Transversal Clifford-Hierarchy Gates via Non-Abelian Surface Codes

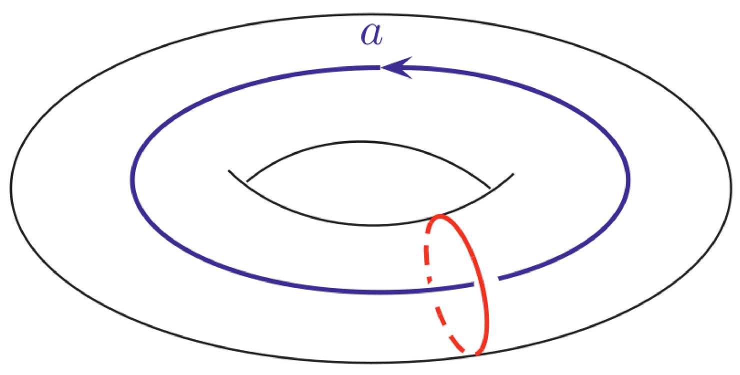
Authors: Alison Warman, Sakura Schäfer-Nameki (University of Oxford)

UNIVERSITY OF OXFORD

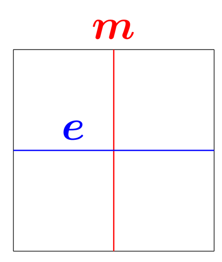
Mathematical Institute

Topological quantum codes

- Idea of using anyons to encode quantum information [Kitaev, '97] (on 2D torus)



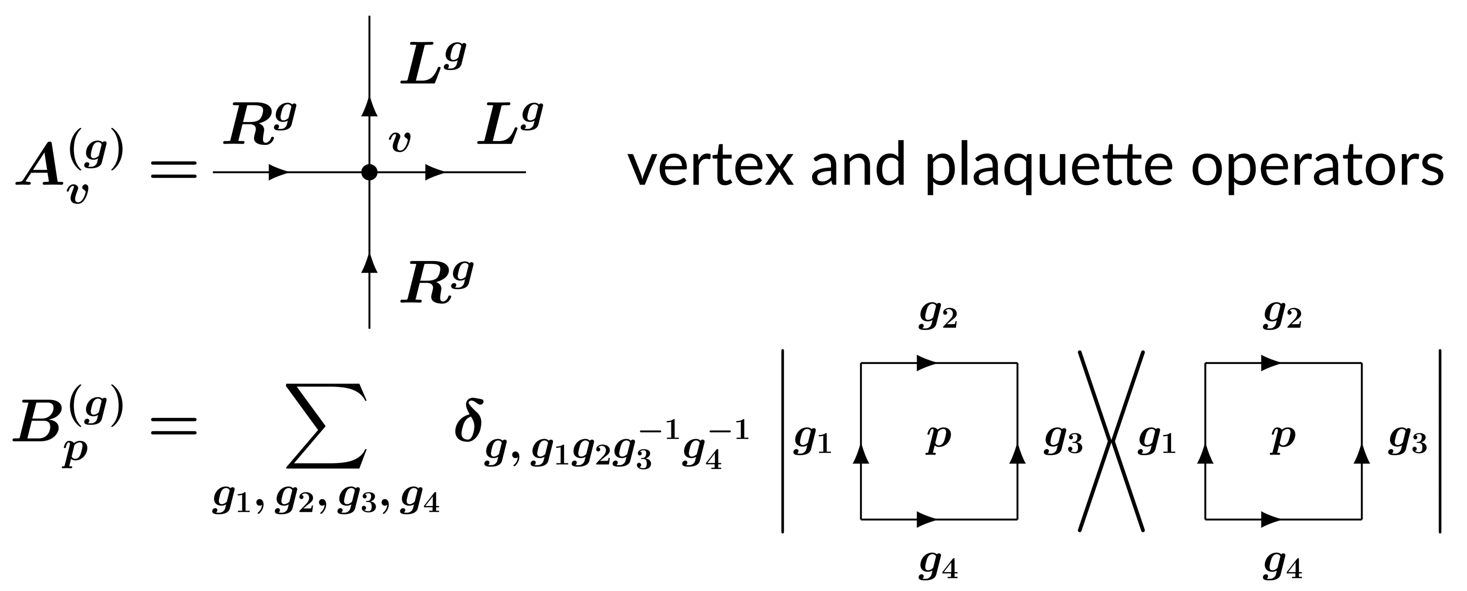
- Later extended to 2D planar surface with boundaries [Beigi, Shor, Whalen, '10]



- Logical states: inequivalent ground states of local Hamiltonian that differ by global anyon configurations
- Quantum information is stored non-locally and is topologically-protected

Group surface codes

- Local Hilbert on each edge of the lattice $\{|h\rangle \;:\; h \in G\}$
- Multiplication operators $L^g\,|h\rangle = |gh\rangle\,,\quad R^g\,|h\rangle = |hg^{-1}\rangle$



- Hamiltonian on patch with boundaries labeled by $K_i \subseteq G$

$$H_G = -\sum_v A_v - \sum_p B_p^{(\mathrm{id})} - \sum_{s_i} (A_{s_i}^{K_i} + B_{s_i}^{K_i})$$

Qubit logical states in $D(D_{4N})$

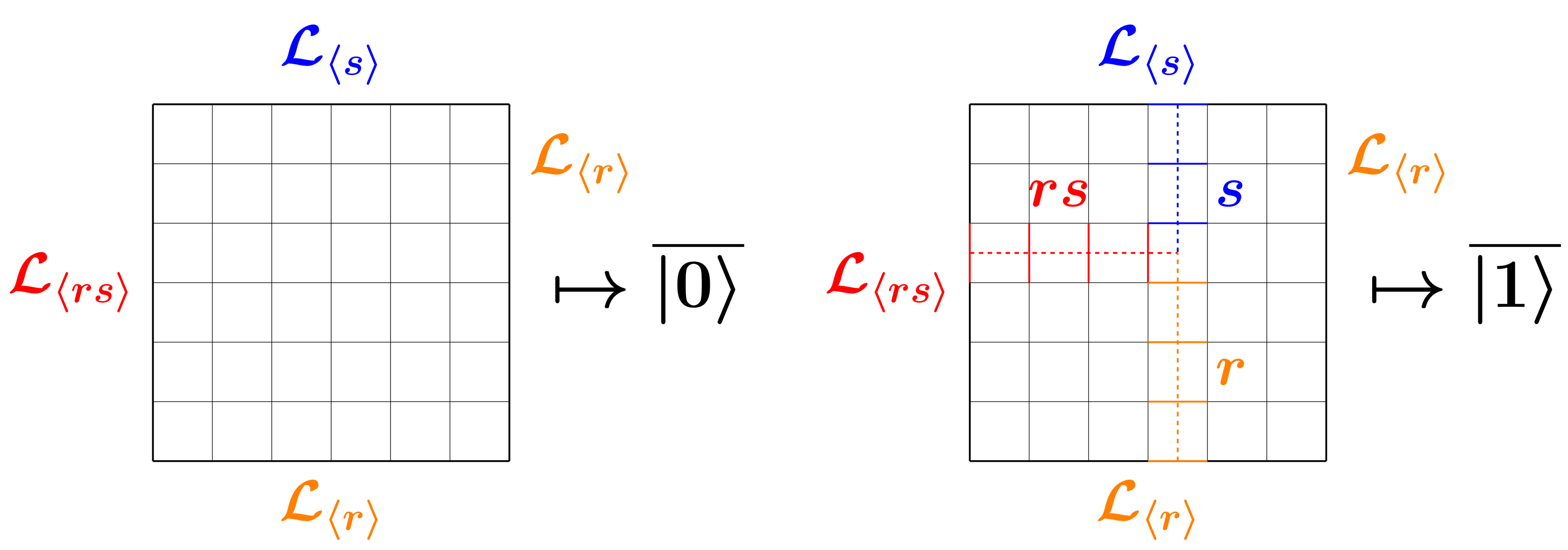
- Group of symmetries of regular $4N$-gon

$$D_{4N} = \langle r, s \,|\, r^{4N} = s^2 = \mathrm{id}\,,\; srs = r^{-1} \rangle$$

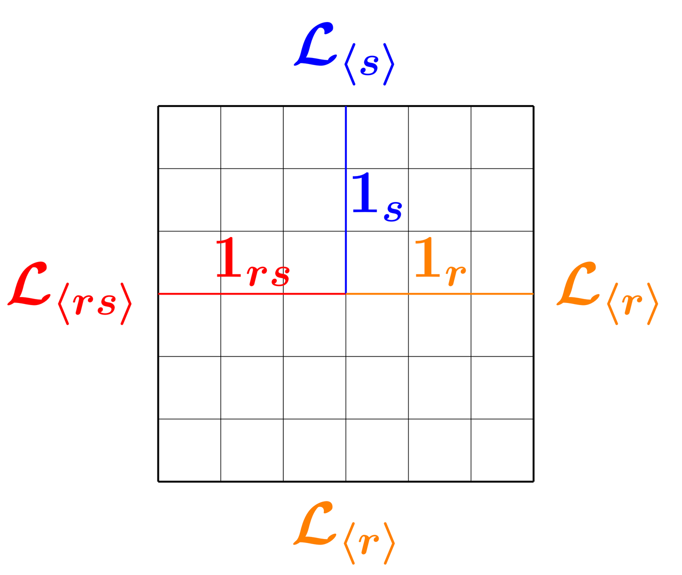
- Irreps of dimension 1, generated by 1_r and 1_s:

$$1_r(r) = +1\,, \qquad 1_r(s) = -1\,,$$
$$1_s(r) = -1\,, \qquad 1_s(s) = +1\,,$$

- Logical qubit basis states:



- $\overline{|0\rangle}$ and $\overline{|1\rangle}$ have eigenvalues $+1$ and -1 under $\overline{Z}$ whose anyons are $(1_{rs}, 1_s, 1_r)$
  $\Rightarrow$ they are logical qubit states



Bounds for topological Pauli stabilizer codes

[Bravyi–König, '12]: consider a topological code:

1. in low spatial dimensions
2. with local, nearest-neighbour Hamiltonian
3. comprised only of Pauli stabilizers

$\Rightarrow$ topologically-protected gates in n-th Clifford hierarchy level require n space dimensions

$\Rightarrow$ only Clifford gates in 2D (not universal) [Gottesman–Knill, '98]

Our results: arbitrary precision phase gates in 2D

Using $D(D_{4N})$ non-abelian codes we construct

$$T^{1/N} = \mathrm{diag}\left(1, e^{i\pi/(4N)}\right)\,, \qquad N \in \mathbb{N}$$

with arbitrary precision purely in 2D! [AW, Schafer-Nameki, '25]

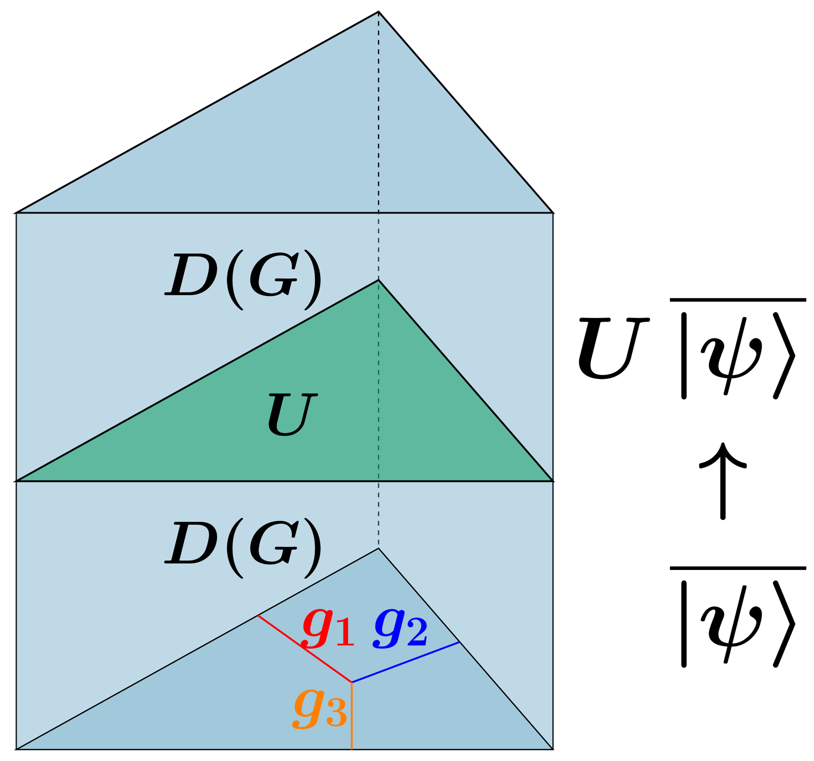
$\Rightarrow$ we bypass Bravyi–König while keeping

1. two spatial dimensions
2. local, nearest-neighbor Hamiltonian

How? Construct logical gate U from $\alpha \in H^2(G, U(1))$ on 2D spatial slice with $\beta^{(i)} : G \to U(1)$ on boundaries

$$U\overline{|0\rangle} = \overline{|0\rangle}$$

$$U\overline{|1\rangle} = \frac{\alpha(g_1,g_2)\beta^{(3)}(g_1g_2)}{\beta^{(1)}(g_1)\,\beta^{(2)}(g_2)}\overline{|1\rangle}$$



For $G = D_{4N}$

- $\overline{|0\rangle}$: $g_1 = g_2 = \mathrm{id} \;\Rightarrow\; U(\mathrm{id},\mathrm{id}) = 1$
- $\overline{|1\rangle}$: $g_1 = rs,\; g_2 = s \;\Rightarrow\; U(rs, s) = e^{i\pi/(4N)}$

Physical qubits and code-switching to $\mathbb{Z}_2 \times \mathbb{Z}_2$

- For $|D_{4N}| = 8N = 2^n$ we can map the local Hilbert space $\mathcal{H} = \{|g\rangle\;,\; g \in D_{2^{n-1}}\}$ to n physical qubits $\Rightarrow \mathcal{H} \cong (\mathbb{C}^2)^n$
- The first $n-1$ qubits correspond to $r^{2^{n-2}}, r^{2^{n-3}}, \cdots, r$, the last qubit corresponds to $s \Rightarrow \left|r^a s^j\right\rangle \mapsto |\mathrm{bin}(a)\rangle\,|j\rangle$
- Switch from/to $\mathbb{Z}_2 \times \mathbb{Z}_2$ by using the map of anyons

$$1 \sim \mathcal{A} = 1 \oplus [r^2] \oplus \cdots \oplus [r^{2N}]$$
$$e_1 \sim 1_s\,\mathcal{A}\,, \qquad m_1 \sim [r]\,\mathcal{A}$$
$$e_2 \sim 1_r\,\mathcal{A}\,, \qquad m_2 \sim [s]\,\mathcal{A}$$
$$e_1e_2 \sim 1_{rs}\,\mathcal{A}\,, \qquad m_1m_2 \sim [rs]\,\mathcal{A}$$

- Start with $\overline{|+\rangle}$ in $\mathbb{Z}_2 \times \mathbb{Z}_2$ patch $\Rightarrow$ obtain magic state

$$T^{1/N}\overline{|+\rangle} = \overline{|0\rangle} + e^{i\pi/(4N)}\overline{|1\rangle}$$

- Summary table:

N	n	D_{4N}	Level of $T^{1/N}$ Logical Gate
1	3	$D_4 = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$	3
2	4	$D_8 = \mathbb{Z}_8 \rtimes \mathbb{Z}_2$	4
4	5	$D_{16} = \mathbb{Z}_{16} \rtimes \mathbb{Z}_2$	5
2^{n-3}	n	$D_{2^{n-1}} = \mathbb{Z}_{2^{n-1}} \rtimes \mathbb{Z}_2$	n