Global Weak Solution to the Cauchy Problem of Spherically Symmetric Compressible Full Navier-Stokes Equations.

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This is a joint work with Gui-Qiang G. Chen(University of Oxford) and Shengguo Zhu(Shanghai Jiao Tong University).

Outline for the presentation

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1. Introduction

1.1 Introduction: Equations

For dimensions $n = 2, 3$, heat-conducting compressible flow is governed by the following system of partial differential equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \vec{U}) = 0\,, \\ \partial_t(\rho \vec{U}) + \operatorname{div}(\rho \vec{U} \otimes \vec{U}) + \nabla P = \operatorname{div}\mathbb{S}\,, \\ \partial_t(\rho E) + \operatorname{div}((\rho E + P)\vec{U}) = \operatorname{div}(\mathbb{S} \cdot \vec{U}) + \operatorname{div}(\kappa_Q \nabla \theta)\,, \end{cases} \tag{CNS}$$

where $\rho \geq 0$ is mass density, $\vec{U} = (U^1, \ldots, U^n) \in \mathbb{R}^n$ is velocity, $P \geq 0$ is pressure, $E := e + \frac{|\vec{U}|^2}{2}$ is total energy per unit mass, e is internal energy per unit mass, θ is temperature, $\kappa_Q \geq 0$ is heat conduction coefficient, and $\mathbb{S}$ is viscous stress tensor.

Two assumptions: (1) Ideal gas; (2) Polytropic, i.e. $e = c_V \theta$. Hence,

1. $P = P(\rho, e) = (\gamma - 1)\rho e$, where $\gamma > 1$.
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1.1 Introduction: Viscous Stress Tensor

The viscous stress tensor, $\mathbb{S}$ is given as follows:

$$\mathbb{S}(\nabla\vec{U}) := \mu(\nabla\vec{U} + \nabla\vec{U}^\top) + \lambda\mathbb{I}_n \mathrm{div}\vec{U},$$

where $\mathbb{I}_n$ is $n \times n$ identity matrix, μ is the shear viscosity coefficient and quantity $\frac{2}{n}\mu + \lambda$ is the bulk viscosity coefficient.

We assume μ, λ to be given constants. Then the term $\mathrm{div}\mathbb{S}$ can be written as the following second order linear differential operator:

$$\mathrm{div}\mathbb{S}(\nabla\vec{U}) = L\vec{U} := \mu\,\triangle\vec{U} + (\mu+\lambda)\,\nabla\mathrm{div}\vec{U}.$$

Moreover, we also impose the following physical condition:

$$\mu > 0 \quad \text{and} \quad \frac{2}{n}\mu + \lambda \geq 0\,.$$

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For spherically symmetric solution, it takes the form:

$$(\rho, \vec{U}, e)(\vec{x}, t) = (\rho(|\vec{x}|, t), u(|\vec{x}|, t)\frac{\vec{x}}{|\vec{x}|}, e(|\vec{x}|, t)), \quad (\vec{x}, t) \in \mathbb{R}^n \times [0, \infty)\,.$$

Denote $r \equiv |\vec{x}|$ and $m \equiv n-1$. Then $(\rho, u, e)(r,t)$ satisfies the equations:

$$\begin{cases} \partial_t \rho + u\partial_r \rho + \rho\Big(\partial_r u + m\dfrac{u}{r}\Big) = 0, \\ \rho\partial_t u + \rho u \partial_r u + \partial_r P = \beta\partial_r\Big(\dfrac{\partial_r(r^m u)}{r^m}\Big), \\ \rho\partial_t e + \rho u\partial_r e + P\dfrac{\partial_r(r^m u)}{r^m} = \mathcal{D} + \kappa\dfrac{\partial_r(r^m\partial_r e)}{r^m}, \end{cases} \tag{SNS}$$

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We call it the **(Spherically Symmetric) Cauchy Problem** if the domain is set in $(r,t) \in [0,\infty) \times [0,\infty)$, with the initial condition:

$$(\rho, u, e)(r, 0) = (\rho_0, u_0, e_0)(r) \quad \text{for} \quad r \in [0, \infty). \tag{ID}$$

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- Kawashima and Nishida 1981 extended the result of Kazhikhov-Shelukhin to the Cauchy Problem in the entire spatial domain $x \in \mathbb{R}$. Moreover,

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2. Main Theorem

2.1 Definition of Weak Solution I

Definition 1.1 (Weak Solution)

Given $T > 0$ *and initial data* $(\rho_0, \vec{U}_0, e_0)(\vec{x})$ *in* $\vec{x} \in \mathbb{R}^n$, *we say* $(\rho, \vec{U}, e)(\vec{x}, t)$ *is a weak solution to the Cauchy Problem* **(CNS)** *and* **(ID)** *in* $(\vec{x}, t) \in \mathbb{R}^n \times [0, T]$ *if*

1. *there exists an upper semi-continuous map* $\underline{r}(t) : [0, T] \to [0, \infty)$ *and a constant* $C_0 = C_0(\rho_0, u_0, e_0) > 0$, *such that*

$$\sup_{t \in [0,T]} \underline{r}(t) \leq C_0 \quad \textit{and} \quad \lim_{t \to 0^+} \underline{r}(t) = 0. \quad \underline{r}(t) \textit{ is called "vacuum radius".}$$

Using $\underline{r}(t)$, *we define the fluid region* $\mathcal{F}$, *and the vacuum region* $\mathcal{V}$ *as:*

$$\mathcal{F} := \{(\vec{x}, t) \in \mathbb{R}^n \times [0, T] : \underline{r}(t) < |\vec{x}| < \infty\}, \quad \mathcal{V} := \mathbb{R}^n \backslash \mathcal{F}.$$

2. $\rho \in L^\infty_{loc}(\mathcal{F})$ *and* $(\vec{U}, e) \in H^1_{loc}(\mathcal{F})$. $\rho(\vec{x}, t) = 0$ *for a.e.* $(\vec{x}, t) \in \mathcal{V}$.
3. *the weak form of continuity equation holds for any* $\Phi \in \mathcal{C}^1([0, T]; \mathcal{C}^1_c(\mathbb{R}^n))$.
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Theorem 1.1 (G.-Q. Chen, S. Zhu, Y. H., 2022, arXiv:2208.05094)

Denote $r = |\vec{x}|$. Let $(\rho_0, \vec{U}_0, e_0)(\vec{x}) = (\rho_0(r), u_0(r)\frac{\vec{x}}{r}, e_0(r))$ be a spherically symmetric initial data in $\vec{x} \in \mathbb{R}^n$ such that, there exists $C_0 > 0$ for which it satisfies the following conditions:

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Then for any $T > 0$, there exists a spherically symmetric weak solution $(\rho, \vec{U}, e)(\vec{x}, t)$ to the problem **(CNS)** *and* **(ID)** *in the domain $(\vec{x}, t) \in \mathbb{R}^n \times [0, T]$.*

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\begin{aligned}
& C_0^{-1} \leq e_0(r), \quad C_0^{-1} \leq \rho_0(r) \leq C_0 \qquad \text{for } r \in [0, \infty), \\
& \int_0^\infty \{|\rho_0 - 1|^2 + |u_0|^4 + |e_0 - 1|^2\}(r) r^m \mathrm{d}r \leq C_0, \\
& \int_0^\infty \big\{\frac{1}{2}\rho_0|u_0|^2 + (\gamma - 1)G(\rho_0) + \rho_0\psi(e_0)\big\}(r) r^m \mathrm{d}r \leq C_0,
\end{aligned}
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where $G(\zeta) := 1 - \zeta + \zeta \log \zeta$ *and* $\psi(\zeta) := \zeta - 1 - \log \zeta$.

Then for any $T > 0$, *there exists a spherically symmetric weak solution* $(\rho, \vec{U}, e)(\vec{x}, t)$ *to the problem* (**CNS**) *and* (**ID**) *in the domain* $(\vec{x}, t) \in \mathbb{R}^n \times [0, T]$.

2.2 Global Existence of Weak Solution I

Theorem 1.1 (G.-Q. Chen, S. Zhu, Y. H., 2022, arXiv:2208.05094)

Denote $r = |\vec{x}|$. *Let* $(\rho_0, \vec{U}_0, e_0)(\vec{x}) = (\rho_0(r), u_0(r)\frac{\vec{x}}{r}, e_0(r))$ *be a spherically symmetric initial data in* $\vec{x} \in \mathbb{R}^n$ *such that, there exists* $C_0 > 0$ *for which it satisfies the following conditions:*

$$
\begin{aligned}
&C_0^{-1} \leq e_0(r), \quad C_0^{-1} \leq \rho_0(r) \leq C_0 \qquad \text{for } r \in [0,\infty),\\
&\int_0^\infty \{|\rho_0 - 1|^2 + |u_0|^4 + |e_0 - 1|^2\}(r) r^m \mathrm{d}r \leq C_0,\\
&\int_0^\infty \big\{\frac{1}{2}\rho_0|u_0|^2 + (\gamma - 1)G(\rho_0) + \rho_0\psi(e_0)\big\}(r) r^m \mathrm{d}r \leq C_0,
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Then for any $T > 0$, *there exists a spherically symmetric weak solution* $(\rho, \vec{U}, e)(\vec{x}, t)$ *to the problem* (**CNS**) *and* (**ID**) *in the domain* $(\vec{x}, t) \in \mathbb{R}^n \times [0, T]$.

Theorem 1.1 (G.-Q. Chen, S. Zhu, Y. H., 2022, arXiv:2208.05094)

1. *For $T > 0$, there exists $C(T) = C(C_0, T) > 0$ such that*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\mathbb{R}^n} \{G(\rho) + \frac{\rho|\vec{U}|^2}{2}\}(\vec{x}, t)\mathrm{d}\vec{x} \leq C(T).$$

Moreover, there exists a positive, continuous, strictly increasing function $g : [0, \infty) \to [0, \infty)$ with $\lim_{y\to 0^+} g(y) = 0$ such that,

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_E \rho e(\vec{x}, t)\mathrm{d}\vec{x} \leq C(T) + g(\int_E \mathrm{d}\vec{x}),$$

for all bounded measurable set $E \subset \mathbb{R}^n$.

2.2 Global Existence of Weak Solution II

Theorem 1.1 (G.-Q. Chen, S. Zhu, Y. H., 2022, arXiv:2208.05094)

1. *For $T > 0$, there exists $C(T) = C(C_0, T) > 0$ such that*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\mathbb{R}^n} \{G(\rho) + \frac{\rho|\vec{U}|^2}{2}\}(\vec{x},t)\mathrm{d}\vec{x} \le C(T).$$

Moreover, there exists a positive, continuous, strictly increasing function $g : [0,\infty) \to [0,\infty)$ with $\lim_{y\to 0^+} g(y) = 0$ such that,

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_E \rho e(\vec{x},t)\mathrm{d}\vec{x} \le C(T) + g\Big(\int_E \mathrm{d}\vec{x}\Big),$$

for all bounded measurable set $E \subset \mathbb{R}^n$.

2.2 Global Existence of Weak Solution III

Theorem 1.1 (G.-Q. Chen, S. Zhu, Y. H., 2022, arXiv:2208.05094)

2. *There exists a continuous map* $\tilde{r}(y,t) : (0,\infty)\times[0,T]\to[0,\infty)$ *s.t.*
   - 2a. $y\mapsto\tilde{r}(y,t)$ *is strictly monotone increasing for all* $t\in[0,T]$,
   - 2b. $\underline{r}(t)=\lim_{y\to 0+}\tilde{r}(y,t)$ *for a.e.* $t\in[0,T]$,
   - 2c. *for a.e.* $y>0$ *and* $t\in[0,T]$,

$$\int_{\underline{r}(t)<|\vec{x}|\leq\tilde{r}(y,t)}\rho(\vec{x},t)\mathrm{d}\vec{x}=y.$$

   Using this, one defines $\mathcal{F}_\varepsilon:=\{(\vec{x},t)\in\mathbb{R}^n\times[0,T]:|\vec{x}|\geq\tilde{r}(\varepsilon,t)\}$.

3. *For any* $\varepsilon>0$, *there exists* $C(\varepsilon)=C(\varepsilon,T,C_0)>0$ *such that for all* $(\vec{x},t)$, $(\vec{y},t)$, $(\vec{x},s)$, $(\vec{x},t)\in\mathcal{F}_\varepsilon$ *with* $0<s<t$,

$$t|e(\vec{x},t)-e(\vec{y},t)|+t^{\frac{1}{2}}|\vec{U}(\vec{x},t)-\vec{U}(\vec{y},t)|\leq C(\varepsilon)|\vec{x}-\vec{y}|^{\frac{1}{2}},$$

$$s|e(\vec{x},t)-e(\vec{x},s)|+s^{\frac{1}{2}}|\vec{U}(\vec{x},t)-\vec{U}(\vec{x},s)|\leq C(\varepsilon)|t-s|^{\frac{1}{4}},$$

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$$\int_{\underline{r}(t)<|\vec{x}|\le\tilde{r}(y,t)} \rho(\vec{x},t)\mathrm{d}\vec{x} = y.$$

Using this, one defines $\mathcal{F}_\varepsilon := \{(\vec{x},t) \in \mathbb{R}^n \times [0,T] : |\vec{x}| \ge \tilde{r}(\varepsilon,t)\}$.

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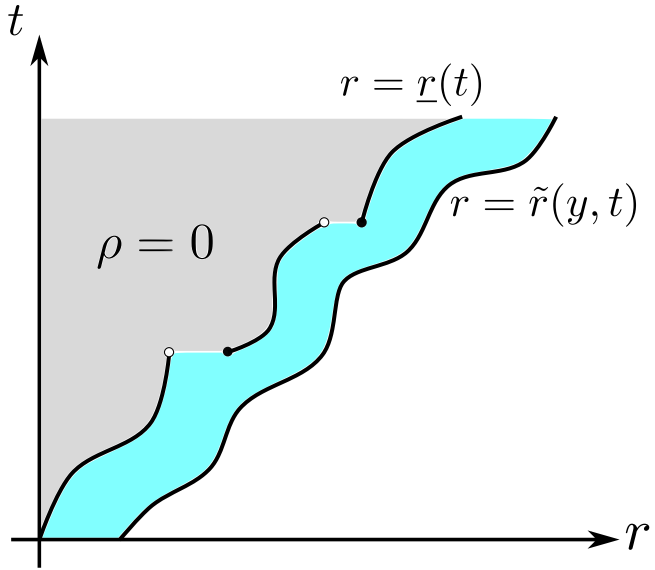
2.2 Vacuum Radius and y-Mass Radius



3. Main Strategy

3.1 Spherically Symmetric Exterior Problem I

If one supposes that the map $\vec{x} \mapsto (\vec{U}, \nabla e)(\vec{x}, t)$ is continuous at the origin $\vec{x} = \vec{0}$. Then the spherically symmetric condition implies that

$$u(0,t) = \partial_r e(0,t) = 0 \ \text{ for all } \ t \in [0,\infty). \qquad (\mathcal{O})$$

The main difficulty is the singular terms such as $m\beta\partial_r\left(r^{-1}u\right)$ and $\kappa r^{-1}\partial_r e$ in the equations.

Motivated by $(\mathcal{O})$, one introduces the **Exterior Problem** with parameter $a \in (0,1)$, by imposing the boundary condition:

$$u(a,t) = \partial_r e(a,t) = 0 \ \text{ for } t \in [0,\infty) \text{ at } r = a.$$

Note that the above condition corresponds to a physical model where there is an insulating solid ball of radius $a \in (0,1)$ centred at the origin, and we call it the **Exterior Boundary Condition**.

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Note that the above condition corresponds to a physical model where there is an insulating solid ball of radius $a \in (0,1)$ centred at the origin, and we call it the **Exterior Boundary Condition**.

Furthermore, we impose the condition near far-field region $|\vec{x}| \to \infty$:

$$\lim_{r\to\infty} (\rho, u, e, \partial_r e)(r,t) = (1,0,1,0) \quad \text{for all} \quad t \in [0,\infty).$$

Combining with the previously mentioned exterior boundary condition, we define the **Eulerian (Spherically Symmetric) Exterior Problem** with radius $a \in (0,1)$ as

$$\begin{cases} \textbf{(SNS)} & \text{in } [a,\infty)\times[0,\infty), \\ u(a,t) = \partial_r e(a,t) = 0 & \\ \lim\limits_{r\to\infty} (\rho, u, e, \partial_r e)(r,t) = (1,0,1,0) & \text{for all } t\in[0,\infty), \\ (\rho, u, e)(r,0) = (\rho_a^0, u_a^0, e_a^0)(r) & \text{for all } r \in [a,\infty). \end{cases} \qquad \textbf{(EE)}_a$$

where $(\rho_a^0, u_a^0, e_a^0)(r)$ is the modified initial data from $(\rho_0, u_0, e_0)(r)$.

3.1 Spherically Symmetric Exterior Problem II

Furthermore, we impose the condition near far-field region $|\vec{x}| \to \infty$:

$$\lim_{r\to\infty} (\rho, u, e, \partial_r e)(r,t) = (1,0,1,0) \quad \text{for all} \quad t \in [0,\infty)\,.$$

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3.2 Lagrangian Coordinate: General MD Case

Let $(\rho, \vec{U}, e)(\vec{y}, t)$ in $(\vec{y}, t) \in \mathbb{R}^n \times [0, \infty)$ be a solution to **(CNS)** such that $C^{-1} \leq \rho \leq C$ for some $C > 0$.

For general multi-dimensional flow, Let $\vec{X} : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ be the characteristic curve satisfying:

$$\begin{cases} \dfrac{\mathrm{d}\vec{X}}{\mathrm{d}t}(\vec{z}, t) = \vec{U}(\vec{X}(\vec{z}, t), t) & \text{for } t \in [0, \infty), \\ \vec{X}(\vec{z}, 0) = \varphi_0(\vec{z}) & \text{for } \vec{z} \in \mathbb{R}^n, \end{cases} \qquad \textbf{(Char)}$$

where $\varphi_0 : \mathbb{R}^n \to \mathbb{R}^n$ is a given diffeomorphism.

Then the **Eulerian** coordinate variables $(\vec{y}, t) \in \mathbb{R}^n \times [0, \infty)$ and **Lagrangian** coordinate variables $(\vec{z}, t) \in \mathbb{R}^n \times [0, \infty)$ satisfy:

$$(\vec{y}, t) = (\vec{X}(\vec{z}, t), t)$$

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Then the **Eulerian** coordinate variables $(\vec{y}, t) \in \mathbb{R}^n \times [0, \infty)$ and **Lagrangian** coordinate variables $(\vec{z}, t) \in \mathbb{R}^n \times [0, \infty)$ satisfy:

$$(\vec{y}, t) = (\vec{X}(\vec{z}, t), t)$$

3.2 Lagrangian Coordinate: General MD Case

Let $(\rho, \vec{U}, e)(\vec{y}, t)$ in $(\vec{y}, t) \in \mathbb{R}^n \times [0, \infty)$ be a solution to **(CNS)** such that $C^{-1} \leq \rho \leq C$ for some $C > 0$.

For general multi-dimensional flow, Let $\vec{X} : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ be the characteristic curve satisfying:

$$\begin{cases} \dfrac{\mathrm{d}\vec{X}}{\mathrm{d}t}(\vec{z}, t) = \vec{U}(\vec{X}(\vec{z}, t), t) & \text{for } t \in [0, \infty), \\ \vec{X}(\vec{z}, 0) = \varphi_0(\vec{z}) & \text{for } \vec{z} \in \mathbb{R}^n, \end{cases} \qquad \textbf{(Char)}$$

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3.2 Lagrangian Coordinate: Spherical Exterior Problem

Let $(\rho_a, \vec{U}_a, e_a)(\vec{y}, t) = (\rho_a(|\vec{y}|, t), u_a(|\vec{y}|, t)\frac{\vec{y}}{|\vec{y}|}, e(|\vec{y}|, t))$ be a symmetric solution to the Exterior Problem in $\{|\vec{y}| \geq a\}$. We denote $r \equiv |\vec{y}|$ and $x \equiv |\vec{z}|$.

Set $\varphi_0(\vec{z})$ as the spherically symmetric map:

$$\varphi_0(\vec{z}) = \tilde{r}_a^0(|\vec{z}|)\frac{\vec{z}}{|\vec{z}|} \quad \text{where} \quad x = \int_a^{\tilde{r}_a^0(x)} \rho_a^0(r) r^m \mathrm{d}r \quad \text{for all } x \geq 0.$$

The choice of φ_0 physically means that $x \geq 0$ amount of initial mass is contained in the annular domain between $r = a$ and $r = \tilde{r}_a^0(x)$.

By uniqueness of the solution to **(Char)**, $\vec{z} \to \vec{X}(\vec{z}, t)$ is spherically symmetric, thus we can define $\tilde{r}_a(|\vec{z}|, t) := \sqrt{|\vec{X}(\vec{z}, t)|^2}$. Suppose $\tilde{r}_a \geq a$, then

$$\begin{cases} \partial_t \tilde{r}_a(x, t) = u_a(\tilde{r}_a(x, t), t) & \text{for } t \in [0, T], \\ \tilde{r}_a(x, 0) = \tilde{r}_a^0(x) & \text{for } x \geq 0. \end{cases}$$

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3.2 Lagrangian Reformulation

Under spherical symmetry, the Lagrangian variables (x,t) is related to the Eulerian variables (r,t) via the relation: $r = \tilde{r}_a(x,t)$.

Denote $\tilde{v}(x,t) := 1/\rho_a(\tilde{r}_a(x,t),t)$, $(\tilde{u},\tilde{e})(x,t) := (u_a,e_a)(\tilde{r}_a(x,t),t)$, and $\tilde{r}(x,t) := \tilde{r}_a(x,t)$. Then equations (**SNS**) can be reformulated as

$$\begin{cases} D_t\tilde{v} = D_x(\tilde{r}^m\tilde{u}) \\ D_t\tilde{u} + \tilde{r}^m D_x p(\tilde{v},\tilde{e}) = \beta\tilde{r}^m D_x\Big(\dfrac{D_x(\tilde{r}^m\tilde{u})}{\tilde{v}}\Big) \\ D_t\tilde{e} - \kappa D_x\Big(\dfrac{\tilde{r}^{2m}}{\tilde{v}}D_x\tilde{e}\Big) = \mathcal{G}(\tilde{r},\tilde{v},\tilde{u},\tilde{e}) \end{cases} \quad \text{in } (x,t)\in[0,\infty)^2, \qquad \textbf{(LNS)}$$

where $\quad \mathcal{G}(r,v,u,e) = \beta\dfrac{|D_x(r^m u)|^2}{v} - p(v,e)D_x(r^m u) - 2m\mu D_x(r^{m-1}u^2),$

and $p(v,e) := (\gamma-1)e/v$. Moreover (**LNS**) is supplemented with the non-linear coefficient $\tilde{r} = \tilde{r}(x,t)$ defined by:

$$\tilde{r}(x,t) := \Big(a^n + n\int_0^x \tilde{v}(z,t)\,\mathrm{d}z\Big)^{\frac{1}{n}}.$$

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3.2 Approximation problems in finite annuli

(1) Construct $(v_{a,k}, u_{a,k}, e_{a,k})$ to the approximation Lagrangian problem:

$$\begin{cases} \textbf{(LNS)} & \\ r_{a,k}(x,t) = \Big(a^n + n\displaystyle\int_0^x v_{a,k}(y,t)\mathrm{d}y\Big)^{\frac{1}{n}} & \text{in } [0,k]\times[0,\infty), \\ u_{a,k}(0,t) = D_x e_{a,k}(0,t) = 0 & \\ u_{a,k}(k,t) = D_x e_{a,k}(k,t) = 0 & \text{for } t\in[0,\infty), \\ (v_{a,k}, u_{a,k}, e_{a,k})|_{t=0} = (v^0_{a,k}, u^0_{a,k}, e^0_{a,k}) & \text{for } x\in[0,k]. \end{cases} \qquad \textbf{(LE)}_{a,k}$$

(2) Convert the approximate solutions into Eulerian coordinate:

$$(\bar\rho_{a,k}, \bar u_{a,k}, \bar e_{a,k})(r,t) = (v^{-1}_{a,k}, u_{a,k}, e_{a,k})(x_{a,k}(r,t),t)$$

where $r \mapsto x_{a,k}(r,t)$ is the inverse of $x \mapsto r_{a,k}(x,t)$.

(3) Extend $(\bar\rho_{a,k}, \bar u_{a,k}, \bar e_{a,k})(r,t)$ into $r\in[0,\infty)$ by cut-off function $\varphi_{a,k}(r)$:

$$\tilde\rho_{a,k} = \bar\rho_{a,k}\varphi_{a,k} + (1-\varphi_{a,k}), \quad \tilde e_{a,k} = \bar e_{a,k}\varphi_{a,k} + (1-\varphi_{a,k}), \quad \tilde u_{a,k} = \bar u_{a,k}\varphi_{a,k}.$$

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3.3 Uniform a-priori estimates: bounds on density I

To take limit $k \to \infty$ and $a \to 0^+$, some uniform estimates on $(v_{a,k}, u_{a,k}, e_{a,k})$ are required. This is achieved by first obtaining the **uniform point-wise upper and lower bounds on density**.

Lemma 2.1 (Entropy Estimate)

Set $\psi(\zeta) := \zeta - \log\zeta - 1$, *and define* $\mathcal{S} := (\gamma - 1)\psi(v) + \psi(e) + \frac{|u|^2}{2}$ *to be the entropy. If* (v, u, e) *is a solution to the approximation Lagrangian Exterior Problem* $(\textbf{LE})_{a,k}$ *then*

$$\int_0^k \mathcal{S}(x,t)\mathrm{d}x + \int_0^t \int_0^k \Big\{\big(\frac{2\mu}{n} + \lambda\big)\frac{|D_x(r^m u)|^2}{ve} + \kappa\frac{r^{2m}|D_x e|^2}{ve^2}\Big\}\mathrm{d}x\mathrm{d}t$$

$$= \int_0^k \mathcal{S}(x,0)\mathrm{d}x.$$

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3.3 Uniform a-priori estimates: bounds on density II

Substitute the continuity equation $D_t v = D_x(r^m u)$ into the momentum equation:

$$D_t u + r^m D_x p = \beta r^m D_x\Big(\frac{D_x(r^m u)}{v}\Big), \quad \text{where } \beta = 2\mu + \lambda > 0,$$

then multiply both sides by r^{-m}, and since $D_t r(x,t) = u(x,t)$, we have

$$D_t\big(\frac{u}{r^m}\big) + m\frac{|u|^2}{r^n} + D_x p = \beta D_t D_x \log v.$$

Integrating the above equation in the region $(y,s) \in [x_1, x_2] \times [0,t]$ and then take exponential on both sides, we get the **representation formula for density**

$$\begin{aligned}&\frac{v_0(x_2)v(x_1,t)}{v_0(x_1)v(x_2,t)}\\ &= \exp\Big(\int_{x_1}^{x_2} \frac{u}{\beta r^m} \mathrm{d}y\Big|_{s=t}^{s=0} + \int_0^t \frac{\gamma-1}{\beta}\frac{e}{v}\mathrm{d}s\Big|_{y=x_2}^{y=x_1} - \int_0^t\int_{x_1}^{x_2} \frac{m|u|^2}{\beta r^n}\mathrm{d}y\mathrm{d}s\Big) \equiv \mathcal{I}.\end{aligned}$$

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$$\frac{v_0(x_2)v(x_1,t)}{v_0(x_1)v(x_2,t)} = \exp\Big(\int_{x_1}^{x_2} \frac{u}{\beta r^m}\mathrm{d}y\Big|_{s=t}^{s=0} + \int_0^t \frac{\gamma-1}{\beta}\frac{e}{v}\mathrm{d}s\Big|_{y=x_2}^{y=x_1} - \int_0^t\int_{x_1}^{x_2}\frac{m|u|^2}{\beta r^n}\mathrm{d}y\mathrm{d}s\Big) \equiv \mathcal{I}.$$

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$\mathcal{I}$ can be bounded by Grönwall's inequality and the entropy estimate, Lemma 2.1.

From this formula for density, we can determine two **explicit** functions $\overline{v}(x,t)$, $\underline{v}(x,t) : [0,k]\times[0,\infty] \to (0,\infty)$, which are independent of (a,k) so that

$$\underline{v}(\varepsilon,T) \le v(x,t) \le \overline{v}(\varepsilon,T) \quad \text{for all } (x,t)\in[\varepsilon,k]\times[0,T], \text{ for each } \varepsilon>0.$$

The restriction $x\in[\varepsilon,k]$ comes from the lower bound of $r(x,t)$. By entropy estimate $\int_0^k \psi(v)\mathrm{d}x \le C_0$, and Jensen's inequality, one has for each $\varepsilon>0$,

$$\Big(n\varepsilon\psi_-^{-1}\big(\frac{C_0}{\varepsilon}\big)\Big)^{\frac{1}{n}} \le r(x,t) \quad \text{in} \quad (x,t)\in[\varepsilon,k]\times[0,\infty).$$

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3.3 Uniform a-priori estimates: bounds on density III

$$\frac{v_0(x_2)v(x_1,t)}{v_0(x_1)v(x_2,t)}$$
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Once the point-wise upper and lower bounds of $v(x,t)$ and $r(x,t)$ are obtained, one can utilise the parabolic structure:

$$D_t u - \beta r^m D_x\Big(\frac{D_x(r^m u)}{v}\Big) = \{\cdots\}, \qquad D_t e - \kappa D_x\Big(\frac{r^{2m}}{v} D_x e\Big) = \{\cdots\},$$

to derive the **uniform a-priori estimates** on $u(x,t)$ and $e(x,t)$.

Lemma 2.2

Assume $(v,u,e)(x,t)$ solves $(\mathbf{LE})_{a,k}$. Then, for each $\varepsilon \in (0,1]$ there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0) > 0$ independent of (a,k) such that $\mathcal{L}_\varepsilon[v,u,e](T) \le C(\varepsilon)$, where $\sigma(t) := \min\{1,t\}$ and

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(1) Since $C(\varepsilon)^{-1} \leq v(x,t) \leq C(\varepsilon)$ is restricted in $x \in [\varepsilon, k]$, it is necessary to incorporate a cut-off function $g_\varepsilon \in C^1([0,\infty])$ such that supp$(g_\varepsilon) \subseteq [\varepsilon, \infty)$. However, integration by parts with g_ε leads to a problematic boundary term:

$$\int_0^T \int_\varepsilon^{2\varepsilon} \{e + |u|^2\}(x,t)\,\mathrm{d}x\mathrm{d}t,$$

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THANK YOU FOR YOUR ATTENTION!