Phases with Non-Invertible Symmetries in (1+1)d: from Categories to Cold Atoms

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- 2310.03786 and 2310.03784: **Categorical Landau Paradigm** and **Gapped Phases with Non-Invertible Symmetries: (1+1)d**, by *LB, LEB, DP, SSN*
- 2312.17322: **The Club Sandwich: Gapless Phases and Phase Transitions with Non-Invertible Symmetries**, by *LB, LEB, DP, SSN*
- 2403.00905: **Hasse Diagrams for Gapless SPT and SSB Phases with Non-Invertible Symmetries**, by *LB, DP, SSN*, **AW**
- 2405.05964 and 2405.05302: **Lattice Models for Phases and Transitions with Non-Invertible Symmetries** by *LB, LEB, SSN, AT*
- 2412.15024: **Categorical Symmetries in Spin Models with Atom Arrays** by **AW**, *Fan Yang, AT, Hannes Pichler, SSN*

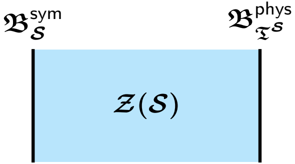
- **SymTFT:** [F. Apruzzi, F. Bonetti, I.G. Etxebarria, S.S. Hosseini, S.S chafer-Nameki '21], [D. Freed, G. Moore, C. Teleman, '22], [A. Chatterjee, W. Ji, X.-G. Wen, '22], [J. Kaidi, K. Ohmori, Y. Zheng, '22], [J. Kaidi, E. Nardoni, G. Zafrir, Y. Zheng, '23], [M. Del Zotto, S. N. Meynet, R. Moscrop, '24]
- **Gapped phases:** [J. Fröhlich, J. Fuchs, I. Runkel, C. Schweigert, '06], [L. Bhardwaj, Y. Tachikawa, '17], [R. Thorngren, Y. Wang, '19-'21], [T.-C. Huang, Y.-H. Lin, S. Seifnashri, '21], [C. Zhang, C. Córdova, '23], [Y. Choi, B. Rayhaun, Y. Sanghavi, S.-H. Shao, '23], [S.-J. Huang, M. Cheng, '23]
- **Gapless phases:** [R. Thorngren, A. Vishwanath, R. Verresen, '20], [A. Chatterjee, X-G.Wen, '22], [L.Li, M. Oshikawa, Y. Zheng, '23], [R. Wen, A. C. Potter, '23]
- **Lattice models:** [D. Aasen, R. Mong, P. Fendley, '20], [L. Lootens, C. Delcamp, G. Ortiz, F. Verstraete, '21], [L. Eck, P. Fendley, '23], [C. Fechisin, N. Tantivasadakarn, V. V. Albert, '23], [N. Seiberg, S. Seifnashri, S.-H. Shao, '24], [S. Seifnashri, S.-H. Shao, '24], [L. Lootens, C. Delcamp, F. Verstraete, '24], [A. Chatterjee, Ö.M. Aksoy, X.-G. Wen, '24], [R. Vanhove, V. Ravindran, D. T. Stephen, X.-G. Wen, X. Chen, '24], [S.D. Pace, H.T.Lam, Ö.M.Aksoy, '24], [L.E. Bottini, S. Schäfer-Nameki, '24], [S.-J.Huang, Y.Chen, '25], [M. Davydova et al., '25]

- **What**: study **gapped** and **gapless phases** in (1+1)d with fusion category symmetry $\mathcal{S}$: $a \otimes b = \sum_c N_{ab}^c \, c$ and F-symbols (associators)

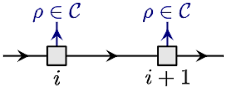
- **Why**: to understand novel gapped and gapless quantum phases of matter
  $\Rightarrow$ **Categorical Landau paradigm**

- **How**:

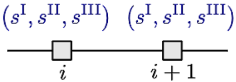
  - using the **SymTFT**: a (2+1)d TFT $\mathcal{Z}(\mathcal{S})$ with a **symmetry** and **physical boundary** on which **anyons** can condense



  - with (1+1)d **lattice models**: **general anyon chain** construction and



  - **simple spin-chains** for specific models suitable for near-future **quantum simulators**

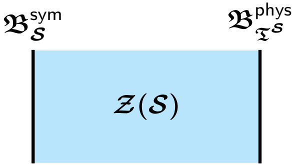


The SymTFT (club) sandwich

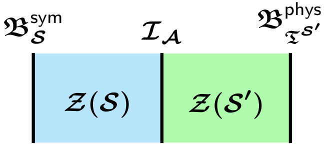
- **Quantum Phases with any Categorical Symmetry $\mathcal{S}$** $\leftrightarrow$ **Anyon condensation on the SymTFT boundaries**

- **Symmetry $\mathcal{S}$** is fixed by a maximal condensation on $\mathfrak{B}^{\text{sym}}$ of the SymTFT:
  - Generators of $\mathcal{S}$ are topological defects with Neumann b.c. on $\mathfrak{B}^{\text{sym}}$
  - Generalized charges of operators follow from linking in the SymTFT bulk

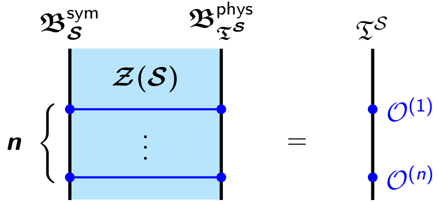
- Maximal condensation of anyons on $\mathfrak{B}^{\text{phys}}$ (SymTFT sandwich) $\Rightarrow$ **gapped phases**



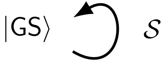
- Non-maximal condensation of anyons: interface $\mathcal{I}_{\mathcal{A}}$ (SymTFT club sandwich) $\Rightarrow$ **gapless phases**



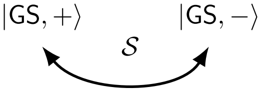
Symmetric gapped phases



- $\mathfrak{B}^{\text{phys}}$ maximal (Lagrangian) algebra $\Rightarrow$ gapped phase
- n = number of anyons ending on $\mathfrak{B}^{\text{sym}}_{\mathcal{S}}$ and $\mathfrak{B}^{\text{phys}}_{\mathfrak{T}^{\mathcal{S}}}$



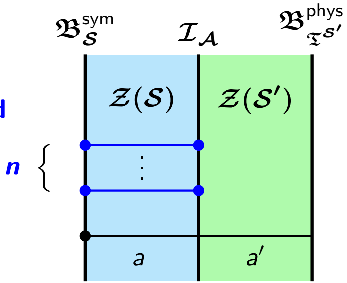
- **SPT**: symmetry protected topological phase
  $n = 1$ symmetric ground state



- **SSB**: spontaneous symmetry breaking phase $n > 1$ ground states related by $\mathcal{S}$
- This classification is valid for **any categorical symmetry** $\mathcal{S}$
  $\Rightarrow$ **Categorical Landau paradigm**

- **Gapless** phases: **non-maximal** SymTFT anyon condensation on $\mathcal{I}_{\mathcal{A}}$

  $\mathcal{Z}(\mathcal{S})/\mathcal{A} = \mathcal{Z}(\mathcal{S}')$ for a reduced symmetry $\mathcal{S}'$



anyons
- end on $\mathfrak{B}^{\text{sym}}_{\mathcal{S}}$ and $\mathcal{I}_{\mathcal{A}}$ $\Rightarrow$ **condensed**
- non-local with $\mathcal{A}$ $\Rightarrow$ **confined**
- local with $\mathcal{A}$ $\Rightarrow$ **de-confined**

- New phase transitions from old:

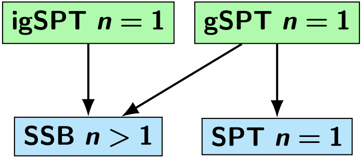
$$\mathfrak{T}^{\mathcal{S}'}_{B} \xleftarrow{\;-\mathcal{O}'\;} \mathfrak{T}^{\mathcal{S}'}_{CFT} \xrightarrow{\;+\mathcal{O}'\;} \mathfrak{T}^{\mathcal{S}'}_{A}$$

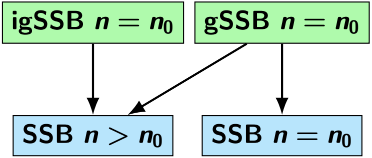
map: $\mathcal{Z}(\mathcal{S}') \to \mathcal{Z}(\mathcal{S})$ of topological defects

$$\Downarrow$$

$$\mathfrak{T}^{\mathcal{S}}_{B} \xleftarrow{\;-\mathcal{O}\;} \mathfrak{T}^{\mathcal{S}}_{CFT} \xrightarrow{\;+\mathcal{O}\;} \mathfrak{T}^{\mathcal{S}}_{A}$$

- **(g)SPT**: (gapless) symmetry preserving phase $n = 1$
- **(g)SSB**: (gapless) spontaneous symmetry breaking phase $n > 1$
- **Intrinsically gapless** (igSPT or igSSB) phases: can only be deformed to gapped phases by spontaneously breaking symmetry





Example: Non-invertible Rep(D_8) symmetry

- D_8 = group of symmetries of a square

$$\boldsymbol{D_8}: \quad \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \mid \underbrace{\boldsymbol{a}^2 = \boldsymbol{b}^2 = \boldsymbol{c}^2 = \mathbf{1}}_{\mathcal{H}=\otimes(\mathbb{C}^2)^3 \text{ on the lattice}},\ \underbrace{\boldsymbol{cac} = \boldsymbol{b}}_{\text{non-abelian}} \rangle$$

$$\mathbf{Rep}(\boldsymbol{D_8}): \quad \mathbf{1},\ \mathbf{1}_a,\ \mathbf{1}_c,\ \mathbf{1}_a\mathbf{1}_c,\ \boldsymbol{E}$$

$$\boldsymbol{E} \otimes \boldsymbol{E} = \mathbf{1} \oplus \mathbf{1}_a \oplus \mathbf{1}_c \oplus \mathbf{1}_a\mathbf{1}_c \quad \leftarrow \quad \textbf{non-invertible fusion}$$

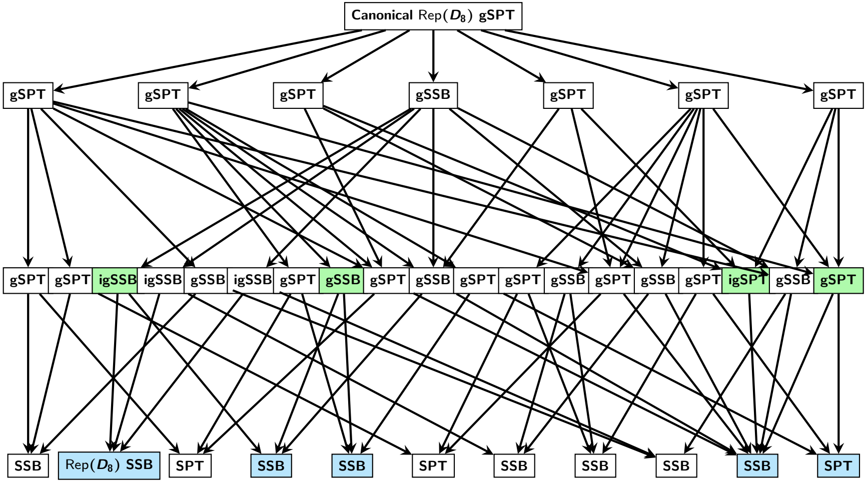
- Anyons in the SymTFT are labelled by $([\boldsymbol{g}], \boldsymbol{R})$.

- The symmetry Lagrangian algebra for Rep(D_8) is:

$$\mathcal{A}^{\text{sym}}_{\text{Rep}(D_8)} = ([\mathbf{1}], \mathbf{1}) \oplus ([\boldsymbol{ab}], \mathbf{1}) \oplus ([\boldsymbol{a}], \mathbf{1}) \oplus ([\boldsymbol{c}], \mathbf{1}) \oplus ([\boldsymbol{ca}], \mathbf{1})$$

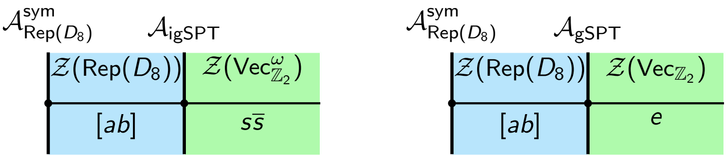
- We computed from the SymTFT all possible condensations of anyons for $\boldsymbol{\mathcal{S}} = \mathbf{Rep}(\boldsymbol{D_8})$ categorical symmetry. There are 38 phases.

Full Rep(D_8) phase diagram

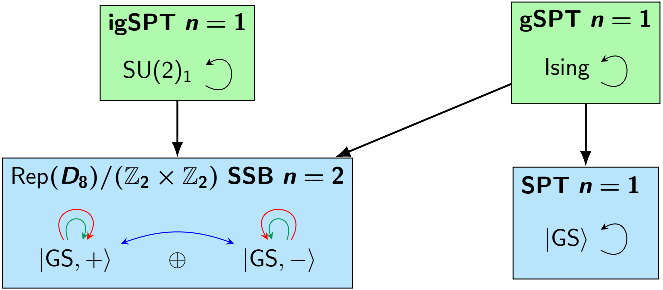


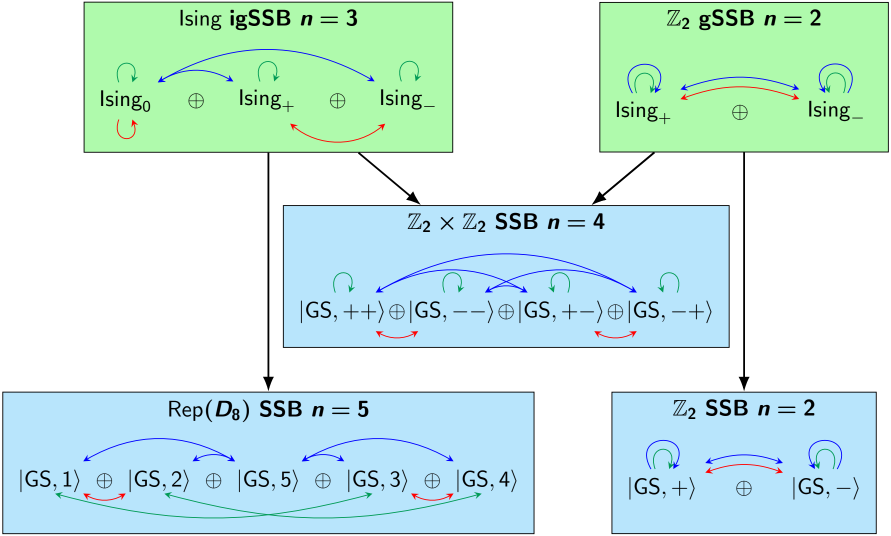
Gapless phases: igSPT vs. gSPT for $\mathrm{Rep}(D_8)$

- The SymTFT gives the map of generalized charges



- For $\mathbb{Z}_2^{\omega}$ input $\mathrm{SU}(2)_1$, for $\mathbb{Z}_2$ the Ising CFT $\Rightarrow$ get $\mathrm{Rep}(D_8)$ gapless phases




Ising igSSB $n = 3$
$\text{Ising}_0 \oplus \text{Ising}_+ \oplus \text{Ising}_-$
$\mathbb{Z}_2$ gSSB $n = 2$
$\text{Ising}_+ \oplus \text{Ising}_-$
$\mathbb{Z}_2 \times \mathbb{Z}_2$ SSB $n = 4$
$|GS, ++\rangle \oplus |GS, --\rangle \oplus |GS, +-\rangle \oplus |GS, -+\rangle$
Rep(D_8) SSB $n = 5$
$|GS, 1\rangle \oplus |GS, 2\rangle \oplus |GS, 5\rangle \oplus |GS, 3\rangle \oplus |GS, 4\rangle$
$\mathbb{Z}_2$ SSB $n = 2$
$|GS, +\rangle \oplus |GS, -\rangle$

- The Categorical Landau paradigm predicts novel phases and transitions
- Can we substantiate the SymTFT predictions by constructing concrete (1+1)d Hamiltonian lattice models exhibiting these phase transitions?
- Can we provide experimental proposals for some models, e.g. in cold atom quantum simulators?

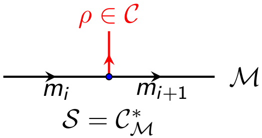
- The Categorical Landau paradigm predicts novel phases and transitions
- Can we substantiate the SymTFT predictions by constructing concrete (1+1)d Hamiltonian lattice models exhibiting these phase transitions?
- Can we provide experimental proposals for some models, e.g. in cold atom quantum simulators?

YES

- Many works on lattice models with categorical symmetry (see slides 1-2)
- An **anyon chain** with any $\mathcal{S}$ is a (1+1)d Hamiltonian lattice model defined by:

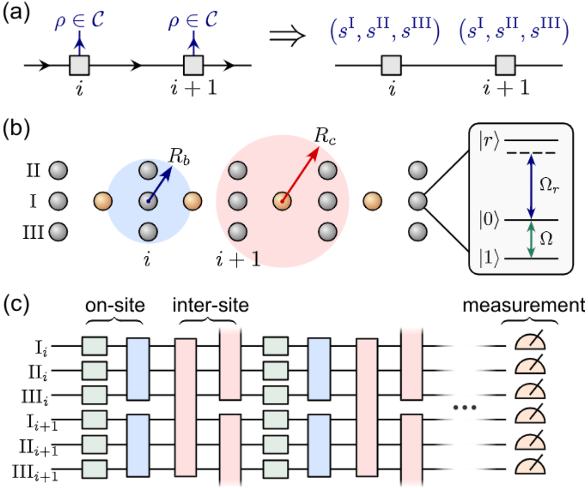
$$\begin{cases} \text{An input fusion category } \mathcal{C}, \\ \text{A } \mathcal{C}\text{-module category } \mathcal{M}. \text{ The symmetry category is } \mathcal{S} = \mathcal{C}^*_{\mathcal{M}} \\ \text{A (generically non-simple) object } \rho \in \mathcal{C} \text{ and } h \in \rho \otimes \rho \to \rho \otimes \rho. \end{cases}$$

- **Hamiltonians** act by changing the basis state "from above", whereas **symmetry operators** act "from below" $\Rightarrow$ the model is $\mathcal{S}$-symmetric



- For certain categories, the anyon chain becomes a **simple spin-chain** on a **tensor-product Hilbert space** acted on by **generalized Pauli** $\boldsymbol{X}$ **and** $\boldsymbol{Z}$, e.g. $\mathcal{S} = \text{Rep}(D_8)$

simple spin-chain model with $\mathcal{S} = \text{Rep}(D_8)$ encompassing all gapped phases and phase transitions, from $D_8 = \langle a, b, c \mid \underbrace{c^2 = a^2 = b^2 = 1}_{\mathcal{H}=\otimes(\mathbb{C}^2)^3 \text{ on the lattice}},\ cac = b \rangle$



- Multiplication operators for generators of D_8:

$$L^a = X^{\mathrm{II}}\,, \qquad R^a = \frac{1}{2}(\mathbb{I} + Z^{\mathrm{I}})\,X^{\mathrm{II}} + \frac{1}{2}(\mathbb{I} - Z^{\mathrm{I}})\,X^{\mathrm{III}}\,,$$

$$L^b = X^{\mathrm{III}}\,, \qquad R^b = \frac{1}{2}(\mathbb{I} + Z^{\mathrm{I}})\,X^{\mathrm{III}} + \frac{1}{2}(\mathbb{I} - Z^{\mathrm{I}})\,X^{\mathrm{II}}\,,$$

$$L^c = X^{\mathrm{I}}\,(\mathrm{Swap})^{\mathrm{II,III}} = \frac{1}{2}(\mathbb{I} + Z^{\mathrm{II}}Z^{\mathrm{III}})\,X^{\mathrm{I}} + \frac{1}{2}(\mathbb{I} - Z^{\mathrm{II}}Z^{\mathrm{III}})\,X^{\mathrm{I}}X^{\mathrm{II}}X^{\mathrm{III}}\,,$$

$$R^c = X^{\mathrm{I}}$$

- Hamiltonians: $H_{(F,\beta)} = -\frac{1}{|F|}\sum_i \sum_{f\in F} (R_\beta^{f^{-1}})_i (L_\beta^f)_{i+1} - \sum_i P_i^{(F)}$,

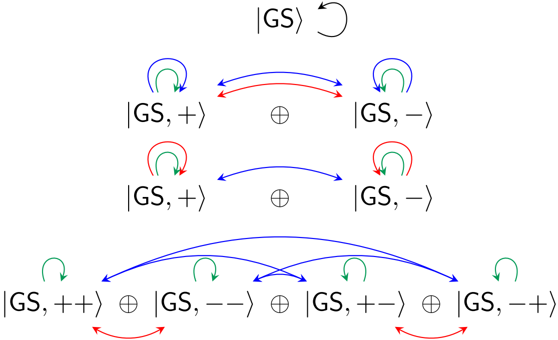
  where $F \subseteq D_8\,, \quad \beta \in H^2(F, U(1))$

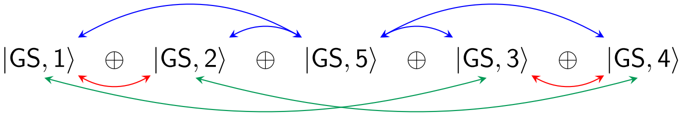
- Symmetry operators:

$$\mathcal{S}_{1_a} = \prod_i (Z^{\mathrm{I}})_i\,, \qquad \mathcal{S}_{1_c} = \prod_i (Z^{\mathrm{II}}Z^{\mathrm{III}})_i, \qquad \mathcal{S}_{1_{ca}} = \prod_i (Z^{\mathrm{I}}Z^{\mathrm{II}}Z^{\mathrm{III}})_i$$

$$\mathcal{S}_E = Z^{1,1}_{E,\mathrm{prod}} + Z^{2,2}_{E,\mathrm{prod}}\,, \qquad Z_E = \tfrac{1}{2}\begin{pmatrix} Q^{1_{ca},+}(S^{\mathrm{I}})^\dagger Z^{\mathrm{III}} & Q^{1_{ca},-}\,(S^{\mathrm{II}})^\dagger S^{\mathrm{III}}(CZ)^{\mathrm{II,III}} \\ Q^{1_{ca},-}\,S^{\mathrm{II}}(S^{\mathrm{III}})^\dagger (CZ)^{\mathrm{II,III}} & Q^{1_{ca},+}\,S^{\mathrm{I}}Z^{\mathrm{III}} \end{pmatrix}$$

- 3 SPTs with $\boldsymbol{n} = \mathbf{1}$
- 3 $\mathbb{Z}_2$ SSBs with $\boldsymbol{n} = \mathbf{2}$
- 1 $\frac{\mathbf{Rep}(\boldsymbol{D}_8)}{(\mathbb{Z}_2 \times \mathbb{Z}_2)}$ SSB with $\boldsymbol{n} = \mathbf{2}$
- 3 $\mathbb{Z}_2 \times \mathbb{Z}_2$ SSBs with $\boldsymbol{n} = \mathbf{4}$
- 1 Rep($\boldsymbol{D}_8$) SSB with $\boldsymbol{n} = \mathbf{5}$





- $H_1 = -\sum_i \left[\mathbb{I} + \frac{1}{8}(\mathbb{I} + Z^{\text{I}})(\mathbb{I} + Z^{\text{II}})(\mathbb{I} + Z^{\text{III}})\right]_i$

  unique Rep(D_8)-symmetric tensor-product ground state

  $$|\text{GS}\rangle_{\text{Triv}} = \bigotimes_i |000\rangle_i \quad \Rightarrow \quad \text{Trivial Rep}(D_8)\text{-SPT}$$

- $H_{(\mathbb{Z}_2^a \times \mathbb{Z}_2^b)^-} \approx -\frac{1}{2}\sum_i \left[X_i^{\text{II}}(X^{\text{II}}Z^{\text{III}})_{i+1} + (Z^{\text{II}}X^{\text{III}})_i X_{i+1}^{\text{III}} + (\mathbb{I}_i + Z_i^{\text{I}})\right]$

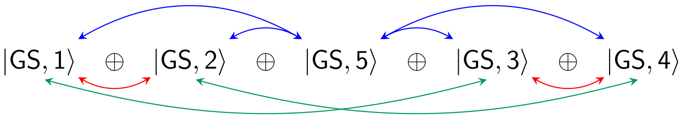
  unique Rep(D_8)-symmetric entangled ground state

  $$|\text{GS}\rangle_{\text{SPT}} = \prod_i \frac{1}{\sqrt{2}}\left[\mathbb{I}_i\mathbb{I}_{i+1} + (Z^{\text{II}}X^{\text{III}})_i X_{i+1}^{\text{III}}\right] \bigotimes_i |0,+,0\rangle_i$$

  Non-trivial Rep(D_8)-SPT detected by string order parameters, e.g. $X_{i_0}^{\text{II}}(\prod_{k=1}^{N-1} Z_{i0+k}^{\text{III}})Z_{i0+N}^{\text{III}}X_{i0+N}^{\text{II}}$, that have unit vev

- The ground state of $H_{(\mathbb{Z}_2^c \times \mathbb{Z}_2^{ab})^-}$ is similarly a non-trivial Rep(D_8)-SPT

Rep(D_8) SSB



- One can write the Rep(D_8) SSB ground states as follows:

$$|\mathrm{GS},1\rangle = \bigotimes_i \sum_{g\in D_8} \frac{1}{\sqrt{8}} |g\rangle_i \ ,$$

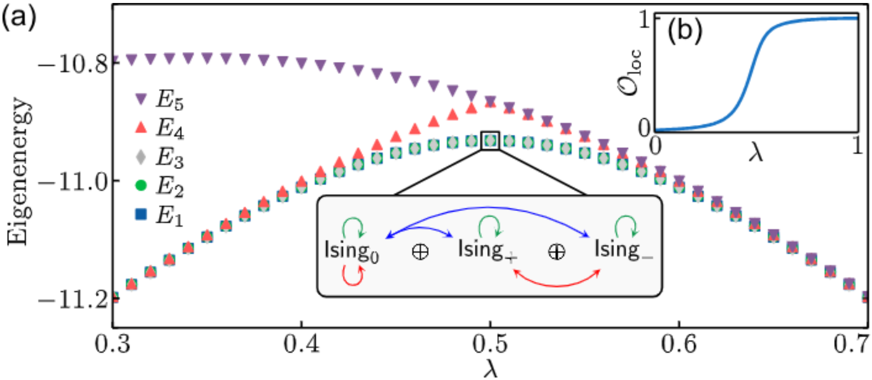
$$|\mathrm{GS},2\rangle = \mathcal{S}_{1_c} |\mathrm{GS},1\rangle \ , \qquad |\mathrm{GS},3\rangle = \mathcal{S}_{1_a} |\mathrm{GS},1\rangle \ ,$$

$$|\mathrm{GS},4\rangle = \mathcal{S}_{1_{ca}} |\mathrm{GS},1\rangle \ , \qquad |\mathrm{GS},5\rangle = \mathcal{S}_{E} |\mathrm{GS},1\rangle$$

- $\mathcal{S}_E |\mathrm{GS},5\rangle = |\mathrm{GS},1\rangle + |\mathrm{GS},2\rangle + |\mathrm{GS},3\rangle + |\mathrm{GS},4\rangle \ \longleftarrow$ Non-invertible action
- $|\mathrm{GS},p\rangle$ for $p = 1,2,3,4$ are tensor-product states, whereas $|\mathrm{GS},5\rangle$ has non-zero entanglement

Indeed, for the 2-dim irrep E, the character of the product of group elements along the spin-chain is not the product of characters.

$H(\lambda)$ is the linear interpolation between a $\mathbb{Z}_2 \times \mathbb{Z}_2$ SSB and the Rep(D_8) SSB



The critical point is an Ising igSSB:

3 copies of the Ising CFT related by spontaneously broken Ising category

Summary

- Categorical Landau paradigm for phases with any fusion category symmetry $\mathcal{S}$
- The SymTFT gives:

  - ground state degeneracy and symmetry-breaking properties
  - local and string order parameters
  - new gapless phases and phase transitions
  - Hasse phase diagram: all possible symmetry-preserving deformations

  $\Rightarrow$ identify intrinsically gapless phases
- Lattice models can be constructed to explicitly verify SymTFT predictions, e.g. spin-chain with $\mathcal{S} = \text{Rep}(D_8)$ for quantum simulators with cold atoms
- The SymTFT is similarly powerful in (2+1)d:

  [2408.05266, 2502.20440, 2503.12699, by L. Bhardwaj, $\pm$ Y. Gai, $\pm$ S.-J. Huang, $\pm$ K. Inamura, $\pm$ D. Pajer, S. Schäfer-Nameki, Apoorv Tiwari, **A.W.**]

  (Apoorv Tiwari's talk and related works, also on the lattice)