A new approach to the mean-field limit of Vlasov-Fokker-Planck equations

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Introduction

Our first objective is to obtain the mean-field limit for kinetic models such as the Vlasov-Poisson-Fokker-Planck, while keeping the full singularity.

More generally we wish to understand better the statistical properties of large systems of agents/particles with realistic, singular interactions:

- Can we control how agents/particles may concentrate or aggregate in a given region? This is critical for the mean-field limit but also of interest on its own.
- What can be said of such systems in a transient regime when away from molecular chaos, when correlations cannot be neglected?
- Are there physical quantities that can be propagated, at least over some time scales, to answer this?

$\rightarrow$ When diffusion is present, it is possible to weight observables with the energy to do just that.

From very large particles: Galaxies

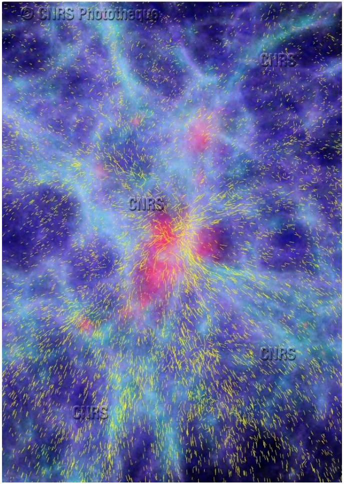

Figure: Credits: CNRS, France; Numerical simulation of the formation of large scale structures in the universe: Dynamics of galaxies moving to the central concentration.

To very small agents: Biological neurons

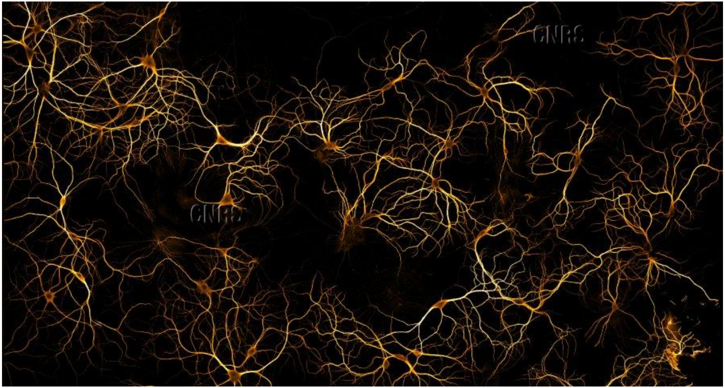

Figure: Credits: CNRS Bordeaux, France; 2D reconstruction of rat hippocampus, marked for cytoskeleton protein.

Particles or agent are everywhere

Many-particle or multi-agent systems are used in a widespread range of applications

- Plasmas: Particles are ions or electrons.
- Astrophysics: Particles are dark matter particles, galaxies or galaxy clusters...
- Fluids: Point vortices, suspensions...
- Bio-mechanics: Medical aerosols in the respiratory tract, suspensions in the blood...
- Bio-Sciences: Collective behaviors of animals, swarming or flocking, but also dynamics of micro-organisms, chemotaxis, cell migration, neural networks...
- Social Sciences: Opinion dynamics, consensus formation...
- Economics: Mean-field games...

Just as much variation in the number of particles

What is N the number of particles or agents under consideration?

- In cosmology/astrophysics, N ranges from 10^{10} to $10^{20} - 10^{25}$; some models of dark matter even predict up to 10^{60} particles.
- In plasma dynamics, N is typically of order $10^{20} - 10^{25}$. This is the typical order of magnitude for physics settings.
- When used for numerical purposes (particles' methods...), the number is of order $10^{9} - 10^{12}$.
- In biology or Life Sciences, typical population of micro-organisms include between 10^{6} and 10^{12}.
- In other applications such as collective dynamics, Social Sciences or Economics, numbers can be much lower of order 10^{3}.

The inside of the future Tokamak at ITER

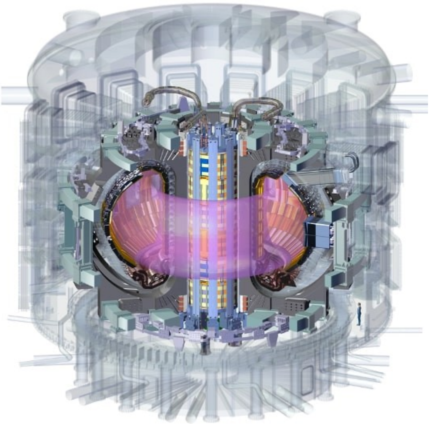

Figure: Credits: ITER, France.

A guiding example: The dynamics of point charges

Consider ions or electrons in a plasma when their velocities is small enough w.r.t. the speed of light. Denote by

m_i = Total mass of particle $\#i$, q_i = Total charge of particle $\#i$,
$X_i(t)$ = position of the center of mass at time t,
$V_i(t)$ = velocity of the center of mass at time t.

Then we have the following system of coupled SDE's

$$\frac{d}{dt}X_i(t) = V_i(t), \quad m_i\, dV_i(t) = \sum_{j\neq i} q_i\, q_j\, K(X_i - X_j) + \sigma\, dW_i, \quad (1)$$

with the electrostatic force derived by Coulomb in 1785

$$K(x) = \frac{x}{|x|^3} \quad \text{in dimension } 3, \qquad K(x) = \frac{x}{|x|^d} \quad \text{in dimension } d.$$

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where the W_i are independent Brownian motions, representing collisions against a random background (electrons against the background of ions for example).

Our model under the mean-field scaling

We consider the following multi-agent/many-particle system

$$\frac{d}{dt}X_i = V_i, \quad X_i(0) \in \Pi^d,$$
$$dV_i = S(X_i)\,dt + \frac{1}{N}\sum_{j\neq i} K(X_j - X_i)\,dt + \sigma\,dw_i, \quad V_i(0) \in \mathbb{R}^d.$$

We consider indistinguishable particles or agents, leading to an exchangeable system. This is a classical assumption that makes sense for some settings (electrons...) and less for others. However our method would extend to non-exchangeable systems under reasonable assumptions.

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We assume the mean-field scaling, which formally makes the interaction sum of order 1.
As long as K is homogeneous, this is equivalent to fixing the scales and in particular the time scale.

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The kernel K represents a general two-body interaction term. Typically for us, K is unbounded and singular and we should try to require as few assumptions as possible on it.
One may also include a self-interaction force $S(x)$ which can be an external magnetic fluid or self-propulsion.
It would also be possible to add a friction term $-V_i\,dt$ in the force term.

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For simplicity, we assume that the positions X_i are on the torus Π^d. The case of bounded domains with proper boundary conditions could be handled in a similar manner. Taking X_i in the whole $\mathbb{R}^d$ would require adjustments.

The velocities V_i are a priori unbounded in the whole space.

Brief overview of the existing literature

The rigorous derivation of mean-field limit for Vlasov-Poisson-Fokker-Planck had remained fully open, in spite of many efforts:

- The case of Lipschitz interactions $K(x)$ was handled by McKean 67 and Sznitman 91 for the stochastic setting and by Braun-Hepp 77, Dobrushin 79 in the deterministic setting. Still important to further understand the framework. See for example Golse 16, Golse-Mouhot-Ricci 13, Hauray-Mischler 14, Mischler-Mouhot 13...
- Mild singularities $K(x) << |x|^{-1}$ were handled in Hauray-Jabin 09 and 15.
- Truncated kernels (essential for numerics) in Boers-Pickl 16, Lazarovici-Pickl 17, Pickl 19 and in Huang-Liu-Pickl with diffusion.
- Singularity not at the origin: Carrillo-Choi-Hauray-Salem 18 for swarming models.

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- Deriving Vlasov-Poisson and Vlasov-Poisson-Fokker-Planck in dimension 1 seems to be more accessible, as per Hauray-Salem 19, Guillin–Le Bris–Monmarché.
- Derivations of fluid equations or first order macroscopic systems directly from second order models are also known, see Duerinckx–Serfaty 20 and Han-Kwan–Iacobelli 21.
- Marginals also play a key role in understanding fluctuations, Lacker 21, and corrections to the mean-field limit as in Duerinckx–Saint-Raymond 21.
- Deriving collisions models is even harder, also relies on controlling the marginals (without diffusion!). See Lanford 75 and more recently Gallagher–Saint-Raymond–Texier 14, Bodineau–Gallagher–Saint-Raymond 17, Bodineau–Gallagher–Saint-Raymond–Simonella 20 or Pulvirenti–Saffirio–Simonella 14, Pulvirenti–Simonella 17

Marginals or observables

The statistical information about the system is contained in the various marginals or observables:

$$f_k(t, x_1, v_1, \dots, x_k, v_k) = \text{Law at time } t \text{ of } X_1, V_1, \dots, X_k, V_k.$$

For example f_1 is the 1-particle distribution, while f_2 contains information about correlations between particles.
The various marginals are nested in a natural hierarchy

$$f_k(t, x_1, v_1, \dots, x_k, v_k) = \int_{\Pi^d \times \mathbb{R}^d} f_{k+1}(t, x_1, v_1, \dots, x_{k+1}, v_{k+1})\, dx_{k+1}\, dv_{k+1}.$$

Marginals control concentrations of particles

Consider for example a small region $\Omega \subset \Pi^d \times \mathbb{R}^d$. The average proportion of particles concentrated Ω is directly given by integrating f_1. To control concentrations, we need to bound

$$\int_\Omega f_1(t,x,v)\,dx\,dv << 1, \quad \text{if } |\Omega| << 1.$$

If Ω is a ball or spherically symmetric, we can sometimes use the potential energy. Otherwise, any L^p bound on f_1 would allow to quantify this with for example

$$\int_\Omega f_1(t,x,v)\,dx\,dv \leq |\Omega|^{1/2}\,\|f_1(t,.,.)\|_{L^2(\Pi^d\times\mathbb{R}^d)}.$$

However if we only look at concentrations in positions with $\Omega = \omega \times \mathbb{R}^d$, we would also require control of the tails in velocity.

How can we get bounds on the marginals?

There are only two already known ways of obtaining such L^p bounds on the marginals:

- Through some strong propagation of chaos. This means assuming that the (X_i^0, V_i^0) are independent and identically distributed, or i.i.d., and proving that at time t, the (X_i, V_i) are almost i.i.d. as well. This allows to use the mean-field limit to estimate concentrations. However we are instead hoping that the bounds on the marginal will help with propagation of chaos.
- Make use of the Gibbs entropy of the system. This is straightforward (see next slide) but provides a very weak estimate

$$\int_\Omega f_1(t,x,v)\,dx\,dv \le \frac{C}{\log 1/|\Omega|}.$$

The Gibbs entropy

Because the interactions are divergence free, the Gibbs entropy of the full joint law is decreased

$$\frac{1}{N}\int f_N(t,x_1,v_1,\dots,x_N,v_N)\,\log f_N\,dx_1\,dv_1\dots dx_N\,dv_N$$
$$\leq \frac{1}{N}\int f_N^0(x_1,v_1,\dots,x_N,v_N)\,\log f_N^0\,dx_1\,dv_1\dots dx_N\,dv_N.$$

Moreover since the entropy is sub-additive then

$$\int f_1(t,x_1,v_1)\,\log f_1\,dx_1\,dv_1$$
$$\leq \frac{1}{N}\int f_N(t,x_1,v_1,\dots,x_N,v_N)\,\log f_N\,dx_1\,dv_1\dots dx_N\,dv_N.$$

Unfortunately, it is the only simple quantity satisfying those properties...

The BBGKY hierarchy

Each marginal solves a linear PDE

$$\partial_t f_k + \sum_{i=1}^{k} v_i \cdot \nabla_{x_i} f_k + \sum_{i\leq k} \left(S(x_i) + \frac{1}{N} \sum_{j\leq k} K(x_i - x_j) \right) \cdot \nabla_{v_i} f_k$$
$$+ \frac{N-k}{N} \sum_{i\leq k} \nabla_{v_i} \cdot \int_{\Pi^d \times \mathbb{R}^d} f_{k+1}\, K(x_i - x_{k+1}) dx_{k+1} dv_{k+1}$$
$$= \frac{\sigma^2}{2} \sum_{i\leq k} \Delta_{v_i} f_k. \tag{2}$$

Unfortunately each equation involves the next marginal f_{k+1}; more precisely and even worse because of unbounded velocities, it involves

$$\int_{\mathbb{R}^d} f_{k+1}\, dv_{k+1}.$$

Propagating bounds on f_k

In general the issue when trying to propagate L^p bound on f_k is that it would require to bound

$$\left\| \nabla_{v_i} \int_{\mathbb{R}^d} f_{k+1} \, dv_{k+1} \right\|_{L^p} .$$

This leads to unrealistic assumptions as the control on f_1 requires a control on $\nabla_v f_2$, then $\nabla_v^2 f_3$ and so on...
However when using the regularizing effect of the diffusion, it is possible to improve this and only require

$$\left\| \int_{\mathbb{R}^d} f_{k+1} \, dv_{k+1} \right\|_{L^p} .$$

This only leaves the issue of unbounded velocities...

Our new result

We need to use the energy reduced to k particles by defining

$$e_k(x_1, v_1, \ldots, x_k, v_k) = \sum_{i \leq k} (1 + |v_i|^2) + \frac{1}{N} \sum_{i,j \leq k} \phi(x_i - x_j),$$

for the potential ϕ s.t. $K = -\nabla\phi$.

Observe that e_k is conserved by the reduced interactions

$$\sum_{i=1}^{k} \left(v_i \cdot \nabla_{x_i} + \frac{1}{N} \sum_{j=1}^{k} K(x_i - x_j) \cdot \nabla_{v_i} \right) e_k = 0.$$

For repulsive interactions, $\phi \geq 0$ and we may use e_k as a modified weight to control Gaussian decay in velocity

$$\int_{\Pi^{dk} \times \mathbb{R}^{dk}} e^{\lambda e_k} \, |f_k|^2 \, dx_1 \, dv_1 \ldots dx_k \, dv_k.$$

A simple differential inequality

Denote

$$X_k(t)=\int |f_{k,N}|^q\, e^{\lambda(t)\, e_k},\quad \lambda(t)=\frac{1}{\Lambda\,(1+t)}.$$

Then we have that

$$X_k(t)\leq X_k(0)+k\,L\int_0^t X_{k+1}(s)\,ds,$$

for some $L\sim \|K\|_{L^p}^q$.
It is straightforward to solve this hierarchy and obtain appropriate bounds provided that

$$X_k(0)\lesssim F_0^k,\quad X_N(t)\lesssim F^N.$$

Quantitative bounds on the marginals

Theorem

Assume S, $K \in L^p(\Pi^d)$ for some $p > 1$, $K = -\nabla\phi$ with $\phi \geq 0$. Define

$$\lambda(t) = \frac{1}{\Lambda\,(1+t)},$$

for a positive constant Λ, depending only on p, q, d and σ. Assume that

$$\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_k^0|^q \, e^{\lambda(0)\, e_k} \leq F_0^k,$$

for some $F_0 > 0$ and q such that $2 \leq q < \infty$, with $1/q + 1/p \leq 1$. Then, one has that

$$\sup_{t\leq T} \int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_{k,N}|^q \, e^{\lambda(t)\, e_k} \leq 2^k\, F_0^k,$$

for some $T \sim 1$ independent of N.

Implying the mean-field limit

Corollary

Under the assumptions of the previous theorem, let f be the unique smooth solution to the limiting equation with initial data $f^0 \in C^\infty(\Pi^d)$. Assume moreover that the initial marginals $f^0_{k,N}$ converges weakly in L^1 to $(f^0)^{\otimes k}$ for each fixed k for some $M > 0$ and for all $k \leq N$. Then there exists T^* depending only on M, $\|K\|_{L^p}$ and $\|(\operatorname{div} K)_-\|_{L^\infty}$ such that the marginals $f_{k,N}$ weakly converge to $f_k = f^{\otimes k}$ in $L^q_{loc}([0,\ T^\star] \times \Pi^{kd})$ for any k, and any $q < \infty$.

This result can easily be made quantitative if $K \in L^p$ with $p > 2$ and provides

$$\|f_{k,N} - f^{\otimes k}\|_{L^q} \leq \frac{C_{T,k}}{N}.$$

Conclusions

- Novel, straightforward quantitative estimates with minimal assumptions on the interaction kernel.
- Fits with the expected scaling of molecular chaos where $f_k = f^{\otimes k}$ but valid in any regime.
- Only holds for short times, in line with known blow-up in velocity moments for Vlasov-Poisson in dimension $d \geq 4$.
- Provide a convergence in $O(1/N)$ of the marginals in the mean-field limit (cf. Duerinckx, Lacker) vs. the stochastic fluctuations in $O(1/\sqrt{N})$ (see for example Fernandez-Méléard 97).

Even more straightforward for systems on bounded domains

Consider

$$\frac{d}{dt}X_i(t) = \frac{1}{N}\sum_{j\neq i} K(X_i - X_j)\,dt + \sigma\,dW_i,, \quad X_i(t=0) = X_i^0,$$

fully on the torus Π^d.

The mean-field limit is similar

$$\partial_t f + (K \star_x f)\cdot\nabla_x f = \frac{\sigma^2}{2}\Delta_x f.$$

Because this system does not involve unbounded velocities, many technical difficulties in our proofs actually vanish: We do not need to impose Gaussian decay or have K derive from a repulsive potential...

Theorem

Assume that

$$K \in L^p(\Pi^d) \quad \text{for some} \quad p > 1, \qquad (\operatorname{div} K)_- \in L^\infty(\Pi^d),$$

where x_- denotes the negative part of x. Let f be the unique smooth solution to the limiting equation with initial data $f^0 \in C^\infty(\Pi^d)$. Assume that the initial marginals $f^0_{k,N}$ converges weakly in L^1 to $(f^0)^{\otimes k}$ for each fixed k and that

$$\|f^0_{k,N}\|_{L^\infty(\Pi^{dN})} \leq M^k,$$

for some $M > 0$ and for all $k \leq N$. Then there exists T^* depending only on M, $\|K\|_{L^p}$ and $\|(\operatorname{div} K)_-\|_{L^\infty}$ such that the marginals $f_{k,N}$ weakly converge to $f_k = f^{\otimes k}$ in $L^q_{loc}([0,\ T^\star] \times \Pi^{kd})$ for any k, and any $q < \infty$.