence and quantum error correction

41.5 Reading List

1. Beyond the Quantum Horizon by D. Deutsch and A. Ekert, Scientific American, Sep 2012.
2. Less reality more security by A. Ekert, Physics World, Sep 2009.
3. The Limits of Quantum Computers, by S. Aaronson, Scientific American, Mar 2008.
4. A Do-It-Yourself Quantum Eraser by R. Hillmer and P. Kwiat, Scientific American, May 2007.
5. Quantum Seeing in the Dark by P. Kwiat et al, Scientific American, Nov 1996
6. Physical Limits of Computation by C.H. Bennett and R. Landauer, Scientific American, Jul 1985.

42 C7.5 General Relativity I

42.1 General Prerequisites

ASO: Special Relativity, *B7.1 Classical Mechanics*, and *B7.2 Electromagnetism.*

42.2 Overview

The course is intended as an introduction to general relativity, covering both its observational implications and the new insights that it provides into the nature of spacetime and the structure of the universe. Familiarity with special relativity and electromagnetism as covered in the Part A and Part B courses will be assumed. The lectures will review Newtonian gravity, special relativity (from a geometric point of view), and then move on to cover

Differential geometry, Riemannian geometry, physics in curved space time, and the Einstein equations. These will then be used to give an account of planetary motion, the bending of light, the existence and properties of black holes and elementary cosmology.

42.3 Learning Outcomes

By the end of the course, students will understand the tension between special relativity and gravitation, and appreciate the physical considerations (such as the equivalence principle) which motivate the Einstein equations. They will understand tensors and tensor calculus, including notions of covariance and curvature, leading to an understanding of the Einstein equations. They will be able to derive simple physical consequences of the Einstein equations, such as the bending of light, the varying speeds of clocks in gravitational fields. They will be able to interpret the Schwarzschild solution, either as describing the exterior of a spherical body, or as a black hole, and they will understand some simple cosmological solutions and their properties, including the big bang.

42.4 Synopsis

Review of Newtonian gravity and special relativity. Difficulties in reconciling Special relativity with gravity, and the equivalence principle. Curved space time: elements of differential and Riemannian geometry; connections, curvature and geodesic deviation. The Einstein equations, and other physical laws in curved spacetime. Planetary motion and the bending of light. Introduction to black hole solutions; the Schwarzschild solution. Introduction to cosmology: homogeneity and isotropy, and the Friedman-Robertson-Walker solutions.

42.5 Reading List

1. S. Carroll, *Space Time and Geometry: An Introduction to General Relativity* (Addison Welsey, 2003)
2. R.M. Wald, *General Relativity* (Chicago, 1984).
3. Nakahara, *Geometry, Topology and Physics.*

42.6 Further Reading

1. B. Schutz, *A First Course in General Relativity* (Cambridge University Press, 1990).
2. W. Rindler, *Essential Relativity* (Springer-Verlag, 2nd edition, 1990).
3. S. Weinberg, *Gravitation and Cosmology.*
4. J.B. Hartle *Gravity, an introduction to Einstein's general relativity*

43 C7.6 General Relativity II

43.1 General Prerequisites

C7.5 General Relativity I

43.2 Overview

This course will further extend our understanding of general relativity. We begin by studying linearised gravity and gravitational waves before studying stellar collapse of cold stars and the inevitability of the formation of black holes. We discuss singularities, the causal structure of spacetime using Penrose diagrams and give a formal definition of a black hole. We then study larger classes of black hole solutions including the Kerr–Newman and Reissner–Nordstrom solutions and study their causal properties before studying the laws of black hole thermodynamics.

43.3 Learning Outcomes

By the end of the course, students will be able to

1. Understand linearised gravity and gravitational waves.
2. Understand the gravitational collapse of a star to form a black hole.
3. Understand the causal properties of various different spacetimes by constructing Penrose diagrams.
4. Explain the concept of a black hole and identify its mass, angular momentum and electric and magnetic charges.
5. Understand the laws of black hole thermodynamics and the need for a theory of quantum gravity.

43.4 Synopsis

Linearised gravity and gravitational waves. The Tolman–Oppenheimer–Volkoff equations and stellar collapse. Definition of a black hole and Penrose singularity theorems. Causality of spacetime and Penrose diagrams. Charged and rotating black holes. Laws of black holes thermodynamics and the Hawking temperature.

43.5 Reading List

1. S. Carroll, *Space Time and Geometry: An Introduction to General Relativity* (Addison Welsey, 2003)
2. R. M. Wald, *General Relativity*, Univ of Chicago Press (1984).

43.6 Further Reading

1. B. Schutz, *A First Course in General Relativity* (Cambridge University Press, 1990).
2. W. Rindler, *Essential Relativity* (Springer-Verlag, 2nd edition, 1990).
3. S. Hawking and G. Ellis, *The Large Scale of the Universe*, (Cambridge Monographs on Mathematical Physics, 1973).

44 C7.7 Random Matrix Theory

44.1 General Prerequisites

There are no formal prerequisites, but familiarity with basic concepts and results from linear algebra and probability will be assumed, at the level of A0 (Linear Algebra) and A8 (Probability).

44.2 Overview

Random Matrix Theory provides generic tools to analyse random linear systems. It plays a central role in a broad range of disciplines and application areas, including complex networks, data science, finance, machine learning, number theory, population dynamics, and quantum physics. Within Mathematics, it connects with asymptotic analysis, combinatorics, integrable systems, numerical analysis, probability, and stochastic analysis. This course aims to provide an introduction to this highly active, interdisciplinary field of research, covering the foundational concepts, methods, questions, and results.

44.3 Learning Outcomes

Students will learn how some of the various different ensembles of random matrices are defined. They will encounter some examples of the applications these have in Data Science, modelling Complex Quantum Systems, Mathematical Finance, Network Models, Numerical Linear Algebra, and Population Dynamics. They will learn how to analyse eigenvalue statistics, and see connections with other areas of mathematics and physics, including combinatorics, number theory, and statistical mechanics.

44.4 Synopsis

The course is divided into four parts:

1. Introduction (1 Lecture)
   Matrix ensembles: Wigner and Wishart random matrices, the Gaussian and Circular Ensembles. Overview of applications.
2. Empirical Spectral Distribution (6 Lectures)

The method of moments: Derivation of the Wigner Semicircle law and the Marchenko-Pastur law for sample covariance matrix. Non-Hermitian case: the Circular law. Introduction to the Stieltjes transform.

3. Eigenvalue Statistics (6 Lectures)

   Derivation of the joint distribution of eigenvalues for the Gaussian and Circular Ensembles. Eigenvalues as a point process: method of orthogonal polynomials. The spectrum in the bulk and at the edge.

4. Dynamical Approach and Universality (3 Lectures)

   Dyson Brownian Motion. Connections to other problems in mathematics: the longest increasing subsequence problem, distribution of zeros of the Riemann zeta-function, topological genus expansions.

44.5 Reading List

1. T. Tao, *Topics in Random Matrix Theory* (AMS Graduate Studies in Mathematics)
2. GW Anderson, A Guionnet, O Zeitouni, *An Introduction to Random Matrices* (Cambridge Studies in Advanced Mathematics)
3. ES Meckes, *The Random Matrix Theory of the Classical Compact Groups* (Cambridge University Press)
4. G. Livan, M. Novaes & P. Vivo, *Introduction to Random Matrices* (Springer Briefs in Mathematical Physics)
5. J.-P. Bouchaud & M. Potters, *A First Course in Random Matrix Theory for Physicists, Engineers and Data Scientists* (Cambridge University Press)

44.6 Further Reading

1. ML Mehta, *Random Matrices* (Elsevier, Pure and Applied Mathematics Series)
2. G. Akemann, J. Baik & P. Di Francesco, *The Oxford Handbook of Random Matrix Theory* (Oxford University Press)

45 C8.1 Stochastic Differential Equations - Draft

45.1 General Prerequisites

Integration theory: Riemann-Stieljes and Lebesgue integral and their basic properties Probability and measure theory: σ-algebras, Fatou lemma, Borel-Cantelli, Radon-Nikodym, L^p-spaces, basic properties of random variables and conditional expectation, Martingales in discrete and continuous time: construction and basic properties of Brownian motion, uniform integrability of stochastic processes, stopping times, filtrations, Doob's theorems (maximal and L^p-inequalities, optimal stopping, upcrossing, martingale decomposition), martingale (backward) convergence theorem, L^2-bounded martingales, quadratic variation; Stochastic Integration: Ito's construction of stochastic integral, Ito's formula.

45.2 Overview

Stochastic differential equations (SDEs) model evolution of systems affected by randomness. They offer a beautiful and powerful mathematical language in analogy to what ordinary differential equations (ODEs) do for deterministic systems. From the modelling point of view, the randomness could be an intrinsic feature of the system or just a way to capture small complex perturbations which are not modelled explicitly. As such, SDEs have found many applications in diverse disciplines such as biology, physics, chemistry and the management of risk. Classic well-posedness theory for ODEs does not apply to SDEs. However, when we replace the classical Newton-Leibnitz calculus with the (Ito) stochastic calculus, we are able to build a new and complete theory of existence and uniqueness of solutions to SDEs. Ito formula proves to be a powerful tool to solve SDEs. This leads to many new and often surprising insights about quantities that evolve under randomness. This course is an introduction to SDEs. It covers the basic theory but also offers glimpses into many of the advanced and nuanced topics.

45.3 Learning Outcomes

By the end of this course, students will be able to analyse if a given SDEs admits a solution, characterise the nature of solution and explain if it is unique or not. The students will also be able to solve basic SDEs and state basic properties of the diffusive systems described by these equations.

45.4 Synopsis

Recap on martingale theory in continuous time, quadratic variation, stochastic integration and Ito's calculus.

Levy's characterisation of Brownian motion, stochastic exponential, Girsanov theorem and change of measure, Burkholder-Davis-Gundy, Martingale represenation, Dambis-Dubins-Schwarz.

Strong and weak solutions of stochastic differential equations, existence and uniqueness.

Examples of stochastic differential equations. Bessel processes.

Local times, Tanaka formula, Tanaka-Ito-Meyer formula.

45.5 Reading List

1. J. Obloj, *Continuous martingales and stochastic calculus* online notes. Students are encouraged to study all the material up to and including the Ito formula prior to the course.

2. D. Revuz and M. Yor, *Continuous martingales and Brownian motion* (3rd edition, Springer).

45.6 Further Reading

1. I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Graduate Texts in Mathematics 113 (Springer-Verlag, 1988).
2. L. C. G. Rogers & D. Williams, *Diffusions, Markov Processes and Martingales Vol 1 (Foundations) and Vol 2 (Ito Calculus)* (Cambridge University Press, 1987 and 1994).
3. R. Durrett, *Stochastic Calculus* (CRC Press).
4. B. Oksendal, *Stochastic Differential Equations: An introduction with applications* (Universitext, Springer, 6th edition).
5. N. Ikeda & S. Watanabe, *Stochastic Differential Equations and Diffusion Processes* (North–Holland Publishing Company, 1989).
6. H. P. McKean, *Stochastic Integrals* (Academic Press, New York and London, 1969).

46 C8.2 Stochastic Analysis and PDEs

46.1 General Prerequisites

Integration and measure theory, martingales in discrete and continuous time, stochastic calculus. Functional analysis is useful but not essential.

46.2 Overview

Stochastic analysis and partial differential equations are intricately connected. This is exemplified by the celebrated deep connections between Brownian motion and the classical heat equation, but this is only a very special case of a general phenomenon. We explore some of these connections, illustrating the benefits to both analysis and probability.

46.3 Learning Outcomes

The student will have developed an understanding of the deep connections between concepts from probability theory, especially diffusion processes and their transition semigroups, and partial differential equations.

46.4 Synopsis

Feller processes and semigroups. Resolvents and generators. Hille-Yosida Theorem (without proof). Diffusions and elliptic operators, convergence and approximation. Stochastic differential equations and martingale problems. Duality. Speed and scale for one dimensional diffusions. Green's functions as occupation densities. The Dirichlet and Poisson problems. Feynman-Kac formula.

46.5 Reading List

A full set of typed notes will be supplied.

Important references:

1. O. Kallenberg. *Foundations of Modern Probability.* Second Edition, Springer 2002. This comprehensive text covers essentially the entire course, and much more, but should be supplemented with other references in order to develop experience of more examples.
2. L.C.G Rogers & D. Williams. *Diffusions, Markov Processes and Martingales*; Volume 1, Foundations and Volume 2, Itô calculus. Cambridge University Press, 1987 and 1994. These two volumes have a very different style to Kallenberg and complement it nicely. Again they cover much more material than this course.

46.6 Further Reading

1. S.N. Ethier & T.G. Kurtz. *Markov Processes: characterization and convergence.* Wiley 1986. It is not recommended to try to sit down and read this book cover to cover, but it is a treasure trove of powerful theory and elegant examples.
2. S. Karlin & H.M. Taylor. *A second course in stochastic processes.* Academic Press 1981. This classic text does not cover the material on semigroups and martingale problems that we shall develop, but it is a very accessible source of examples of diffusions and things one might calculate for them.

A fuller list of references will be included in the typed notes.

47 C8.3 Combinatorics

47.1 General Prerequisites

B8.5 Graph Theory is helpful, but not required.

47.2 Overview

An important branch of discrete mathematics concerns properties of collections of subsets of a finite set. There are many beautiful and fundamental results, and there are still many basic open questions. The aim of the course is to introduce this very active area of mathematics, with many connections to other fields.

47.3 Learning Outcomes

The student will have developed an appreciation of the combinatorics of finite sets.

47.4 Synopsis

Chains and antichains. Sperner's Lemma. LYM inequality. Dilworth's Theorem.

Shadows. Kruskal-Katona Theorem.

Intersecting families. Erdos-Ko-Rado Theorem. Cross-intersecting families.

VC-dimension. Sauer-Shelah Theorem.

t-intersecting families. Fisher's Inequality. Frankl-Wilson Theorem. Application to Borsuk's Conjecture.

Combinatorial Nullstellensatz.

47.5 Reading List

1. Bela Bollobás, *Combinatorics*, CUP, 1986.

2. Stasys Jukna, *Extremal Combinatorics*, Springer, 2011

48 C8.4 Probabilistic Combinatorics - Draft

48.1 General Prerequisites

B8.5 Graph Theory and *A8: Probability. C8.3 Combinatorics* is not as essential prerequisite for this course, though it is a natural companion for it.

48.2 Overview

Probabilistic combinatorics is a very active field of mathematics, with connections to other areas such as computer science and statistical physics. Probabilistic methods are essential for the study of random discrete structures and for the analysis of algorithms, but they can also provide a powerful and beautiful approach for answering deterministic questions. The aim of this course is to introduce some fundamental probabilistic tools and present a few applications.

48.3 Learning Outcomes

The student will have developed an appreciation of probabilistic methods in discrete mathematics.

48.4 Synopsis

First-moment method, with applications to Ramsey numbers, and to graphs of high girth and high chromatic number.

Second-moment method, threshold functions for random graphs.

Lovász Local Lemma, with applications to two-colourings of hypergraphs, and to Ramsey numbers.

Chernoff bounds, concentration of measure, Janson's inequality.

Branching processes and the phase transition in random graphs.

Clique and chromatic numbers of random graphs.

48.5 Reading List

1. N. Alon and J.H. Spencer, *The Probabilistic Method* (third edition, Wiley, 2008).

48.6 Further Reading

1. B. Bollobás, *Random Graphs* (second edition, Cambridge University Press, 2001).
2. M. Habib, C. McDiarmid, J. Ramirez-Alfonsin, B. Reed, ed., *Probabilistic Methods for Algorithmic Discrete Mathematics* (Springer, 1998).
3. S. Janson, T. Luczak and A. Rucinski, *Random Graphs* (John Wiley and Sons, 2000).
4. M. Mitzenmacher and E. Upfal, *Probability and Computing: Randomized Algorithms and Probabilistic Analysis* (Cambridge University Press, New York (NY), 2005).
5. M. Molloy and B. Reed, *Graph Colouring and the Probabilistic Method* (Springer, 2002).
6. R. Motwani and P. Raghavan, *Randomized Algorithms* (Cambridge University Press, 1995).

49 C8.7 Optimal Control

49.1 General Prerequisites

None, but some basic knowledge of stochastic calculus (as would be obtained from B8.2 Continuous Martingales and Stochastic Calculus or B8.3 Mathematical Models of Financial Derivatives) will be assumed. Familiarity with applied PDE (as would be obtained from B5.2 Applied Partial Differential Equations, B6.1 Numerical Solution of Partial Differential Equations, or B7.1 Classical Mechanics) would also be beneficial.

49.2 Overview

Optimal control is the question of how one should select actions sequentially through time. The problem appears, in various forms, in many applications, from industrial problems and classical mechanics through to problems in biology and finance. This course will study the mathematics required to understand these problems, both in discrete and continuous time, and in settings with and without randomness. The two main perspectives on control – dynamic programming and the Pontryagin principle – will be explored, along with how these perspectives lead to equations that describe the optimal action. The numerical solution of these equations will also be considered.

49.3 Learning Outcomes

The students will develop an understanding of the classical theory of optimal control, and be able to determine optimal controls within mathematical models. They will be familiar with manipulating the corresponding PDEs, and with reinforcement learning techniques to solve them.

49.4 Synopsis

Dynamic programming in discrete time, the Bellman equation and value function. Iteration methods for discrete systems and variations from reinforcement learning. Continuous deterministic systems and the Hamilton–Jacobi equation. Pontryagin maximum principle for deterministic systems. Feynman–Kac theorem and values in stochastic systems, the Linear-Quadratic-Gaussian case. Martingale characterization of optimality and the Hamilton–Jacobi–Bellman equation, maximum principle and verification theorem. Examples from finance and engineering.

49.5 Reading List

There are a large number of textbooks which cover the course material with a varying degree of detail/rigour. Recommended reading includes:

- Sutton and Barto *Reinforcement Learning: An Introduction*, MIT 1998
- Bertsekas and Shreve, *Stochastic Optimal Control: The Discrete-time case*, Athena Scientific, 1996
- Bensoussan, *Estimation and Control of Dynamical Systems*, Springer 2018
- Yong and Zhou *Stochastic Controls: Hamiltonian Systems and HJB equations*, Springer 1999
- Fleming and Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer 2006

More advanced texts include:

- Touzi, *Optimal Stochastic Control, Stochastic Target Problems and Backward SDE*, Fields Lecture Notes 2010
- Krylov, *Controlled Diffusion Processes*, Springer 1980
- Pham, *Continuous-time Stochastic Control and Optimization with Financial Applications*, Springer 2009

50 CCD Dissertations on a Mathematical Topic

50.1 Overview

STUDENTS MUST OFFER A DOUBLE-UNIT CCD DISSERTATION ON A MATHEMATICAL TOPIC OR A COD DISSERTATION IN THE HISTORY OF MATHEMATICS.

Students may offer a double-unit dissertation on a Mathematical topic for examination at Part C. A double-unit is equivalent to a 32-hour lecture course. Students will have a supervisor for their dissertation and will meet with them 4-5 times during Michaelmas and Hilary terms, together with the other students offering that dissertation topic. The group size will be between 2 and 5 students. The first meeting will take place in week 7 or 8 of Michaelmas term to provide them with the information needed to start work on their dissertation over the Christmas vacation and to agree the pattern of project supervision in Hilary term.

Students considering offering a dissertation should read the *Guidance Notes on Extended Essays and Dissertations in Mathematics* available at:

`https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/projects`.

Application The list of potential dissertation topics will be published on Friday of week 0 of Michaelmas term, following the Dissertation Information Session. For each potential topic there will be a short abstract outlining the topic, details of prerequisite knowledge, suggested references and possible avenues of investigation. There will be a limit on the number of students each supervisor is able to supervise and this information will also be provided.

You will be asked to submit a ranked list of dissertation choices via an online form by 12 noon on Friday of week 3. You will need to submit 6 choices, and will be given the opportunity to explain if there is a particular reason why you would like to do a specific topic. For example, you may like to undertake a dissertation in an area in which you are hoping to go on to further study.

You are not expected to make contact with the dissertation supervisor(s) before submitting your choices but if you have a question about a dissertation topic you should feel free to email the supervisor for further information.

Projects Committee will meet in week 4 to decide upon the allocation of dissertation topics and will seek to ensure that students receive one of their top choices as far as possible. You will be notified of which project you have been allocated at the start of week 5.

Assessment Dissertations are independently double-marked, normally by the dissertation supervisor and one other assessor. The two marks are then reconciled to give the overall mark awarded. The reconciliation of marks is overseen by the examiners and follows the department's reconciliation procedure (see `https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/projects`).

Submission An electronic copy of your dissertation should be submitted via the Mathematical Institute website to arrive no later than **12 noon on Monday of week 1, Trinity term 2024**. Further details may be found in the *Guidance Notes on Extended Essays and*

Dissertations in Mathematics.

51 COD Dissertations on the History of Mathematics

51.1 Overview

STUDENTS MUST OFFER A DOUBLE-UNIT CCD DISSERTATION ON A MATHEMATICAL TOPIC OR A COD DISSERTATION IN THE HISTORY OF MATHEMATICS.

Students may offer a double-unit dissertation on a topic related to the History of Mathematics for examination at Part C. A double-unit is equivalent to a 32-hour lecture course. Students will have a supervisor for their dissertation and will meet with them 4-5 times during Michaelmas and Hilary terms, together with the other students offering that dissertation topic. The group size will be between 2 and 5 students. The first meeting will take place in week 7 or 8 of Michaelmas term to provide them with the information needed to start work on their dissertation over the Christmas vacation and to agree the pattern of project supervision in Hilary term. In addition there are lectures on writing mathematics and using LaTeX in Michaelmas and Hilary terms. See the lecture list for details.

Candidates considering offering a dissertation should read the *Guidance Notes on Extended Essays and Dissertations in Mathematics* available at:

`https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/projects`.

Application Students wishing to do a dissertation based on the History of Mathematics should contact Dr Christopher Hollings at christopher.hollings@maths.ox.ac.uk by Wednesday of week 1 with a short draft proposal. Dr Hollings will contact you to arrange a short informal interview to discuss the proposal further. All decisions made by Dr Hollings will be communicated to students by the end of week 2.

All proposals supported by Dr Hollings will then be referred to Projects Committee who meet in week 4 for final approval. With the support of Dr Hollings students must submit a COD Dissertation Proposal Form to Projects Committee by the end of week 3. The form can be found in the Dissertation Guidance - `https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/part-c-students/teaching-and-learning/dissertations`.

Students whose proposal is not supported by Dr Hollings will be given the option to to submit a ranked list of dissertation choices via an online form by 12 noon on Friday of week 3. You will need to submit 6 choices, and will be given the opportunity to explain if there is a particular reason why you would like to do a specific topic. This is optional for Part C students and compulsory for OMMS students.

Projects Committee will meet in week 4 to decide upon the allocation of dissertation topics and will seek to ensure that students receive one of their top choices as far as possible. You will be notified of which project you have been allocated at the start of week 5.

Assessment Dissertations are independently double-marked, normally by the dissertation supervisor and one other assessor. The two marks are then reconciled to give the overall mark awarded. The reconciliation of marks is overseen by the examiners and follows the department's reconciliation procedure (see `https://www.maths.ox.ac.uk/members/students/undergraduate-courses/teaching-and-learning/projects`).

Submission An electronic copy of your dissertation should be submitted via the Mathematical Institute website to arrive no later than **12 noon on Monday of week 1, Trinity term 2024**. Further details may be found in the *Guidance Notes on Extended Essays and Dissertations in Mathematics.*

52 An Introduction to LaTeX

52.1 General Prerequisites

There are no prerequisites. The course is mainly intended for students writing a Part B Extended Essay or a Part C Dissertation but any students are welcome to attend the two lectures given in Michaelmas Term. Note that there is no assessment associated with this course, nor credit for attending the course.

52.2 Overview

This short lecture series provides an introduction to LaTeX.

LaTeX is a markup language, released by Donald Knuth in 1984 and freely sourced, for the professional typesetting of mathematics. (It is based on the earlier TeX released in 1978.) A markup language provides the means for rendering text in various ways - such as bold, italicized or Greek symbols - with the main focus of LaTeX being the rendering of mathematics so that even complicated expressions involving equations, integrals and matrices and images can be professionally typeset.

52.3 Learning Outcomes

Following these introductory lectures, a student should feel comfortable writing their own LaTeX documents, and producing professionally typset mathematics. The learning curve to producing a valid LaTeX document is shallow, and students will further become familiar will some of the principal features of LaTeX such as chapters, item lists, typesetting mathematics, including equations, tables, bibliographies and images. Then, with the aid of a good reference manual, a student should feel comfortable researching out for themselves further features and expanding their LaTeX vocabulary

52.4 Reading List

The Department has a page of LaTeX resources at `https://www.maths.ox.ac.uk/members/it/faqs/latex` which has various free introductory guides to LaTeX.