Pricing and Hedging
Life Insurance Guarantees
in Incomplete Market Setting

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This thesis is dedicated to my wife and daughter Heidi
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Abstract

Recent evolution in the annuity markets suggest an evolution towards simplified products design and less complex embedded derivatives. In a competitive market largely driven by sales, the techniques used for pricing and risk management proved inefficient to adequately capture the dynamics of the various risk factors. In particular the assumption of market completeness for products with a large actuarial risk component and with embedded options on non-traded assets is problematic. Uncertainty around actuarial decrements such as mortality, lapse and partial withdrawal and basis risk resulting from the impossibility to hedge with the underlying asset make the classical option valuation theory inadequate for such products. This dissertation proposes an innovative methodology to price, structure and hedge the different market risks in equity-linked life insurance contracts consistently with realistic assumptions. The original contribution of this study lies in the introduction of the concept of implied utility which allows an intuitive treatment of the game option. Implied utility is also applied to the design and consistent pricing of tailor-made policies. Optimal asset allocation is calculated subject to the desired consumption rate, returns expectations and risk appetite of the insured person.
Introduction

The equity-linked life insurance market has been hit hard by the last financial crisis, exposing the inadequacy of the analytical tools employed for the risk management of these products. With tumbling equity markets, declining interest rates and unprecedented volatility, the severe market turbulence has created the worst possible environment for the hedging of variable annuity guarantees. Variable annuities can be defined as investments embedded in a life insurance contract. They promise the insured person a minimum return on investment either upon death of the policyholder, called guaranteed minimum death benefits, or at a predetermined maturity of the contract. Guarantees with predetermined maturity are known as living benefits and also frequently referred to as riders. Annuity contracts feature a large variety of different eligible benefit types and consist generally of a combination of a death and a living benefit. Originally, death benefits were simply principal guarantees, while most living benefits offered a minimum guaranteed interest on investments in bond funds. In the last decades, the combination of low interest rate environment and bullish stock markets made equity-linked contracts more appealing. In an increasingly competitive market, insurers have been challenged to provide a wider range of products and to design feature aimed at reducing the lapsation. For both types of benefits, a large variety of flavours have been made available. In some contracts, the guaranteed amount is periodically increased at a predetermined rate or reset depending of the investment’s performance. These benefits are known respectively as rollup, and reset. Sometimes the benefit is readjusted only under the condition that the level of the fund at the reset date is higher than the current benefit level. This particular type of profit locking reset is called high-water-mark or ratchet, and can be seen as a discrete look-back option. Riders can also vary in the way benefits are paid to the policy-holders. While accumulation benefits guarantee a lump-sum due at maturity, income benefits are paid in the form of an annuity with a credited interest rate that is guaranteed as well. Other commonly offered types of riders include withdrawal benefits which allow the policy-holder to make periodical withdrawals of a predetermined amount,
independently of the remaining amount invested. From a risk management point of view, the major difference between death benefits and living benefits is the possibility of optimal option exercise in the latter. Another significant source of financial risk results from the unlimited changes in asset allocation that these contracts generally allow. The effectiveness of the hedging for this type of insurance products depends on the ability of a model to capture actuarial risks such as morbidity, mortality and lapse behaviour on the one hand; and financial risks such as fluctuations in the assets value, interest rate movements, along with policy-holder’s investment decisions over time on the other hand.

In this study we propose a general framework to price, design and hedge equity-linked life insurance guarantees in an environment where certain sources of risk cannot be hedged entirely. The work will be structured as follows. In the first part we will formalise the problem of pricing distinct types of options embedded in life-insurance contracts according to the theory of arbitrage. We recall some fundamental results of the theory, we distinguish between the following three categories of claims to be evaluated: guaranteed amount payable upon death of the policy-holder, survivor benefits and the game option. The latter is a real-option that allows the client to take current gains or losses and surrender the contract. A particular attention will be paid to the modelling of the policy-holder’s option exercise behaviour. We propose an original lapse model that features some ideas borrowed from mortgage prepayment and energy derivative models. An important discussion about the construction of an appropriate probability measure for pricing will follow. The second part addresses the limitation of the arbitrage argument only to price equity-linked life insurance. We describe the viability of the specific life-insurance market specifically in a way that excludes obvious arbitrage opportunities for the issuers, but allows the buyer to have an option exercise behaviour that is suboptimal from a financial point of view. We proposes an innovative marginal-utility based approach to the valuation problem from the policy-holder’s perspective, whereby optimality of investment and consumption plan replaces the arbitrage argument, largely irrelevant to the average client. We apply some well establish stochastic optimisation results and estimate the utility function implied by the buyer’s elected benefits and initial asset allocation. We show how this set-up may improve forecasts of lapse decisions and hence narrows down the set of martingale measures for pricing. Another useful application of our utility based model is that it is also able provide optimal asset allocation strategies. Thus, policies may be structured and priced consistently with the risks involved with
a built-in natural hedge. We end the chapter with a comparison of the model performances of classical portfolio insurance. The question of choosing an optimal hedging strategy in incomplete market will be addressed in the third chapter. In life-insurance derivatives, incompleteness results from the uncertainty surrounding mortality and policy-holder’s option exercise decisions. Most life insurance contracts offer investment vehicles which are not exchange traded, giving rise to another important source of basis risk. Due to the impossibility of duplicating claims, typically contingent on mutual funds returns, with liquid financial instruments, classical option pricing schemes are not applicable for this type of instruments. The fourth chapter presents a numerical analysis. We show some results on the hedge effectiveness of incomplete market models discussed in chapter 3 in comparison with the simplistic assumptions applied in the first chapter.
Chapter 1

Embedded options valuation

When it comes to valuing the claims embedded in insurance contracts, a number of distinct modelling problems must be considered. The defining element of the life insurance contract is the death benefit. Its value depends on a combination of financial elements and a class of risk factors, that will be referred to as actuarial risks in the sequel, including mortality and morbidity as well as other risks related to the economic behaviour of the insured persons. This chapter is organised as follows. After refreshing some of the key results of the arbitrage theory, we will formalise the problem of valuing death benefits in a general finite time setting. We will then explore some simplifying assumptions on the mortality process and derive some rather explicit solutions for the price of death benefits. The second part of the second section presents a more realistic model for mortality risk. The model builds upon a square-root mean reverting model proposed by Møller and Dahl (2006) and inspired by a well-established model of the term structure of interest rates. We will prove the applicability of the analytical solutions borrowed from interest rate theory without further substantiation in Møller’s paper for the purpose of modelling survival probability. We will also suggest a more practical implementation of the model’s parameters estimation. In the third section, we turn to the question of valuing the living benefits. These benefits have characteristics of plain financial options and have, in general, a higher financial value. The complexity of these products resides mostly in the multiple processes underlying the guarantees and some interference of actuarial risk-factors. We conduct an informal discussion about selecting appropriate equity, foreign-exchange and interest rates models, and carry out a sequence of changes of probability measures that translates our problem to a simple formulation. Finally, we define a model for the surrender behaviour inspired by a mortgage-prepayment model sometimes used by banking institutions. A distinction is made between structurally observed lapse rate, and reduction in surrender intensity due to the financial value
of the contract. An implementation is displayed for a risk-neutral lapse model. We then discuss the theoretical validity of applying historical estimates in the risk-neutral world, and extend our model definition to a restricted information setting.

1.1 Arbitrage and hedging

Consider a financial market in a finite time horizon \( T \) where the fluctuations in stocks value is given by an \( \mathbb{R}^d \)-valued stochastic process \( S_t = (S_t)_{t \in [0,T]} \), defined on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\). \( \mathcal{F}_t \) is the augmentation of the \( \sigma \)-algebra generated by the process \( S \) up to time \( t \). Defining \( \mathcal{N}_t \) as the filtration generated by all subsets of the null-sets of the \( \sigma \)-algebras \( \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_t, \mathcal{F}_0 = \{\emptyset, \Omega\} \), \( \mathcal{F}_t \) is defined by

\[
\mathcal{F}_t = \mathcal{F}_t \vee \mathcal{N}_t \quad (1.1)
\]

This purely technical definition is necessary to ensure the right-continuity of the filtration\(^1\). The construction of the filtration is slightly more detailed than usual as a preliminary to the theoretical validity of the extended model in the third chapter. Define the set \( \mathcal{A} \) of admissible \((d)\)-dimensional trading strategies \( \pi_t \triangleq (\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(d)})^\top \), where the superscript \( \top \) to denote transposition of a matrix.

\[
\mathcal{A} \triangleq \left\{ \int_0^\infty \pi_u^\top \pi_u d\langle S \rangle_u < \infty \forall t \in [0, T] \right\} \quad (1.2)
\]

We further assume the existence of a riskless asset \( B_t \), the money-market account, whose value process at time \( t \in [0, T] \) is given by

\[
B_t = \exp \left\{ \int_0^t r_u du \right\} \quad (1.3)
\]

Here, \( r \) is the progressively measurable bounded instantaneous risk-free interest rate process. With these notations, the wealth process \( X_t = (X_t)_{t \in [0,T]} \) may be stated in terms of \( d \)-dimensional vectors \( \pi_t \) and \( S_t \) and the scalar \( c_t \) denoting the instantaneous consumption at time \( t \)

\[
X_t = v_t B_t + (\pi_t)^\top S_t \quad (1.4)
\]

\(^1\)See (Karatzas and Shreve, 1991, Problem 3.4.16)
Proposition 1.1.1  The value process of a self financing portfolio follows the dynamics
\[ X_t = X_0 + \int_0^t \pi_u X_u du - \int_0^t c_u du \] (1.5)

Remark  This results from the definition of the self financing property. At the beginning of a new period, when the portfolio is rebalanced, only the new value of the old portfolio can be reinvested. Hence the value of the new portfolio must equal the old portfolio’s new value less the amount used for consumption during that time frame.

\[ X_{t+1} = \sum_{i=1}^{d} \pi_{t+1}^i S_{t+1}^i = \pi_{t+1} S_{t+1} = \pi_t S_{t+1} - c_t \] (1.6)

Subtracting the time t value of the old portfolio on both side of the equation yields
\[ \pi_{t+1} S_{t+1} - \pi_t S_t = \pi_t [S_{t+1} - S_t] - c_t \] (1.7)

Letting \( dt \) getting arbitrarily close to zero, we find the dynamics of a self financing portfolio in differential notation as
\[ dX_t = \pi_t dS_t - c_t \] (1.8)

Thus, we see that the self-financing condition implies that the portfolio value evolves according to equation (1.5).

In the context of hedging the option with a set of appropriately chosen exchange-traded instruments, the evolution of the market value of the replicating portfolio claim in terms of market is the most important measure of risk. It is well known that a necessary condition for a financial market to be viable is that prices do not admit arbitrage opportunities. The definition of arbitrage opportunity can be formalised as follows

Definition 1.1.2 (Arbitrage) An arbitrage is a self-financing strategy (\( \pi \in A \)) whose value process \( X \) satisfies the conditions

(i) \( X_0^\pi \leq 0 \) and \( X_t^\pi \geq 0 \) with probability one

(ii) \( X_t^\pi > 0 \) with strictly positive probability
The pioneering work by Brennan and Schwartz (1976), Boyle and Schwartz (1977) was the first attempt to compute the fair value of the embedded options according to the paradigm of arbitrage theory which foundations were laid by Bachelier (1900), Samuelson (1965), Black and Scholes (1973). Harrison and Kreps (1979) studied the problem in discrete-time setting, Harrison and Pliska (1981) extended this result to the continuous-time case in a finite probability space setting. Kreps (1981) generalised this result to the infinite-dimension case, involving a definition of arbitrage with fairly hard topological considerations, and no clear economic interpretation. Delbaen and Schachermayer (1994) proved that if the market is exempt of arbitrage opportunity, the underlying process is a locally bounded semimartingale. The market is free of arbitrage if and only if there exists an equivalent local martingale measure \( Q \). The number of existing equivalent local martingale measures is however not specified and is closely related to the concept of completeness

**Definition 1.1.3 (Market Completeness)** A market is said to be complete if all claims are attainable

Loosely speaking, market completeness requires that a sufficient amount of assets is traded. When such is the case, the equivalent martingale measure is uniquely determined. Clearly, a necessary condition for the market to be complete is that more assets are traded than there are sources of randomness. However, there is also a requirement for sufficient diversity in asset returns. If some assets can be replicated by a linear combination of others, they are redundant and do not contribute to the completeness of the market. A rigorous mathematical condition for market completeness is given in Davis and Oblój (2008, Theorem 3.2). If some of the risk factors can not be eliminated by some strategy involving some liquid assets, the market is said to be incomplete. In that case, the market does not provide unique contingent claim prices and hedging strategies and exogenous criteria are involved. This will be extensively addressed in our third chapter.

### 1.2 Life contingent benefits

From the perspective of market risk, the particularity of life contingent claim is the absence of optimal exercise. For obvious reason, it would a priori be unreasonable to assume that death decisions could be a function of the financial value of the insurance contract. In the following sections, we formulate the problem of pricing claims

---

2See discussion paper on the work of David Kreps by Schachermayer (2002)
contingent on mortality and market risks. The problem of pricing the policy can be viewed as a put option with random exercise time \( \tau \). Defining \( \tau' \) as the random time of death and the partition \( \mathcal{P} = \{0,T_1,T_2,...\} \) as the set of contractually determined payment dates, \( \tau \) is defined as

\[
\tau \triangleq \inf \{ \in \mathcal{P} : t \geq \tau' \} 
\]  

(1.9)

This seemingly technical definition merely generalises our model to the case where guaranteed amounts are not granted immediately upon death of the policy-holder but deferred to an ulterior date. It is common that life insurance contracts offer death benefits only at a contractually determined time after the death, typically the next policy anniversary following the death of the insured person. We extend our prior filtration \( (\mathcal{F}_t)_{t \in [0,T]} \) and define a new filtration \( \mathcal{F}_t^{\tau} \) which includes all sigma-algebras generated by mortality events

\[
\mathcal{F}_t^{\tau} = \mathcal{F}_t \vee \sigma \{ \tau' \leq t \} 
\]

(1.10)

**Remark** Note that the random exercise time \( \tau \) is a \( \mathcal{F}_t^{\tau} \)-stopping time since \( \forall t \) the sets \( \{ \tau \leq t \} \in \mathcal{F}_t^{\tau} \).

The pay-off function is a mapping of the form \( \chi: \mathbb{R}^+ \times \mathbb{R} \mapsto \Theta \), attainability of the claim implies that \( \Theta \) is a subset of \( \mathcal{L}^2(P_X) \). Clearly \( \chi \) is a function of the realisations of the process \( (X_t)_{t \in [0,T]} \) but it may also admit a deterministic time component. At the random exercise time \( \tau \), the value \( V(\tau, X_\tau) \) of the contingent claim must equal the pay-off of the guarantee, imposing the terminal condition

\[
V(\tau, X_\tau) = \max (K_{\tau,X_\tau} - X_\tau) 
\]

(1.11)

We have used the notation \( K_{\tau,X_\tau} \) with subscript to emphasize that the exercise price of the guarantee may possibly be dependent on time and on the realisations of the process \( (X_t)_{t \in [0,T]} \) up to time \( \tau \). The arbitrage-free claim value \( V(t, X_t) \) reads

\[
V(t, X_t) = \mathbb{E}_Q^Q \left[ \int_t^T \frac{B_u}{B_t} \chi (u, X_u) \mathbf{1}_{\{\tau = u\}} du | \mathcal{F}_t \right] 
\]

(1.12)

The uncertainty due to mortality cannot be fully hedged by trading a liquid financial instrument, hence the market resulting of the combination of financial and actuarial risk is incomplete. In such a setting, the equivalent martingale measure \( Q \) is not uniquely determined, instead we have an infinity of measures \( Q^* \) which can take any value in the set
\[ Q^* \in \left[ \min_{Q \in \mathcal{M}} Q; \max_{Q \in \mathcal{M}} Q \right] \] (1.13)

We are then left with the problem of determining the optimal martingale measure with respect to some optimality criteria. In the third chapter, we will compare different approaches to constructing an optimal measure in a more general problem setting where other sources of incompleteness than mortality are considered. We assume that the market is arbitrage-free in the sense that the set \( \mathcal{M} \) of all equivalent martingale measures is non-empty

\[ \mathcal{M} \neq \{ \emptyset \} \] (1.14)

In the following section the pricing measure is considered as given. The matter of constructing of an equivalent martingale measure for the combined actuarial and financial model will be dealt with in depth in the third chapter. For now, we emphasise that actuarial and financial risks is taken independent both under \( P \) and \( Q \).

**Assumption 1.2.1** The probabilities of survival and asset returns are assumed to be independent

This assumption will be used throughout this study, and its importance will be further stressed below. The implication is that combining the stochastic mortality rate model and the financial market generates a filtration satisfying the usual conditions.

### 1.2.1 Hazard rate

Stating the problem as that of pricing a basket of death claims at portfolio level rather than modelling each individual random time of death separately turns out to yield a more intuitive formulation. The probability of survival can then be interpreted as a percentage of persistence in the portfolio, provided that the portfolio is sufficiently diversified for the population sample’s mortality rates to converge to average rates. Redefining \( V(x, t, X_t) \) as the value of all the claims written to clients of age \( x \), equation (1.12) transforms to

\[ V(x, t, X_t) = \mathbb{E}^Q \left[ \sum_{i=1}^{n_x} \int_t^T \frac{B_t}{B_u} \chi (u, X_u) 1_{\{\tau_i = u\}} du | \mathcal{F}^t \right] \] (1.15)

Let us consider the probability of survival \( q_x = (q(x, t))_{t \in [0, T]} \) given for age \( x \) and at time \( t \) in terms of the hazard rate \( \mu(x) = (\mu(x, t))_{t \in [0, T]} \).
\[ q(x, t) = e^{-\int_0^t \mu(x, u)\,du} \]  

(1.16)

For a highly diversified portfolio, the mortality risk is associated with rate changes more than with their levels. Therefore, we will find convenient to express the hazard rate in terms of the initial curve and the mortality intensity change process \( \zeta(x, t) \)

\[ \mu(x, t) = \mu(x, 0)\zeta(x, t) \]  

(1.17)

In a general case, \( \zeta \) is a random variable.

Let \( n_x \) be the number of policy-holders of age \( x \) in the portfolio, and define the counting process \( N(x) = (N(x, t))_{t \in [0,T]} \) as the number of deaths among the population \( n_x \)

\[ N(x, t) = \sum_{i=1}^{n} 1(\tau_i' \leq t) \]  

(1.18)

The process \( N(x) \) is defined on the probability space \((\Omega^a, \mathcal{F}^a, (\mathcal{F}^a_t)_{t \in [0,T]}, \mathbb{P}^a)\) equipped with filtration \((\mathcal{F}^a_t)_{t \in [0,T]} = \sigma\{N(x, u), u \leq t\}\). Assuming independence, and some distribution of the deaths, we have that \( N \) is a Levy process. The intensity of the death counting process describes the expected evolution of the death count and is given by

\[ \mathbb{E}[dN(x, t)|\mathcal{F}^a_t] = (n_x - N(x, t^-))\mu(x, t)\,dt \]  

(1.19)

Following Møller (1998), let us consider the intrinsic value of liability process \( \tilde{V}(x, t, T) \) as the discounted sum of the benefits paid up to time \( t \) and the value of the remaining liabilities

\[ \tilde{V}(x, t, X_t) = \int_0^t \frac{1}{B_u} \chi(u, X_u)\,dN_u + \mathbb{E}^Q\left[ \sum_{i=1}^{n_x} \int_t^T \frac{1}{S(0)(t, u)} \chi(u, X_u)\,1(\tau_i' = u)\,du | \mathcal{F}^T_{t}\right] \]  

(1.20)

The intrinsic value reflects the evolution of the value of a portfolio of claims. It quantifies the impact of portfolio diversification by including the value of past liability payments made and keeping track of the policy count.
1.2.2 Deterministic mortality improvement

It turns out that taking the mortality improvement factor as a predictable process makes pricing under the classical risk-neutral valuation principles possible. Brennan and Schwartz (1976), Brennan and Schwartz (1979) applied this idea to the problem of pricing equity-linked life insurance contracts and derived solutions for certain types of guarantees for portfolios consisting of large number of policies. Invoking the law of large numbers, they replace the original claim $\chi$, involving a random number of survivors, by a synthetic claim $\chi' = \mathbb{E}[1 - N_T] \frac{\bar{B}_T}{B_T}$. The assumption implicitly made is the completeness of the market under mortality risk, which implies the following conditions

**Assumption 1.2.2 (Completeness under mortality risk)** The market is complete under mortality risk if and only if all of the following conditions hold

(i) the financial market is complete, free of transaction costs and frictionless

(ii) the mortality risk can be entirely diversified

(iii) the process of mortality intensity change is deterministic

(iv) mortality and market risk are independent

Diversification of the insurer’s liability can be achieved by selling a sufficiently large number of policies. The population sample should also be diverse and sufficiently spread geographically to avoid high correlations of mortalities. In this case, the law of large numbers can be used to demonstrate that mortality rates converge to the mean almost surely. Thus, assumptions (ii) and (iv) can be used to justify pricing using average mortality rates. In combination with (i) the market model as a whole is complete under both, market risk and mortality risk.

$$V_t = \mathbb{E}^Q \left[ \int_t^T \frac{B_t}{B_s} e^{-\mu(x+u)(a)} \chi(u, X_u) du | \mathcal{F}_t \right]$$  \hspace{1cm} (1.21)

In the deterministic case, $\mu$ can be inferred from annual mortality rates. The procedure is shown in Appendix A.1.

**Example 1.2.3 (Pricing return-of-premium (ROP) benefits)**

$^3$Equivalently, we may relax this assumption by assuming the existence of a liquid and continuously traded asset whose price is perfectly correlated to death rates
In the following examples, we restrict the class of stochastic process governing the stock process $S_t$ to follow a univariate geometric Brownian motion (GBM). The $\mathbb{Q}$-dynamics read

$$dS_t = S_t r dt + S_t \sigma dW_t^\mathbb{Q} \tag{1.22}$$

An application of Ito’s lemma to the process $\log S_t$ yields a more explicit result. We may express $S_t$ in stochastic exponential form

$$S_t = S_0 e^{(r - \frac{1}{2} \sigma^2)dt + \sigma \epsilon_t^\mathbb{Q}} \tag{1.23}$$

The ROP claim has the pay-off of a plain-vanilla put option which is at-the-money, $K = S_0$ at issue date

$$\chi(S_t) = \max[K - S_t, 0] \tag{1.24}$$

Denoting the normal density of a random variable $z$ function by $\varphi(z)$, we may now insert (1.23) into (1.21) to obtain

$$V(x, t) = \int_t^T \int_{-\infty}^\infty \max[K - S_0 e^{(r - \mu(x,s) - \frac{1}{2} \sigma^2)ds + \sigma \epsilon \sqrt{T-t}}, 0] \varphi(\epsilon) ds \tag{1.25}$$

Following the standard Black-Scholes procedure, the inner integral can be solved in terms of the cumulative normal distribution $\Phi$ and the variable $d_1$ and $d_2$ s.t.

$$\begin{cases}
    d_1 = \frac{\log S_t + (r - \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}} \\
    d_2 = d_1 - \sigma \sqrt{T-t}
\end{cases} \tag{1.26}$$

The claim value can be found as

$$V(x, t) = \int_t^T e^{-(r(T-u) - \mu(x,u)u)K \Phi(d_1) - e^{\mu(x,u)u}S_t \Phi(d_2)} du \tag{1.27}$$

For a general specification of the hazard rate $\mu$, the time integral in equation (1.27) can be approximated by a Riemann sum.

The intrinsic value reads

$$\tilde{V}(x, t) = \int_t^T \frac{1}{S_t} \max(K - S_t, 0)(1 - e^{\mu(x,u)u}) du + \int_t^T e^{-(r(u) - \mu(x,u))(T-u)}(K \Phi(d_1) - e^{r(u)}S_t \Phi(d_2)) du \tag{1.28}$$

\(^{4}\text{See for example Hull (2000, p. 268)}\)
It is not difficult to extend these formulas to the case of an exercise price \( K \), which grows at a predetermined rate.

**Example 1.2.4 (Pricing continuous ratchet)**

Recall that a continuous ratchet put option\(^5\) resets the exercise price to the maximum level reached by the process \((S_u)_{u \in [0,t]}\) for all \( t \in [0,T] \). This is equivalent to a lookback rate option and has known closed form solutions in the Black-Scholes framework. Let us define the running maximum process by

\[
M_t^S = \max_{0 \leq s \leq t} S_u; \quad t \in [0, T].
\]

It can be shown that if \( S_t \) is continuous in time, the running maximum satisfies

\[
M_t = \lim_{n \to \infty} J_n(t)
\]

where \( J_n(t) = \left( \int_0^t (S_s)^n ds \right)^{\frac{1}{n}} \). Applying Itô’s lemma and the usual self financing replication yields the following partial differential equation

\[
\frac{\partial V}{\partial x} + S_t r_t \frac{\partial V}{\partial S} + J_n(t) \frac{\partial V}{\partial J_n} + \frac{1}{2} S_t^2 \sigma^2 S_t \frac{\partial^2 V}{\partial S^2} + rV = 0
\]  

(1.30)

When \( n \) approaches infinity, continuity at \( S_t = M_t \) implies that \( \frac{\partial V}{\partial J_n} \to 0 \). Hence, \( V \) must solve the Black-Scholes equation, with terminal condition \( V_T = (M_T^S - S_T) \). Closed form solutions can be found for the pure claim value

\[
V(x, t) = \int_0^T e^{-(r(t) - \mu(x,t))(T-t)} \left( M_t^S \Phi(d_1) - S_t \Phi(d_2) \right) + 
\]

\[
\frac{S_t \sigma^2}{2r} (M_t^S \Phi(d1) + e^{-rt} \left( \frac{M_t^S}{S_t} \right)^{\frac{2r}{\sigma^2}} \Phi(d3)) dt
\]

(1.31)

\( d_1 \) and \( d_2 \) are obtained by setting \( K = M \) in equation (1.26), and \( d3(t) \) is given by

\[
d_3(t) = \frac{\log \frac{S}{M} - (r - \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}
\]

(1.32)

The intrinsic value then writes

\(^5\)Continuous ratchet options are rarely written in practice, ratchet are most often semi-annually or quarterly reset. This formula can however be useful to find an upper bound for a discrete ratchets with frequent reset dates relative to the time to maturity.
\[ \tilde{V}(x,t) = \int_0^t \frac{1}{S_u} \max(K_u - S_u, 0)(1 - e^{\mu(x+u)})du \]
\[ + \int_t^T e^{-(r(u) - \mu(x,u))(T-u)} \left( M_u S \Phi(d_1) - S_t \Phi(d_2) \right) \]
\[ + \frac{S_u \sigma^2}{2r} \left( M_u S \Phi(d1) + e^{-ru} \left( \frac{M_u}{S_u} \right) \frac{2}{\sigma^2} \Phi(d_3(u)) \right) du \]

(1.33)

### 1.2.3 Mortality under square-root diffusion

It is assumed that all probabilities of death are independent and follow the same distribution rule. Møller and Dahl (2006) implemented a systematic mortality risk model where the intensity change process \( \zeta \) is assumed to be governed by a mean-reverting time-inhomogeneous square-root diffusion, whereby \( \zeta \) reverts to a long-run steady-state level \( \tilde{\zeta} \)

\[ d\zeta(x,t) = \kappa(t)(\tilde{\zeta}(t) - \zeta_t)dt + \sigma^\zeta(t) \sqrt{\zeta(x,t)}dW^\zeta_t \]

(1.34)

This type of process, studied by Feller (1951), exhibits a combination of analytic tractability and realistic properties. Provided that \( \zeta(x,t) \) evolves according to the diffusion process (1.34), an application of Ito’s lemma yields the following dynamics for \( \mu(x,t) \)

\[ d\mu(x,t) = \kappa(t)(\bar{\mu}(x,t) - \mu(x,t))dt + \sigma^\mu(t) \sqrt{\mu(x,t)}dW^\mu_t \]

(1.35)

To alleviate the notations, superscripts and arguments are omitted in the sequel. Empirical evidence suggests that mortality rates revert to a long term average more quickly after shocks of larger size. Another realistic property verified by the square-root model is that under appropriate parametrisation, it is guaranteed that the process remains positive.

**Property 1.2.5** The process defined by (1.34) is strictly positive almost surely if for all \( t \), the functions \( \kappa, \sigma \) and \( \bar{\mu} \) satisfy the relation

\[ \bar{\mu} > \frac{\sigma^2}{2\kappa^2} \]

(1.36)

**Proof** See Appendix A.2
It is worthwhile noting that the square root function does not have Lipschitz continuity which is the usual way to ensure the existence of a strong solution. It is not difficult to verify that Lipschitz condition is not verified around zero. Ikeda and Watanabe (1981, p. 221) have shown that under the less stringent Hölder condition, the square-root model admits a strong solution.

Møller and Dahl (2006) invoke known results from the application of the square-root process in the modelling of interest rates and only refers to Bjork (2004). However, the analogy with mortality is not explained. In particular, the term structure of mortality rates cannot be derived from the arbitrage argument as it is the case for bond prices. Therefore, we present a proof that the Riccati solutions are also applicable to mortality. The starting point of the proof is Lamberton and Lapeyre (1996, Proposition 6.2.4, p. 130), establishing that the square-root process can be characterised by its conditional Laplace transform \( \mathcal{L}_{\tau, \delta\tau}(\varrho, \upsilon) \).

\[
\mathcal{L}_{\tau, \delta\tau}(\varrho, \upsilon) = \mathbb{E} \left[ e^{-\varrho \mu(x, t) dt - \upsilon \int_{t}^{t + \delta\tau} \mu(x, u) du} \bigg| \mathcal{F}_{t}^{\varrho} \right] = e^{m(x, t, \delta\tau) - n(x, t, \delta\tau) \mu(x, t)} \quad (1.37)
\]

Note that \( \mathcal{L} \) is identified as the Laplace transform of a non-central \( \chi \)-squared distribution. By the law of iterated expectation we have for some time \( t' \in [t, t + \delta] \)

\[
\mathbb{E} \left[ e^{-\varrho \mu(x, t + \delta t) dt - \upsilon \int_{t}^{t + \delta t} \mu(x, u) du} \right] = \mathbb{E} \left\{ \mathbb{E} \left[ e^{-\varrho \mu(x, t) dt - \upsilon \int_{t}^{t + \delta\tau} \mu(x, u) du} \bigg| \mathcal{F}_{t'}^{\varrho} \right] \bigg| \mathcal{F}_{t}^{\varrho} \right\} \quad (1.39)
\]

Replacing the inner expectation by the corresponding Laplace transform expression, and taking a sufficiently small time interval \( \Delta t = t' - t \), we may approximate (1.40) by

\[
\approx \mathbb{E} \left[ e^{-\upsilon \mu dt} \mathbb{E} \mathcal{L}_{t', \delta\tau - \Delta t} \big| \mathcal{F}_{t}^{\varrho} \right] \quad (1.41)
\]

The dynamics of \( \mu(x, t) \) are specified in an explicit form by equation (1.35). We may insert this expression into (1.41) so that the last equation can be reformulated as

\[
= \mathbb{E} \left[ e^{-\upsilon \mu dt + \mu \delta t - \Delta t - n_{t} - \Delta t [\mu + \kappa(t)(\bar{\mu} - \mu) dt + \sigma \nu \sqrt{\nu} dW_{t}^{\mu}]} \bigg| \mathcal{F}_{t}^{\varrho} \right] \quad (1.42)
\]

See (Øksendal, 2004, p.68-71) for a proof.
Assuming that the conditions that allow taking all deterministic terms outside of the expectation are satisfied, equation 1.42 can be restated as

$$\approx e^{-\nu \mu dt + m_{\delta t - \Delta t} - n_{\delta t - \Delta t} [\mu + \kappa(t) (\bar{\mu} - \mu)] dt} E \left[ e^{n \sigma \sqrt{\Delta t} W_t^\mu |\mathcal{F}_t} \right]$$

(1.43)

Recalling that the moment generating function of a log-normally distributed variable $Z_t$ with standard deviation $s$ is given by $E[exp(Z_t)|Z_0 = z] = e^{\frac{1}{2} \delta^2 s^2 z}$, the last equality transforms to

$$\approx e^{-\nu \mu dt + m_{\delta t - \Delta t} - n_{\delta t - \Delta t} [\mu + \kappa(t) (\bar{\mu} - \mu)] dt + \frac{1}{2} n^2 \sigma^2 \mu}$$

(1.44)

Using the expression for exponential affine survival probability, identifying terms and letting the time differential tend to zero will eventually leave us with the system

$$\begin{align*}
\frac{\partial m}{\partial t}(x, t, T) - \kappa \bar{\mu} n(x, t, T) &= 0 \\
\frac{\partial n}{\partial t}(x, t, T) - \kappa n(x, t, T) + \frac{1}{2} \delta^2 n^2(x, t, T) - 1 &= 0
\end{align*}$$

(1.45)

This type of equations is known as Riccati equation and can be solved efficiently by powerful algorithms. For constant parameters, Riccati equations can be solved explicitly subject to the initial condition $q(x, 0) = 1$. The procedure is rather lengthy and we simply give the result.

$$\begin{align*}
n(x, \delta t) &= \frac{2 e^{\gamma \delta t} - 1}{(\gamma + \kappa)(e^{\gamma \delta t} - 1) + 2 \gamma} \\
m(x, \delta t) &= \frac{2 \gamma e^{\frac{\kappa + \gamma}{\gamma + \kappa} \delta t}}{(\gamma + \kappa)(e^{\gamma \delta t} - 1) + 2 \gamma}
\end{align*}$$

(1.46)

To alleviate the notations, we have introduced the variable $\gamma = \sqrt{\kappa^2 + 2 \sigma^2}$. The probability of survival $q(x, t)$ at time $t$ for age $x$ is given by

$$q(x, t) = e^{m(x, t) - n(x, t) \mu(x, 0)}$$

(1.47)

Obviously, the expected mortality decrement between three dates $0$, $t$ and $T$ so that $0 \leq t \leq T$ must be so that deaths expectation on the intervals $[0, t]$ and $[t, T]$ add up to the overall decrement over the time interval $[0, T]$. This implies that forward mortality is given by

$$q(x, t, T) = -\frac{\partial \log q(x, t)}{\partial T}(t, T)$$

(1.48)

Thus, conditional forward volatility curves may be obtained in the square-root process as the result from the following equality

---

7Solutions to Riccati equations are discussed in Reid (1972)
\[ q(x, t, T) = -\frac{\partial m}{\partial T}(x, t, T) + \frac{\partial n}{\partial T}(x, t, T)\mu(x, t) \]  

(1.49)

Forward mortality curves are of importance for the computation of conditional probability of survival until maturity. As we will see in chapter 3, the latter are involved within dynamic hedging strategies in incomplete market setting. Forward mortality curves may also occur in the evolution of the price of living benefits, which are often paid as annuity rather than as lump-sum.

1.2.4 Remarks

It is legitimate to question the realism of assumption(1.2.1) from an economic perspective. It may be objected that under extreme circumstances, the correlation, possibly positive or negative, between mortality and market movements could increase. In the sequel we will not take this correlation into account. Since the scope of this study is primarily dynamic hedging, we restrict to models that allow for some analytic solutions and relative ease of calibration to observed market data. A realistic representation of the tail-correlation in extreme market circumstances would be computationally expensive and result in difficult parameter estimation, and in additional model risk. It is therefore less likely to significantly improve the overall quality of the hedge. It is a common industry practice to calculate capital reserves for disaster recovery independently of the hedge, using exogenous models. However, there are more important limitations to the applicability of the formulas. Even in the absence of significant mortality risk, the presence of real options in the contract makes pricing in a simple European claim type setting an inexact approximation. One allows the policy-holder to modify investment choices over time while the other makes it possible to surrender the policy at any moment during the lifetime of the contract. The value of the both of these options depend heavily on the realisations of the wealth process \( X_t \). Hence, it will affect the value of death and survivor benefits and should be dynamically modelled. We will come back to this matter after a discussion on the pricing of survivor benefits, equally sensitive to these problems.

1.3 Survivor benefits

From a market risk point of view, survivor benefits are rather conventional types of claim in the sense that the value of such guarantees depend entirely on the benefit of exercising the embedded option when the contract arrives at maturity. A principal
guarantee is merely a European style put option on the underlying basket of mutual funds. The possibility of optimal exercise makes survivor benefits more expensive, but also easier to model based on the usual rational decision setting. Survivor benefits are hence more sensitive to different market risks, we will therefore need to specify the different market risks types explicitly. Life insurance guarantees are invested in a basket of domestic and foreign funds including equity and fixed income elements. In the next section, we propose an informal discussion about dynamics assumptions for the different asset classes, and narrow down the family of processes in order to get a more explicit formulation.

### 1.3.1 Underlying funds dynamics

Since some of the premium may be invested in foreign assets, we have to distinguish three categories of underlying values for the assets including equity, interest rate, and foreign currency. In the first section, we mentioned that absence of arbitrage implies that the equity process belongs to the fairly general class of semi-martingales. Every semi-martingale can be defined by a canonical decomposition of the form

\[ S_t = S_0 + A_t + M_t \]

\[ A_t = (A_t)_{t \in [0,T]} \] is a right continuous process with left limit (henceforth càdlàg), \[ M_t = (M_t)_{t \in [0,T]} \] has continuous paths. This type of processes include pure diffusions processes, stochastic volatility, stochastic interest rate, jump diffusion or Levy process for instance. The class of processes under which the financial market is complete is however more restricted. If the size of the jumps is not known, the market cannot be completed by trading an option with underlying value \( S_t \). Given the exotic nature of many life insurance guarantees, and the subsequent dependence on forward skew, plain GBM assumptions are to simple to capture the true dynamics of the volatility smile. Stochastic volatility model yield more accurate prices for lookback type options. Adding jumps to the process can also help adjusting the skew to the market level. Here we omit all the technicalities involved with hedging in jump-diffusion and stochastic volatility setting and simply give the funds dynamics as a multi-dimensional GBM. Under the physical probability measure \( \mathbb{P} \), we have

\[ dS_t = \text{Diag}(S_t) \alpha_{S,t} \mathbf{1} dt + S_t \sigma_{S,t} dW_t^\mathbb{P} \]
Under some technical conditions\(^8\), the equivalent martingale measure \(Q\) can be constructed by a Girsanov transformation such that the process \(W^Q_t = W_t + \int_0^t \lambda_s \, ds\) is a \(Q\)-martingale. The new measure is defined by the Radon-Nikodým derivative.

\[
\frac{dQ}{dP} = \xi_t \quad (1.52)
\]

where \(\xi_t\) denotes the Doléans integral \(\varepsilon_t(-\lambda_t W_t)\)

\[
\varepsilon_t(-\lambda_t W_t) = \exp \left\{ -\frac{1}{2} \int_0^t \| \lambda_s \|^2 ds - \int_0^t \lambda_s dW_s \right\} \quad (1.53)
\]

By definition, discounted asset prices are martingales under \(Q\) which supposes the \(Q\)-dynamics

\[
dS_t = \text{Diag}(S_t) R^0_t 1 dt + S_t \sigma_S dW^Q_t \quad (1.54)
\]

Here, \(R^0\) denotes the domestic short rate. Hence \(\lambda\) is given by

\[
\lambda_t = \frac{\alpha_{S,t} - R^0_t}{\sigma_{S,t}} \quad (1.55)
\]

Without loss of generality, we may assume that all investments are made in the same currency. It comes down to expressing the value of all investments in terms of the local currency. Supposing that the foreign exchange rate \(F_t\) is also governed by GBM dynamics

\[
dF_t = \alpha_{F,t} dt + \sigma_{F} dW^P_t \quad (1.56)
\]

It can be shown\(^9\) that, in the domestic risk-neutral measure \(Q\), the last equation may be restated in terms of the foreign short rate process \(R_t\) as

\[
dF_t = F_t (R_t - R^0_t) dt + F_t \sigma_F dW^Q_t \quad (1.57)
\]

We may apply Ito’s lemma to find the dynamics of \(S\) when we take \(F\) as the numéraire as

\[
d \left( \frac{S_t}{F_t} \right) = \frac{S_t}{F_t} (\alpha_{S,t} - \alpha_{F,t} + \sigma_{S,t}^T \sigma_{S,t} - \sigma_{S,t}^T \sigma_{F,t}) dt + \frac{S_t}{F_t} (\sigma_{S,t} - \sigma_{F,t})^T dW^P_t \quad (1.58)
\]

\(^8\)Application of Girsanov theorem requires that the Novikov condition, \(E\left[\exp \int_0^T \lambda_u^2 du \right] < \infty\), be satisfied

\(^9\)See for example (Bjork, 2004, Proposition 12.2)
A probability measure $Q^F$ in which the process $\frac{\tilde{S}_t}{F_t}$ is a martingale can be constructed by means of the Radon-Nikodým derivative

$$\frac{dQ^F}{dP} = \varepsilon(-\lambda_t^F . W_t) \quad (1.59)$$

The martingale property in equation (1.58) implies that

$$(\sigma_S - \sigma_F)^\top \lambda_t^F = \alpha_S - \alpha_F + \sigma_S^\top \sigma_S - \sigma_S^\top \sigma_F \quad (1.60)$$

Hence the following relation must hold between the market price of risk processes $\lambda_t^F$ and $\lambda_t^F$, and the volatility of the numéraire

$$\lambda_t^F = \lambda_t - \sigma_t^F \quad (1.61)$$

Premia received from life-insurance contracts may also be invested in different fixed-income funds, including bonds funds and mortgage-backed and asset-backed securities. In the next section, we turn to specify the dynamics followed by the interest rate process.

### 1.3.2 Interest rate dynamics

Changes in interest rate levels affect the value of policies in several ways. Even for claims without any sort of interest rate guarantees, the long maturities observed in life insurance contracts makes the value more interest-sensitive than typical exchange-traded options. A significant proportion of underlying funds offered as investment vehicles are portfolios of government or corporate bonds and therefore have returns that are closely tied to the fluctuations of risk-free rates. In addition, many claims are paid as an annuity rather than a lump sum, often with interest rate guarantees.

A frequently encountered model for the short rate process $R_t^0$ is the time inhomogeneous extended Vasicek model proposed by Hull-White (1991) which represents the short rate as a mean reverting gaussian process

$$dR_t^0 = \kappa(R(t) - R_t^0)dt + \sigma_R dW_t^Q \quad (1.62)$$

The long run mean function $\bar{R}$ can be set so as to fit the yield curve observed at time zero

$$\bar{R}(t) = \frac{1}{\kappa} \frac{\partial^2 \log S_t^{(0)}}{\partial t^2} + \frac{\partial \log S_t^{(0)}}{\partial t} + \frac{\sigma^2}{2\kappa^2}(1 - e^{2\kappa t}) \quad (1.63)$$
This model was first applied to the pricing of unit-linked life insurance contracts by Bacinello and Ortu (1993). Practitioners often value its ability to replicate the yield curve and sometimes inaccurately equate this property to the guarantee of no arbitrage opportunities in the model. This static fit is in fact little more than a parametrisation of the bond prices, the model being arbitrage free is a more dynamic feature related to the dynamics of the replicating portfolio. The choice of a Gaussian interest rate model is mainly motivated by its analytical tractability. Normality of the rate ensures that the bond price is lognormal and makes it more simple to derive closed form solutions for equity options when the interest rate is stochastic. The zero-coupon bond maturing at the expiration date $T$ of the option can be used as the numéraire in a similar procedure as the one described in section 1.3.1. Prices may be computed in the forward measure defined by

$$
\frac{dQ_T}{dP} = \varepsilon(-\lambda^T_t . W_t)
$$

(1.64)

with

$$
\lambda^T = \lambda - \sigma^2 S^0_{t,T}
$$

A known drawback of the Vasicek model is that it doesn’t prevent interest rates from being negative. The Hull-White extension makes it easy to retrieve the $Q$-dynamics directly using equation (1.63), but doesn’t lean itself to estimations of a stable Sharpe ratio. In the context of fair-valuation in incomplete markets, as will be seen in chapter 3, excess return is an important factor. Considering the various natures of the interest rate risk exposure, it is natural to look for a model that describes short rate movements by credible dynamics rather than focusing on replicating the current yield curve. Moreover, the market concerned in this analysis contains many sources of noise, and a great deal of the liabilities have maturities beyond longest term interest rates that can be observed on money-markets. Since some of the randomness cannot be effectively replicated, the pay-off of a hedged portfolio will typically be stochastic. The fair price in this case does depend on expected returns. Unlike in the usual risk-neutral valuation, the market price of risk is a parameter that should be estimated explicitly. The above arguments discounts the tractability of calibrated models and draw our attention more towards equilibrium type models, thus the time-homogeneous model square root model introduced by Cox et al. (1985) and described in the preceding section displays several features of a good candidate.

$$
\frac{dR^0_t}{(\theta - \kappa R_t)dt + \sigma R^0_t dW_t}
$$

(1.65)
The $i^{th}$ foreign short rate process $R^i_t$ reads

$$dR^i_t = (\theta(t) - aR^0_t - \sigma^{F^i} \sigma^R_t)dt + \sigma_R \sqrt{R^i_t}dW^Q_t$$ (1.66)

It is of course easily understood that the yield curve fit should be sufficiently accurate. Large differences in bonds prices between the market and the model may lead to discrepancies between theoretical price and hedge costs and are inappropriate in an arbitrage-free setting. It turns out that multiple factor implementations provide more flexibility for a better replication of the term structure of interest rates. In addition, multiple factors represent a more realistic economy that is not governed by one single variable. Changes in the shape of term structure such as yield curve twist can be reproduced. It is however much more difficult to obtain a robust calibration for the multiple factors.

### 1.3.3 Amortised Swap Martingale measure

An additional source of complexity that needs to be considered arises from the fact that claims are often guaranteed in the form of an annuity rather than as a lump sum. The value of the guarantee at maturity $T$ depends on the the difference between the guaranteed annuity rate and the yield curve at time $T$, and the expected decrease in principal due to mortality. The annuity factor $\ddot{a}$ is defined as

$$\ddot{a}_t = \delta T \sum_{i=1}^{n} (1 - M_{\delta_T}) \left( \frac{1}{B_{t,i}} \right)^{T_i}$$ (1.67)

We may overcome the additional challenge by choosing a measure $Q^{\ddot{a}}$ in which the annuity is a martingale, the swap martingale measure introduced by Jamshidian (1997) has been extended to account for amortising of the principal due to mortality. The value of the guarantee can be expressed in terms of $Q^{\ddot{a}}$ as

$$V_g = \ddot{a}_t \mathbb{E}^{Q^{\ddot{a}}} \left[ \max (K_t - X_T, 0) \right]$$ (1.68)

This change of numéraire converts the pricing of guaranteed annuity rates to that of a plain put option. The validity of this annuity as numéraire is tied to the assumption that mortality intensity and lapse rates are known. However, in certain cases it happens to be an acceptable approximation for the valuation of the claim. In particular when time to maturity is sufficiently long.
1.4 Game option

Most life insurance contracts provide the policy-holder with a game option allowing to terminate the contract at any moment during the life-time of the contract. The right to surrender the policy results in a permanent erosion of the asset under management that in insurance jargon is referred to as lapsation. Lapsation has the effect of reducing the value of liabilities and are in this sense valuable for the insurer on the one hand, and to limit the fees income which are the very source of profitability of the company on the other hand. Underestimating the risk of early surrendering will result either in overpricing the claims and overhedging, while overestimation will result in expected losses after hedging. Lapsation will also affect the forecast of expected fees income, and can therefore work in both directions. Generally, insurers discourage early surrenders either by charging a surrender fee if lapse occurs within an initial period\(^\text{10}\) of a given length, or by taking an upfront commission when the contract is entered. The insured has then a higher incentive to keep the policy until the initial fee is at least partially compensated by increase in the value of the assets. One of the most challenging aspects of hedging models for life insurance policies is to find a model that accurately mimics the policy-holder’s lapse decisions. In the absence of any correlation between lapse decisions and financial value of the option, average lapse rates can be used and the explicit solutions derived above remain valid. In practice, it turns out that lapse behaviour is closely related to the value of the options embedded in the contract. As a result, the closed-form solutions that we have found only have limited applicability.

1.4.1 Qualitative analysis

Empirical evidence suggests that decisions to surrender a policy can be partially explained by financial motivations. The observed lapsation is substantially lower when guaranteed amounts are significantly above the current level of the assets value. The presence of such a correlation makes the treatment of lapsation fundamentally different from that of mortality. Recall that one of the building blocks of our combined financial and actuarial model is the assumption of independence between the two markets\(^\text{11}\). Besides the financial component, a number of other factors can influence the lapse behaviour. It is observed that no matter how the investment funds perform, a certain proportion of policies do lapse while some always remain.

\(^{10}\)The option to leave for a given penalty is sometimes referred to as Israeli option Kühn (2002)

\(^{11}\)The exact implication of this independence will be further explained in the third chapter
1. **Cap and floor** Lapse probability is neither null nor total but moves between a range of values in some subset of the interval (0, 1).

2. **Seasoning** Lapse behaviour differs depending on the age of the contracts and its remaining time to maturity.

3. **Seasonality** Expected lapse rates may have periodical fluctuations depending on the time of the year.

4. **Burnout** policy-holders who assign a higher utility to surrender tend to exercise the lapse option earlier resulting in a decrease of their relative proportion in the portfolio. Hence lapsation tends to reduce over time.

5. **Spike lapse** A sudden jump in lapse probability occurs at the end of surrender charge periods.

6. **Spike lapse deferral** The spike lapse is delayed when the guarantee is deep in-the-money.

Practitioners often make use of models based on a combination of statistical estimation of historical option exercise in combination with risk-neutral pricing techniques. This approach is comparable to the one often used to value the prepayment option in mortgage-backed securities\(^{12}\). Since it provides a relatively simple way to correlate the value of the embedded option to the market value of the hedge instruments.

### 1.4.2 Model specification

The method consists of observing the pattern in past decisions and approximating it with a mathematical function of the relevant financial value. Typically, the function will return a percentage of a lapses as a function of the value underlying the guarantee. This can be formalised by a mapping of the form \( \ell : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+ \). It seems a priori reasonable to use a smooth function that converges asymptotically to a maximum and a minimum. Therefore, we propose a double asymptotic function of the form

\[
\ell(X_t, t) = \theta_0(t) - \frac{\theta_1}{1 + e^{\theta_2 + \theta_3 \phi(X_t)}}
\]  

\(^{12}\)See for example Schwartz and Torous (1989)
The interpretation of the function $\phi$ is that it reflects the incentive to "stick" to the contract, in a sense it is the value\textsuperscript{13} of the guarantee from the buyer's perspective. It will therefore be more accurate to generalise it as $\phi(t, T, K_t, X_t)$. The base lapse function $\theta_0(t)$ represents the structural component of the lapse behaviour, i.e. the probability of surrender independently of the economic environment. $\theta_0(t, T) = \sup_{X_t \in \mathbb{R}^+} \{\ell(X_t, t)\}$ in the most simple case can be modelled as a constant.

Better fits can be obtained by allowing deterministic time dependence to mimic burnout effect, spike lapse and periodicity. To mimic the seasonality it is natural to turn to trigonometric functions. Inspired by models applied to energy markets by Coulon (2009), we suggest a function of the form

$$\theta_0(t) = e^{-\alpha_0(T-t)} [\alpha_1 \cos (2\pi t + \alpha_2) + \alpha_3 \sin (4\pi t + \alpha_4)] + (\alpha_5) 1_{t=t_{sc}} + \varepsilon_t \tag{1.70}$$

The uncertainty of lapse rates is entirely captured by the independent error term $\varepsilon$. $\alpha_5$ determines the size of the jump that occurs at $t_{sc}$, the expiration date of the surrender charge period. The parameter $\theta_1$ controls the maximum lapse reduction. Taking $\theta_1$ constant does not induce any loss of generality, it comes down to modelling all time dependence by the base lapse, avoiding overparametrisation of the function.

Some of the convenient properties of the lapse function include continuity and twice differentiability. The latter will guaranty a more stable behaviour of the hedge

\textsuperscript{13}A rigorous meaning for the value in this sense will be treated in depth the second chapter
Figure 1.2: Double asymptotic lapse function including burnout effect and periodical component
ratios around inflection points. The choice of estimator $\phi$ for the surrender behaviour is a very sensitive question. It is not solely dictated by the need to predict lapses as accurately as possible, but also governed by hedging and fitting considerations. The function should be easily correlated to the value process of hedging instruments. As we will see in the following section, specifying the function $\phi$ has some important implication for the theoretical soundness of the valuation procedure.

1.4.3 Physical measure adjustment

Once all lapse model parameters have been estimated and the best estimator $\phi$ identified. One may be tempted to plug in the estimated lapse function into a risk-neutral simulation and discount liability cashflows at the risk free rate to compute present and hedge ratios. This procedure is however incorrect in a general case\textsuperscript{14}. The question whether it is adequate to project a function with parameters estimated in the objective probability measure $\mathbb{P}$ in the equivalent martingale measure $\mathbb{Q}$ may be challenging. The answer depends mostly on some characteristics of explanatory variable $\phi$. It turns out that the problem may be considerably simplified by specifying the conditional lapse rates $\ell$ as an $\mathcal{F}_t$ predictable process. In other words, the information available at time $t$ is sufficient to determine the time $t$ probability of surrender. If such is the case, the exercise is trivial and $\mathbb{P}$-measure estimates are suitable for pricing in the $\mathbb{Q}$-measure. One way to think about it is to view $\phi$ as an American-type claim with an exotic pay-off scheme. The pay-off is received in the real world, but provided that the claim is attainable, only the risk-neutral dynamics are further relevant.

Intuitively, it seems more realistic to consider that the lapse reduction $\phi$ also depends on expected future outcomes. The question of the appropriate choice of probability measure remains. We have to compute a conditional expectation that should be taken in either one of the measures $\mathbb{P}$ or $\mathbb{Q}$, the modelling approach can be either one of the following two alternatives

(i) Use the Radon-Nikodým derivative $\frac{d\mathbb{P}}{d\mathbb{Q}}$ in order to keep track of physical probabilities within computation in the pricing measure.

(ii) Estimate the lapse function directly under $\mathbb{Q}$. This approach can lead to some technical challenge\textsuperscript{15}, but can be dramatically simplified by taking $\phi$ as a func-

\textsuperscript{14}See (Bjork, 2004, Section 17.2 p. 253) for a discussion about the use of historical estimates in pricing.\n
\textsuperscript{15}An understanding of the difficulty of this approach can be given by an analogy with the problem of estimating the exercise price of an American option given historical data on exercise times.
tion of an explanatory variable that is directly observed in the $Q$-measure. Let’s say an option price for instance.

Both alternatives are in principle conceptually valid, provided that estimation and valuation are performed consistently. The existence of the process $\frac{dP}{dQ}$ for the specific market in scope will be established in the first section of the next chapter. In the sequel, we will adopt the first alternative. As will be confirmed in the next chapter, it allows for an a priori more reasonable estimator.

1.4.4 Restricted information

One of the possible explanations for the observed suboptimality in the policy-holder’s lapse behaviour is the inaccuracy and incompleteness of the information flow they base their decision upon. Partial and approximative information available to the public result in an apparent delay between the optimal and actual exercise dates. Only a small proportion of policy-holders would react promptly to the latest developments on the financial markets. Øksendal and Sulem (2008); Schweizer (1994b) studied the matter of pricing contingent claims under partial information. In this model, investors decisions are based on a filtration $(\mathcal{G})_{t \in [0,T]}$ which is a subset of the $(\mathcal{F})_{t \in [0,T]}$ such that

$$\mathcal{G}_t \subset \mathcal{F}_t$$

(1.71)

It turns out that in our set-up, the modelling of restricted information simplifies considerably. The main reason for this simplicity is that we are pricing liabilities from the insurer’s perspective, while the access to information is restricted to the buyer. We are hence left with the problem of evaluating a $\mathcal{F}_t$-predictable inefficiency in the policy-holders lapsation. The most simple way to model the delay in exercise is by using a lagged filtration. I.e. for surrender estimations, expected utilities are to be taken with respect to the $\sigma$-algebra generated by the random process $X_t$ up to time $t - \Delta t$

$$\mathcal{G}_t = \mathcal{F}_{t-\Delta t}$$

Most insurers include contractual restrictions on surrenders such as upfront costs, protection periods during which lapses are not permitted, or surrender charges when lapse occurs prior to a preset date. These restrictions have the effect of reducing the value of the option to surrender in the early years of the contract. We must hence be careful to decrease the value of surrender function $\phi$ accordingly.
Chapter 2

Marginal Utility Based Buyer Price

In the first chapter, we made use of well-established arbitrage theory and found solutions to the death-contingent claim price. We’ve also shown that the pricing of living benefits translates to a plain option valuation problem after suitable changes of probability measure. The surrender model described in the first chapter was left partly unspecified. We approached matter of defining the explanatory variable $Φ$ from a theoretical point of view and concluded that lapse function estimates can be used without adjustment only if the conditional lapse rate $ℓ$ is an $ℱ_t$-adapted process. We only gave an implementation for $ℓ$ as a function of the current moneyness of the option. This setting exhibits a number of convenient properties, such as compatibility with classic pricing models, intuitive hedging formulas, and ease of calibration. It is however nor supported by experience, neither by the market viability argument in the usual sense, based solely on the simple arbitrage criterion. Unlike exchange traded assets, the insurance liability market displays a permanent asymmetry between buyers and sellers. policy-holders do not have the capital necessary to diversify their risks and therefore have to assign a higher price to them. Investments objectives of the average insurance policy-holder differ from the ones pursued by professional investors in that the policy-holder is mostly interested in affordable security. Optimising the value of the embedded option purchased will not be part of the client’s strategy. Experience shows that policy-holders value mostly expected returns and therefore tend to choose relatively safe investments vehicles despite the security provided by their contract. From the insurer’s perspective, the market viability argument implies that a positive net profit is expected after hedging of the claims. Operational costs, fund management fees, advertisement and distribution expenses, capital requirements and unhedgeable risks should be compensated for in order for the company to persist in the business.
Moreover, insurance contracts cannot be bought or sold in a liquid and frictionless market, so that most apparent arbitrage opportunities are not exploitable. On the other hand, competition ensures that free lunch opportunities should vanish quickly. A definition of arbitrage which is consistent with all the above is needed. It turns out that the definition from Delbaen and Schachermayer (1994) of 'no free lunch with vanishing risk' is sufficient in this context. In the present chapter, we propose to approach the problem from the insured person’s point of view. The first section starts from basic assumptions on client’s optimal investment-consumption plan and derivation of the relation with the equivalent martingale measure is shown. Some classic results of stochastic optimisation are then presented, and we will see how they can be used to infer explicit values for an implied utility function. An a priori more realistic formulation of lapse decision problem is proposed in terms of client’s utility and we show that it is indeed a better predictor. Another useful by-product of our implied utility definition resides in the efficient dynamic allocation schemes it offers to control the risk-return trade-off. Annuity benefits can be structured so as to fit clients preference and priced consistently with the risk profile selected.

2.1 Buyer’s utility

The policy-holder’s financial objective is to maximise the utility of consumption over time while securing enough wealth at the end of the projection period. The wealth process $X_t$ can be viewed as a consumption portfolio\(^1\), and can be evaluated in terms of the $N$-dimensional vector $S_t$ as

$$X_t = \pi^0_t S_t^0 + \int_0^t \sum_{i=1}^N \pi^i_t S^i_s - \int_0^t c_s ds$$  \hspace{1cm} (2.1)

The policy-holder attempts to achieve his purpose by adjusting two control variables throughout the lifetime of the contract: his instantaneous consumption $c_t$, and his asset allocation represented by the scalar $\pi^0_t$ and the vector $\pi_t$. We further assume that he is rational and risk averse in the sense that his preferences are represented by a strictly increasing and concave utility of consumption $U(c_t)$.

$$\begin{align*}
\frac{\partial}{\partial c_t} U(c_t) &> 0 \\
\frac{\partial^2}{\partial c^2_t} U(c_t) &< 0, \forall c_t \geq 0
\end{align*}$$

\(1\)I.e. any additional premium invested can be seen as a distinct self-financing portfolio with consumption
Assuming concave utility, the maximisation problem can then be solved by focusing solely on the first order condition. Another important property to be satisfied by the marginal utility \( \frac{\partial U}{\partial c_t} \) is provided by the Inada conditions.

**Property 2.1.1 (Inada condition)**

\[
\begin{align*}
\lim_{c_t \to 0} \frac{\partial U}{\partial c_t} &= +\infty \\
\lim_{c_t \to +\infty} \frac{\partial U}{\partial c_t} &= 0
\end{align*}
\] (2.3)

The mathematical implication is that the existence of a unique equilibrium is enforced. The Inada conditions also have a clear economic interpretation. The marginal utility becomes infinite when the consumption rate approaches zero, this property prevents the investor from setting \( c_t = 0 \).

Supposing that the cumulative utility \( u \) is time additive, the problem reads

\[
u(t, c_t, X_t) = \max_{\pi \in \mathcal{A}} \mathbb{E} \left[ \int_0^T e^{-\delta t} U(c_t) dt + \Phi(X_T) | G_t \right]
\]

Here, \( T \) denotes the time to maturity while \( \Phi(X_T) \) is the bequest function representing the utility of having some wealth left when the contract arrives at maturity. Clearly, the bequest function is closely related to the value of the option. It follows from the very definition of optimality that any deviation from the optimal asset allocation plan will result in a loss of utility. Hence, the following inequality must hold

\[
\mathbb{E} \left[ \int_t^{t+\Delta t} e^{-\delta s} U(c_s - \frac{\varepsilon S_t^i}{\Delta t}) ds + \int_{t'}^{t'+\Delta t} e^{-\delta s} U(c_s + \frac{\varepsilon S_t^i}{\Delta t}) ds \right] \\
\leq \mathbb{E} \left[ \int_t^{t+\Delta t} e^{-\delta s} U(c_s) ds + \int_{t'}^{t'+\Delta t} e^{-\delta s} U(c_s) ds \right]
\]

In the example above, the investor finances an increased investment of \( \varepsilon \) units of asset \( S_t^i \) by reducing his average instantaneous consumption during a time period \( \Delta t \) and shifting it to a later time period starting at \( t' \geq t + \Delta t \). Rearranging all terms within the time integrals, we have

\[
\mathbb{E} \left[ \int_t^{t+\Delta t} e^{-\delta s} \left\{ U(c_s - \frac{\varepsilon S_t^i}{\Delta t}) - U(c_s) \right\} ds + \int_{t'}^{t'+\Delta t} e^{-\delta s} \left\{ U(c_s + \frac{\varepsilon S_t^i}{\Delta t}) - U(c_s) \right\} ds \right] \leq 0
\]

Dividing by \( \varepsilon \) and letting \( \varepsilon \to 0 \), invoking the fundamental theorem of integral calculus, this inequality becomes
\[
\mathbb{E} \left[ -\frac{S^i_t}{\Delta t} \int_t^{t+\Delta t} e^{-\delta s} \frac{dU(c_s)}{dc_s} ds + \frac{S^i_{t'}}{\Delta t} \int_{t'}^{t'+\Delta t} e^{-\delta s} \frac{dU(c_s)}{dc_s} ds \right] \leq 0 \quad (2.4)
\]

Now letting \( \Delta t \to 0 \) yields
\[
\mathbb{E} \left[ -e^{-\delta t} S^i_t \frac{dU(c_t)}{dc_t} + e^{-\delta t'} S^i_{t'} \frac{dU(c_{t'})}{dc_{t'}} \right] \leq 0 \quad (2.5)
\]

Conversely, the investor may have decided to increase his consumption during a time period \( \Delta t \) and to finance this transaction by decreasing his investment in asset \( X_i \) which would similarly result in a loss of utility. Hence the opposite relation must hold too, and we have equality
\[
S^i_t = \mathbb{E} \left[ e^{-\delta (t'-t)} \xi S^i_{t'} \right] \quad (2.6)
\]

Where, \( \xi(t, t') = \frac{dU(c_{t'})}{dc_{t'}} \left( \frac{dU(c_t)}{dc_t} \right)^{-1} \). These utility theory results are well known and can be traced back to the work of Daniel Bernoulli. The quantity \( \xi(t, t') = \frac{B_t}{B_{t'}} \xi(t, t') \) is commonly referred to as marginal rate of substitution or as state-price deflator. It can be shown that the existence of the deflator is equivalent to that of an equivalent martingale measure \( Q \).

**Corollary 2.1.2 (Martingale optimality equivalence)** The buyer’s optimal martingale measure \( \mathbb{B} \) is defined in terms of the pricing measure \( \mathbb{Q} \) by the Radon-Nikodým derivative \( \xi = \frac{d\mathbb{B}}{d\mathbb{Q}} \).

The importance of this result is that it relates the buyer’s optimal investment-consumption plan to the equivalent martingale measure used for hedging. As discussed in section 1.4.3 above, conditional lapse calibrated using historical data must be calculated in the physical probability measure unless the lapse rate process \( \ell \) is assumed to be \( \mathcal{F}_t \)-adapted. The existence of \( \xi \) makes it possible to compute the value of surrendering the policy from the client’s perspective, within the risk-neutral expectation relevant to the insurer willing to hedge the guarantees sold. The martingale measure enforcing the absence of arbitrage is given by the liquid financial market.

### 2.2 Hamilton-Jacobi-Bellman Equation

At this stage, we have derived an expression for the present value of the policy in terms of the marginal utility and are left with the problem of determining a more explicit value for the utility of consumption, and the bequest function \( \Phi \). The present
section recalls some standard results of stochastic optimal control theory. In a seminal contribution, Merton (1969), Merton (1971) found explicit results for the optimal consumption and investment problem. The optimal value function $J$ is defined as

$$
J(t, x) \triangleq \sup_{\pi \in A, c} E[u(t, c_t, X_t)]
$$

(2.7)

where $u$ is the cumulative life-time utility defined in terms of the instantaneous utility $U$ and the bequest function $\Phi$ as

$$
u(t, c_t, X_t) = \int_t^T U(c_s)ds + \Phi(X_T)
$$

(2.8)

The bequest $\Phi$ represents the utility of having some wealth remaining at the end of the projection period of length $(T - t)$. $J$ can be found as the solution to the Hamilton, Jacobi and Bellman (henceforth HJB) equation

$$
\begin{cases}
\frac{\partial J}{\partial t}(t, x) + \sup_{\pi \in A, c} \{e^{\delta t}u(c_t) + \mathcal{G}\pi J(t, x)\} = 0 \\
J(T, X_T) = \Phi(X_T)
\end{cases}
$$

(2.9)

Here, we have used the notation $\mathcal{G}$ to represent the infinitesimal generator of the process $X_t$ defined as

$$
\mathcal{G}\pi = \left\{\pi_t X_t (\alpha - \beta) + [1 - \pi_t] r_t - c_t\right\} \frac{\partial}{\partial x} + \frac{1}{2} \pi_t X_t \sigma^2 \frac{\partial^2}{\partial x^2} 
$$

The procedure to follow in order to derive the HJB result resembles closely the one used here to define the buyers measure (2.6), and can be found in Bjork (2004, Section 14.3). Before we turn to deriving more explicit results, we need to specify the dynamics followed, in the insured person’s view, by both underlying processes, the equity investments funds and the interest rate. It seems reasonable to assume that a policy-holder expects a constant rate of return and constant volatility from his investments. Hence, we model the underlying assets dynamics as a Geometric Brownian Motion. The value of the money market account is assumed to grow at a constant risk-free rate $r$

$$
\frac{dS_t}{S_t} = (\alpha - \beta) dt + \sigma dW_t^B \\
\quad dB_t = rB_t dt
$$

(2.10)

Where $\beta$ denotes the insurance charge and fund management fees and $\mathcal{B}$ is a subjective probability measure representing the view taken by the buyer on the market.
For the sake of ease of exposition, the dynamics are given in univariate expression. The following results can be extended to the multidimensional case without major complications\(^2\).

### 2.2.1 Constant Relative Risk Aversion (CRRA)

The HJB equation(2.9) is typically a highly non-linear equation. Even numerical solutions will not converge. There is however a restricted number of cases for which it can be solved\(^3\). It turns out that the most practical choice is the power utility. Indeed, it allows us to freely specify the boundary conditions whereas the exponential utility function imposes one since it must be solved in terms of a steady-state solution. Let us define the relative risk aversion in terms of the utility of consumption \(U\) by

\[
\gamma \triangleq -c_t \frac{U'(c_t)}{U''(c_t)}
\]  

This first order linear ODE can be solved fairly easily and yields a solution for \(U\) of the form

\[
U(c_t) = (c_t)^\gamma
\]  

We may model the investor’s time preference by discounting the utility at a subjective rate \(\delta\). In addition, we assume that the bequest function \(\Phi\) has a similar functional form

\[
U(c_t) = e^{\delta t}(c_t)^\gamma \\
\Phi(X_T) = \Psi e^{-\delta T}(X_T)^\gamma
\]  

In order to use the HJB equation, we need to make an educated guess as to the functional form of the optimal value function. It seems appropriate to assume that the value function inherits some properties of the instantaneous utility function as well as the bequest.

\[
\begin{align*}
\mathcal{J}(t, X_t) &= e^{-\delta t} h(t)(X_t)^\gamma \\
V(X_T) &= \Psi e^{-\delta T}(X_T)^\gamma
\end{align*}
\]  


\(^3\)See Merton (1971)
\( \Psi \) is an arbitrary constant. Plugging these results into equation (2.9), and performing the static optimisation with respect to \( \pi \) and \( c \), we find the optimal controls as:

\[
\pi = \frac{\lambda}{\sigma \gamma}, \quad c_t = x h(t)^{-\frac{1}{1-\gamma}} \tag{2.14}
\]

Inserting this function into (2.9), the HJB equation then reduces to an analytically convenient form. The procedure is tedious but straightforward and we simply give the result

\[
\begin{cases}
x^\gamma \frac{\partial J}{\partial t} h(t, x) + \Psi h(t) + (1 - \gamma) h^{-\frac{1}{1-\gamma}} = 0 \\
h(T) = \Psi e^{\delta T}
\end{cases} \tag{2.15}
\]

This type of differential equations is known as Bernoulli equation, and can be solved explicitly. The procedure consists of substituting \( v(t) = h(t)^{1+\frac{1}{1-\gamma}} \) into (2.15). The resulting linear ODE may then be solved by the integrating factor technique, and we find the following solution for \( h \)

\[
h(t) = \left\{ \frac{1 - \gamma}{C} \left( e^{-C[1+(1-\gamma)(1+t) - 1]} - 1 \right) + Const \right\}^{\frac{1}{1+(1-\gamma)^{-1}}} \tag{2.16}
\]

### 2.2.2 Constant Absolute Risk Aversion (CARA)

Another known case of utility that allows for solving equation (2.9) in closed-form is the case that absolute risk aversion is constant. The absolute risk aversion is defined as

\[
\gamma \equiv -\frac{U'(c_t)}{U''(c_t)} \tag{2.17}
\]

Solving the ODE is straightforward and including the subjective instantaneous discount rate \( \delta \) yields a utility function of the form

\[
U(c_t) = \frac{1}{\gamma} e^{\delta t + \gamma c_t} \tag{2.18}
\]

Again, we have to guess the optimal value function, and it is natural to assume that it inherits some properties of the utility function

\[
\mathcal{J}(t, X_t) = h(t) e^{-\delta t} (X_t)^\gamma \tag{2.19}
\]
Equation (2.9) for the exponential utility can be solved in terms of a steady-state solution

\[ f_\infty = -\exp \left\{ 1 - \log(R_t\gamma) - \frac{\mu - R_t^2}{R_t\sigma} - \delta \right\} \]  

(2.20)

The optimal investment-consumptions strategy for constant absolute risk aversion reads

\[
\begin{align*}
\pi_t &= \frac{\lambda}{\sigma R_t X_t} \\
ct &= -\frac{1}{\gamma} \log(-R_t\gamma f_\infty) + R_t X_t
\end{align*}
\]  

(2.21)

### 2.3 Implied Utility

At this stage, we have derived an expression for the present value of the policy in terms of the marginal utility and are left with the problem of determining a more explicit value for the utility of consumption, and the bequest function \( \Phi \). In this section, we show how information can be extracted from the policy in order to infer the utility function parameters. The assumption implicitly made is that life-insurance markets are sufficiently efficient for the buyer to find his optimal investment-consumption plan in the set of products offered. It is also assumed that the policy-holder is rational, in the sense that investments choices made are an accurate reflection of the optimal financial objectives.

**Definition 2.3.1 (Implied utility function)** *The implied utility function for a specified contract is defined as the function satisfying property (2.1.1) and conditions (2.2), and for which optimal controls are given by contractual withdrawal rates and current asset allocation.*

This approach may be compared to the use of investor’s Black-Scholes option formula to estimate the implied volatility parameter from the option market prices. Likewise, we may use the contractual guarantee price and elected withdrawal plan to infer policyholder’s risk preferences. Although the annuity market can not be considered a liquid market, it seems reasonable to assume a sufficiently large number of distributors available for the clients to select a policy that meets their optimality criteria. Thus, we may think of the utility parameters estimates as market implied values. policy-holder’s current asset allocation, periodical withdrawals and the price he agreed to pay for the guaranteed benefit may be used as input to calculate the various utility function parameters. To this effect we will need to specify a particular
functional form for the instantaneous utility function \( u(c_t) \), and the bequest function \( \Phi(X_T) \). It will prove convenient to choose a form that allows closed-form solutions for the HJB equation. As pointed out above, and as will be shown below, we will aim at setting the bequest function so as to match the value of the guaranteed benefit.

### 2.3.1 Matching the optimal consumption plan

The fact that the client’s utility of consumption \( U \) in this model is not an assumption but obtained as result of an optimisation makes it a particularly attractive application of stochastic optimal control theory. One of the most difficult practical challenges of dynamic programming is the difficulty of finding objective estimates for the utility parameters. Clearly, the choice of insurance contract is an indication of the policy-holder’s risk aversion. In the present model, we argue that elected benefits selected, choice of investment funds and scheduled withdrawals can be used to approximate their risk appetite with a tractable utility function. The policy-holder’s utility time-preference and risk-aversion parameters, respectively \( \delta \) and \( \gamma \), may be found as the solution to the constrained optimisation problem

\[
\min_{\delta, \gamma} \int_0^T c_u du - \sum_{i=0}^{\pi} w_i X_t \\
\pi_0 = \pi^* \quad (2.22)
\]

Where \( \pi^* \) is the observed initial percentage of notional invested in equity and \( w^*_i \) denotes the withdrawal rate that is our targeted optimal consumption plan.

Figure (2.1) displays the result of the non-linear optimisation performed with MATLAB.

### 2.3.2 Matching the bequest with the purchase price

If we suppose that the buyer’s relative risk aversion is constant, the bequest function \( \Phi \) is fully specified by the value of the constant \( \Psi \) can be determined by the method of utility indifference pricing. From the theory of option valuation in incomplete markets, we borrow the utility indifference pricing procedure defined by Hodges and Neuberger (1989) as

**Definition 2.3.2 (Utility indifference price)** The utility indifference price \( p \) for a claim is the unique price such that

\[
p_t = \inf \{ q \in \mathbb{R} : u^{(n)}(X_t + nq_t) > u^{(0)}(X_t)|\mathcal{F}_t \}
\]

\( (2.23) \)
Figure 2.1: Utility parameters optimisation

<table>
<thead>
<tr>
<th>Utility parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^2$</td>
</tr>
<tr>
<td>$\gamma$</td>
</tr>
<tr>
<td>$\delta$</td>
</tr>
</tbody>
</table>

Table 2.1: Utility parameters estimation

Where $n$ is the number of claims purchased and $U^{(\cdot)}$ denotes the utility associated with a portfolio containing a number $(\cdot)$ of claims. Setting $n = 1$ we obtain the buyer price, while $n = -1$ gives the seller price. In the present case, we seek to compute the buyer’s indifference price $p$, solving

$$u^{(n)}(X_t + p_t) = u^{(0)}(X_t) \quad (2.24)$$

The purchase price of the embedded guarantee is known to the insurer, and we may use the argument that if the insured had no interest in terminal wealth whatsoever, the utility $\Phi(X_t)$ assigned to terminal wealth would be zero.

Using equation (2.6) to calculate the discounted expected claim value, with expectation taken in the $\mathbb{B}$-measure. Thus, we seek the value $\Psi = \Psi^*$ such that the difference in claim price $p_t^{\Psi^*}$ and $p_t^0$ yields the price at which the guarantee was purchased. We may proceed by first setting $\Psi$ to an arbitrary value $\Psi_0$ and use Newton-Raphson iterations until we match the option cost.
2.4 Implied utility based forecast of dynamic lapse

The exercise of the option to surrender the contract is triggered the first time that the utility of immediate consumption of the present value of the assets exceeds the deflated expected cumulative utility. The first hitting time $\tau^0$ is defined as

$$
\tau^0 \triangleq \inf \left\{ t \in [0, T] : U(X_t) > \mathbb{E}^{\mathbb{B}} \left[ \int_0^T e^{-\delta t} U(c_t) dt + \Phi(X_T) | \mathcal{G}_t \right] \right\}
$$

(2.25)

The use the utility of consumption $U(X_t)$ rather than the value of terminal wealth $\Phi(X_T)$ on the left-hand side of the inequality may seem arbitrary. The reasoning is that the decision is made between using the wealth for immediate consumption or remaining in the contract. In our model, the bequest function $\Phi$ reflects the value of death and living benefits. It is observed that holders of policies including death and living benefits value consumption more than legacy.

**Remark** An interesting question is whether immediate consumption is as accurate an estimator as, let’s say, expected returns from investments in the asset. A convenient by-product of the martingale optimality equivalence from corollary (2.1.2) is that this question is no longer relevant since

$$
\mathbb{E}^{\mathbb{B}}[\hat{X}_T | \mathcal{G}_t] = X_t
$$

(2.26)

where $\hat{X}_T = e^{-\delta(T-t)}X_T$ is the deflated wealth process.

Recall from the preliminary analysis of lapsation in the first chapter, we must take into account the non-financial components affecting the exercise decision of the right to surrender the policy. As in the case of mortality, the formulation of the problem at portfolio level provides a more comprehensive representation in which probability of lapse takes the statistical interpretation of expected rate of decrease in notional due to surrenders. We use the dynamic lapse reduction formula $\ell(t, T, X_t)$ proposed in the first chapter

$$
\ell(t, T, X_t) = \theta_0(t) + \frac{\theta_1}{1 - e^{\theta_2 + \theta_3 \phi(t, T, X_t)}}
$$

(2.27)

The estimator for the lapsation is a specified surrender value function $\phi(t, T, X_t)$ which we express in terms of the difference in utility between immediate withdrawal of the notional and expected utility of the funds returns and claim value.
Utility based Moneyness based

<table>
<thead>
<tr>
<th></th>
<th>Utility based</th>
<th>Moneyness based</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^2$</td>
<td>82.2 %</td>
<td>95.6 %</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>0.107</td>
<td>0.175</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>2.0</td>
<td>1.4</td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>-152</td>
<td>-253</td>
</tr>
</tbody>
</table>

Table 2.2: Lapse parameters estimation

\[
\phi(t, T, X_t) = U(X_t) - \mathbb{E} \left[ \int_0^T e^{-\delta t} U(c_t) dt + \Phi(X_T) \big| \mathcal{F}_t^B \right]
\]  

(2.28)

Here, the expectation is to be taken with respect to the buyers probability measure $\mathbb{B}$. From the policy issuers perspective, the corresponding arbitrage-free price associated with a claim of maturity $T$ can be computed as

\[
V_t = \mathbb{E}^{Q} \left[ \int_t^T \frac{B_s}{B_t} e^{-\mu(x,s) - \ell(s,T,d_B X_s)} \chi(T, X_T) ds \big| \mathcal{F}_t \right]
\]  

(2.29)

An explanatory variable based upon the policy-holder’s return expectation and tolerance to risk seems a priori more reasonable than an estimator depending solely on the instantaneous difference between guaranteed amount and current funds level. It turns out that it is indeed a more accurate predictor and significantly reduces the standard deviation of forecasts with respect to observed lapse rate.

The numerical analysis is based on deseasonalised lapse data observed on a time frame of 5 years. A regression was performed on the same sample of policies using respectively the moneyness and the difference in utility of consumption as the explanatory variable. The regression results are displayed in table 2.2. The $R^2$ of the regression is significantly lower than when utility difference fit is used. Also interesting that we obtain a significantly higher parameter $\theta_3$ with the utility based procedure. $\theta_3$ determines the steepness of the function. Which indicates a stronger dependence to the financial value as defined. Figure(2.2)displays a graphical comparison of the two approaches.
Figure 2.2: Fit to historical lapse using moneyness vs. utility difference
2.5 Implied utility based dynamic asset allocation

The risk exposure arising from writing guarantees on unit-linked life insurance policies may also be reduced by actively managing the asset allocation. Investments can be rebalanced so as to obtain an optimal risk-return profile. The classical tool used in the insurance industry to limit the down-side risk is a principal protection mechanism referred to as portfolio insurance. The most popular variant of portfolio insurance is known as constant proportion portfolio insurance (henceforth CPPI). CPPI prescribes a rebalancing scheme subject to a given level of protection $\Delta X$ against decrease in account value

$$\pi^{cppi} = \frac{1}{\Delta X} \left( 1 - \frac{K (1 + \beta)}{X_t (1 + R_t)} \right)^{T-t} \tag{2.30}$$

It can be seen from equation (2.30) that decreases in the asset value translates into transfers from the risky asset to the money market account. This has the effect of limiting potential losses, but also implies high transaction costs on an unfavourable ’buy-high and sell-low’ basis$^4$.

In the preceding section, we have shown that the policy-holder’s utility function can be inferred from initial asset allocation, and elective benefits chosen. It is also possible to use stochastic optimisation in order to determine the optimal investments according to clients risk appetite. The policy can be ’tailor made’ and design so as to match the policy-holder’s desired level of protection, expected return on investment and withdrawal rate. The non-linear optimisation to be performed will then reads slightly differently, since the investment strategy process $\pi$ is now a control variable that will be adjusted so as to optimise the utility of consumption. The policy-holder’s utility parameters $\Psi$, $\delta$ and $\gamma$ may be found as the solution to the linear programming problem

$$\min_{\Psi,\delta,\gamma} \int_0^T c_u du - \sum_{i=0}^{\pi} w_i X_t$$

Where $w_i$ denotes the withdrawal rate which is our targeted optimal consumption plan. Table 2.5 displays the obtained parameters and regression $R^2$

$^4$Some analysts hold portfolio insurance strategies responsible for precipitating the black Monday crash of Wall Street in 1987
<table>
<thead>
<tr>
<th>CRRA</th>
<th>CARA</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^2$</td>
<td>91.7 %</td>
</tr>
<tr>
<td>$\delta$</td>
<td>1.03 %</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>24.9 %</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>2.21</td>
</tr>
</tbody>
</table>

Table 2.3: Utility parameters estimation

<table>
<thead>
<tr>
<th>All Equity</th>
<th>CPPI</th>
<th>CRRA</th>
<th>CARA</th>
</tr>
</thead>
<tbody>
<tr>
<td>min%</td>
<td>-2.49</td>
<td>2.85</td>
<td>-1.03</td>
</tr>
<tr>
<td>99%</td>
<td>11.58</td>
<td>11.49</td>
<td>10.24</td>
</tr>
<tr>
<td>50%</td>
<td>6.24</td>
<td>15.64</td>
<td>5.90</td>
</tr>
<tr>
<td>1%</td>
<td>0.47</td>
<td>2.97</td>
<td>1.32</td>
</tr>
<tr>
<td>Expd return</td>
<td>6.15</td>
<td>5.80</td>
<td>5.90</td>
</tr>
<tr>
<td>ROP value</td>
<td>$184</td>
<td>$0</td>
<td>$17</td>
</tr>
<tr>
<td>Ratchet value</td>
<td>$3645</td>
<td>$0</td>
<td>$254</td>
</tr>
</tbody>
</table>

Table 2.4: P&L Percentiles for $\sigma = 10 \%$

<table>
<thead>
<tr>
<th>All Equity</th>
<th>CPPI</th>
<th>CRRA</th>
<th>CARA</th>
</tr>
</thead>
<tbody>
<tr>
<td>min%</td>
<td>-11.73</td>
<td>2.68</td>
<td>-7.34</td>
</tr>
<tr>
<td>99%</td>
<td>16.68</td>
<td>16.27</td>
<td>30.03</td>
</tr>
<tr>
<td>50%</td>
<td>4.69</td>
<td>3.14</td>
<td>4.97</td>
</tr>
<tr>
<td>1%</td>
<td>-6.29</td>
<td>2.75</td>
<td>-3.27</td>
</tr>
<tr>
<td>Expd return</td>
<td>4.87</td>
<td>4.84</td>
<td>6.00</td>
</tr>
<tr>
<td>ROP value</td>
<td>$16,876</td>
<td>$0</td>
<td>$6,884</td>
</tr>
<tr>
<td>Ratchet value</td>
<td>$7645</td>
<td>$0</td>
<td>$460</td>
</tr>
</tbody>
</table>

Table 2.5: P&L Percentiles $\sigma = 20\%$
Figure 2.3: P&L distributions for different asset allocation schemes
Figure 2.4: Dynamic asset allocation performance. Stochastic up scenario
The outcome of Monte-Carlo simulation of the respective strategies P&L is given below. 10,000 scenarios have been generated on a period of 20 years with discrete monthly time steps. The policy initial account value $100,000 with a return of premium (ROP) living benefit and a ratchet death benefit. In order to isolate the effect of asset allocation, mortality, lapse and partial withdrawals have been disabled.

The simulation results demonstrate how optimal controls act as a natural hedge for the guarantee by efficient dynamic asset allocation strategies. One of the key results is how the exponential utility succeeds in controlling the funds value fluctuation, considerably diminishing the price of ratchet benefits. In particular, using absolute risk aversion proves particularly effective in a high volatility environment, and yields higher average returns than investments in the risky asset. After a drop in account value, portfolio insurance tends to lock-in losses, while utility based strategies will take the necessary risk to compensate incurred losses. Inada conditions ensures that maximal risk will be taken when the value of the asset approaches zero. Clearly this approach is more sensible than saving a negligible remaining amount at the risk-free rate. It is observed that constant absolute risk aversion yields on average better returns at lower risk than CRRA when the volatility is high with respect to the asset returns. On the other hand, CARA makes high returns less likely and the left tail it produces is relatively fat. This is due to the low and large proportion of wealth invested in the risky assets when the account value is, respectively, large and low.
Chapter 3

Hedging in Incomplete Market

In the first chapter, we pointed out that life insurance guarantees are not attainable in the presence of actuarial risks such as mortality and lapse. In the preceding chapter, we introduced the concept of implied utility and succeeded in considerably reducing the uncertainty around dynamic policy-holder behaviour. Not sufficiently however to consider observed deviations from the lapse function as negligible. An important achievement of our lapse model is to separate the financially motivated dynamic lapse from the structural base lapse, independent of asset returns. We hence maintained a reasonable economic model where the independence assumption 1.2.1 of the actuarial and financial probability spaces is still satisfied. Another source of incompleteness arises from the impossibility of trading the asset underlying the claim directly. Unlike within the ideal Black-Scholes framework, absence of arbitrage is not sufficient in this context to determine contingent claim prices and hedging strategies uniquely. In an incomplete market, replication using self-financing strategies only is in general not possible. Instead, the cumulative cost process associated with a hedging strategy is a stochastic process and the classical risk-neutral valuation will fail to determine unique hedging strategies and arbitrage-free prices. The arbitrage argument leads to an infinite number of possible martingale measures and one is left with the problem of selecting the one that is optimal according to some optimality criteria. Multiple approaches have been proposed in order to extend the arbitrage pricing theory to the incomplete market setting. In this chapter we selected three of the most celebrated approaches and present the main hedging results in a nutshell. The problem of hedging in incomplete markets is an active area of research and it turns out that different definitions tend to converge on several points, and exhibit similarities with classical actuarial premium pricing principles.
3.1 Combined model

In the first chapter, we pointed out that the mortality risk leads to a combined financial and actuarial model which is incomplete. We now turn to give an explicit construction of the pricing measure under assumption 1.2.1. Møller (2003) suggests a product space model to construct a class of equivalent martingale for pricing actuarial and financial risk components in a combined model. We need to extend the definition of the actuarial filtration \( \mathcal{F}_a^t \) \( t \in [0, T] \) in order to account for the actuarial risk arising from lapses. The counting process defined in section 1.2.1 becomes

\[
N(x, t) = \sum_{n=0}^\infty 1_{(\tau_i^+ \vee \tau_0^i) \leq t}
\]

\[
\mathbb{E} [dN(x, t) | \mathcal{F}^a_t] = (n_x - N(x, t^-)) (\mu(x, t) + \theta_0(t)) dt
\]

(3.1)

\( \theta_0(u) \) is the base lapse function introduced in equation (1.69) and specified by equation (1.70), dynamic lapse depends on the realisations of the wealth process will be modelled in the financial component. With definitions as above the actuarial filtration is defined as

\[
\mathcal{F}_a^t \triangleq \sigma \{ N_u; 0 \leq u \leq t \}
\]

(3.2)

The claim value process \( V_t \) is adapted to the filtration \( (\mathcal{F}_t)_{t \in [0, T]} \) obtained by combining the two probability spaces, actuarial and financial, into a product space. The \( \sigma \)-algebra \( \mathcal{N} \) represents the filtration generated by all subsets of the null-sets of the combined model \( \mathcal{F}^f \otimes \mathcal{F}^a \)

\[
\mathcal{N} = \sigma \{ \mathbb{F} \subseteq \Omega^f \times \Omega^a \mid \exists G \in \mathcal{F}^f \otimes \mathcal{F}^a : F \subseteq G, (\mathbb{P}^a \otimes \mathbb{P}^f)(G) = 0 \}
\]

(3.3)

Lemma 3.1.1 Consider the \( \sigma \)-algebra defined by

\[
\mathcal{F}_t = \{ \mathcal{F}^a \otimes \mathcal{F}^f \} \vee \mathcal{N}
\]

(3.4)

and define the \( \sigma \)-algebras on the product space by

\[
\mathcal{F}_t^1 = (\mathcal{F}^f \otimes \{ \emptyset, \Omega^a \}) \vee \mathcal{N}, \quad \mathcal{F}_t^2 = (\{ \emptyset, \Omega^f \} \otimes \mathcal{F}^a) \vee \mathcal{N}
\]

(3.5)

(i) The filtrations \( \mathbb{F}^1 = (\mathcal{F}_t^1)_{t \in [0, t]} \) and \( \mathbb{F}^2 = (\mathcal{F}_t^2)_{t \in [0, t]} \) satisfy the usual conditions of completeness and right-continuity.

(ii) \( \mathbb{F}^1 \) and \( \mathbb{F}^2 \) are independent

49
(iii) The filtration \((\mathcal{F}_t)_{t\in[0,T]}\) defined by \(\mathcal{F}_t = \mathcal{F}^1_t \lor \mathcal{F}^2_t\) satisfy the usual conditions of completeness and right-continuity. And \(\mathcal{F}_t = \{\mathcal{F}^a \otimes \mathcal{F}^j\} \lor \mathcal{N}\)

**Proof** Having extended the actuarial filtration to account for the non-financial lapse component, the proof follows the same arguments as Møller (2003)

(i) The completeness is trivial, since \(\forall t \in [0,T]\) we have \(\mathcal{N} \subseteq \mathcal{F}^1_t\) and \(\mathcal{N} \subseteq \mathcal{F}^2_t\)
In order to prove the right-continuity of \(\mathcal{F}_t\), we define \(\mathcal{D}_t\) as

\[
\mathcal{D}_t = \left\{ F_1 \times \Omega^a | F_1 \in \mathcal{F}^j_t \right\}
\]

(3.6)

By definition, \(\sigma(\mathcal{D}_t) = \mathcal{F}^j_t \otimes \{\emptyset, \Omega^a\}\), since \(\mathcal{D}_t\) is also a \(\sigma\)-algebra, we have

\[
\mathcal{D}_t = \mathcal{F}^j_t \otimes \{\emptyset, \Omega^a\}
\]

(3.7)

Hence,

\[
\bigcap_{\epsilon > 0} \mathcal{F}^1_{t+\epsilon} = \bigcap_{\epsilon > 0} (\mathcal{D}_{t+\epsilon} \lor \mathcal{N}) = \left( \bigcap_{\epsilon > 0} \mathcal{D}_{t+\epsilon} \right) \lor \mathcal{N}
\]

(3.8)

From the right continuity of \(\mathcal{F}^j\), the last term equals \(\mathcal{D}_t \lor \mathcal{N}\) and we have proved that \(\mathcal{F}^1\) is right-continuous. Right-continuity of \(\mathcal{F}^2\) can be established following a similar procedure.

(ii) Consider \(\mathcal{F}_1 = \mathcal{F}^j_1 \times \mathcal{O}_2\) and \(\mathcal{F}_2 = \mathcal{O}_1 \times \mathcal{F}^a_2\) where \(\mathcal{F}^j_1 \in \mathcal{F}^j\), \(\mathcal{F}^a_2 \in \mathcal{F}^a\) and \(\mathcal{O}_1 \in \{\emptyset, \Omega^j\}\), \(\mathcal{O}_2 \in \{\emptyset, \Omega^a\}\). Then,

\[
P(\mathcal{F}_1 \cap \mathcal{F}_2) = P((\mathcal{F}^j_1 \times \mathcal{O}_2) \lor (\mathcal{O}_1 \times \mathcal{F}^a_2)) = P((\mathcal{F}^j_1 \lor \mathcal{O}_1) \times (\mathcal{F}^a_2 \lor \mathcal{O}_2))
\]

\[
= P^j(\mathcal{F}^j_1 \lor \mathcal{O}_1) P^a(\mathcal{F}^a_2 \lor \mathcal{O}_2) = P^j(\mathcal{F}^j_1) P^j(\mathcal{O}_1) P^a(\mathcal{F}^a_2) P^a(\mathcal{O}_2)
\]

\[
= P(\mathcal{F}_1) P(\mathcal{F}_2)
\]

(3.9)

As required, we have that \(\forall \mathcal{F}_1 \in \mathcal{F}^1_T\) and \(\forall \mathcal{F}_2 \in \mathcal{F}^2_T\) the following equality holds

\[
P(\mathcal{F}_1 \cap \mathcal{F}_2) = P(\mathcal{F}_1) \cap P(\mathcal{F}_2)
\]

(3.10)

Thus, the independence property is verified.
(iii) Since $\mathbb{F}_1$ and $\mathbb{F}_2$ are independent and both satisfy the usual conditions, a general result of probability theory shows that $\mathbb{F}$ satisfies the usual conditions. The equality is trivial.

The importance of the independence assumption (1.2.1) between the actuarial and financial risk factors relies in the possibility of constructing a tractable combined model as above. The matter of choosing optimality criteria for the equivalent martingale measure is discussed in the following sections.

### 3.2 Quadratic Risk Minimisation

The classical approach towards pricing contingent claims in incomplete markets consists of minimising a particular risk measure. Föllmer and Sondermann (1986) introduced the notion of risk minimisation and applied it to the problem of determining optimal hedging strategies for fixed payment claims. Consider the cumulative cost process $C_t = (C_t)_{t \in [0,T]}$ associated with a square-integrable trading strategy $\pi$

$$C_t = V_t(\pi) + \int_0^t \pi_s d\hat{X}_s \quad (3.11)$$

$\hat{X}_s$ is the discounted wealth process, and $V_t(\pi) \triangleq \pi X_t + \nu_t$ denotes the value process associated with trading strategy $\pi$. The optimisation problem proposed is that of minimising the quadratic risk process

$$R_t(\pi) \triangleq \mathbb{E} \left[ (C_t(\pi) - C_T(\pi))^2 | \mathcal{F}_t \right] \quad (3.12)$$

A natural requirement is to restrict to strategies that do not involve costs after all liabilities are paid, hence the following definition of the set $\mathcal{A}_0$ of 0-admissible strategies

$$\mathcal{A}_0 \triangleq \{ \pi \in \mathcal{L}^2(\mathbb{P}_X) : \pi_T X_T + \nu_T = 0 \} \quad (3.13)$$

The risk-minimising strategy may be defined as the one that minimises the cumulative quadratic risk process at maturity

**Definition 3.2.1** (Föllmer and Sondermann (1986)) An $\mathcal{F}_t$-adapted strategy $\pi \in \mathcal{A}_0$ is called risk-minimising if for all $\hat{\pi} \in \mathcal{L}^2(\mathbb{P}_X)$ the following relation holds

$$R_t(\hat{\pi}) \geq R_t(\pi) \quad (3.14)$$
Föllmer and Sondermann (1986) defined a cost process which is independent of the issuer’s liability and proved that the cost process associated with risk-minimising strategy is a martingale. Due to this martingale property, risk-minimising strategies are said to be mean-self financing. The definition of the hedging cost process from Föllmer and Sondermann (1986) restricts to strategies which replicate the claim at maturity and exclude trading after payment of the liabilities. This framework was later extended by Møller (1998) to account for more general types of liability pay-offs where intermediate liability payments can occur.

\[
C_t = V_t(\pi) + \int_0^t \pi_s dX_s + L_t \tag{3.15}
\]

\(L_t\) representing the forward cost process. Any martingale \(V^*\) can be uniquely represented by Kunita-Watanabe projection in the space \(\mathcal{M}\) of square integrable martingales. The Galtchouk-Kunita-Watanabe decomposition theorem gives \(V^*\) as

\[
V^*_t = \mathbb{E}^Q [B^{-1}(t, T) \chi(X_T)|\mathcal{F}_t] = V_0 + \int_0^t \theta_u dX_u + L_t \tag{3.16}
\]

\(L_t\) is a martingale orthogonal to \(X, \theta \in L^2(P_X)\) is as usual a progressively measurable process. Assuming stock dynamics as in 1.22, the Galtchouk-Kunita-Watanabe for the intrinsic value function \(\tilde{V}\) defined by equation (1.20) is found as

\[
\tilde{V}_t = V_0 + \int_0^t \pi_u dX_u + \int_0^t \pi_u^0 dM(x, u) \tag{3.17}
\]

Where the compensated counting process \(M(x, t) \triangleq N(x, t) - \mathbb{E}[dN(x, t)|\mathcal{F}_t]\) is an \(\mathcal{F}_t^g\)-martingale. In terms of the arbitrage-free price \(V^f\) of the claim given the filtration \(\mathcal{F}^{f1}\) the risk-minimising strategy follows

**Proposition 3.2.2 (Quadratic risk-minimising strategy)** The unique quadratic risk-minimising hedge is given by the following mean-self financing strategy

\[
\begin{cases}
\pi_t = [n_x - N_{t^-}] \int_t^T V^f(S_t, u) q(x+t) \mu_{x+u} du \\
v_t = \int_0^t B_u^{-1} \chi(u, u) dN(x, u) + (n_x - N_{t^-}) \int_t^T B_t^{-1} V^f(S_t, u) q(x+t) \mu_{x+u} du - \tilde{\pi}_t S_t
\end{cases}
\]

Föllmer and Schweizer (1991) extended the results obtained by Föllmer and Sondermann (1986) for local martingales only to the more general semi-martingale setting. They obtained a uniquely determined decomposition from Doob-Meyer decomposition and Girsanov transformation. The resulting strategy is found in terms of

\footnote{In this context, that is the price given by Black-Scholes formula}
the minimal martingale measure, which turns out to occur naturally in the context of dual optimisation pricing as we will see below. The minimal martingale measure $\mathbb{P}^M$ may be found as the one minimising the relative entropy $\mathcal{H}$ with respect to $\mathbb{P}$ among all martingales measures with fixed expectation $\mathbb{E}\left[\int_0^T \pi_u d\langle X \rangle_u\right]$

\[
\mathcal{H}(Q, P) = \begin{cases} 
\int \log \frac{dQ}{dP} \, dQ, & \text{if } Q \ll P \\
+\infty & \text{otherwise}
\end{cases}
\]

The resulting variance-optimal hedging strategy is given by $\pi_t^* = \frac{d(\mathbb{E}^M[\chi | \mathcal{F}_t], X)}{d(X)}$. Different characterisations of the minimal martingale measure can be found in Schweizer (1995). In particular it happens that it also minimises the process $\sqrt{\text{Var} \frac{dQ}{dP}}$.

Schäl (1994) and Schweizer (1996) studied an $\mathcal{L}^2$-approximation approach to pricing in the variance-optimal measure. The procedure consists of finding the initial capital and trading strategy $\pi$ that approximates $\chi$ with respect to the distance in $\mathcal{L}^2$. Schweizer (1994a) imposed a special structure for the process $X = (X_t)_{t \in [0,T]}$ that admits a decomposition of the form $X = X_0 + M + \int \alpha_u d\langle X \rangle_u$, with a stringent condition on the relation of $\alpha$ and $M$. This unnatural restriction was later released by Pham et al. (1998); Rheinländer and Schweizer (1997), who found solutions for a general $\mathcal{F}_t$-adapted claim $\chi \in \mathcal{L}^2$. Further developments of the variance-optimal pricing framework are presented in Schweizer (1997). Prices are derived according to some of the classical actuarial pricing principles, and the close relation with utility theory is exploited to derive utility indifference prices\(^2\). For the variance principle, the utility function is given by

\[
U(X_t) = \mathbb{E}[X_T] - \alpha \text{Var}^{\beta}[X]
\]

The major drawback of quadratic risk measure is that gains and losses are equally sanctioned. This leads to unreasonable hedging strategies whereby investors seek as much protection against possible profits as losses.

### 3.3 Super-replication

The concept of super-hedging was first introduced by El Karoui and Quenez (1995). The basic idea is to find the cheapest self-financing strategy that dominates the pay-off of the claim except perhaps on a set of measure zero.

\(^2\)Bielecki et al. (2005) used such quadratic utility functions and found solutions for the HJB equation when the short rate of interest is governed by a CIR diffusion
\[
\inf \left\{ V_0 | \exists \pi \in \mathcal{L}^2(P_X) : V_0 + \int_0^T \pi_s d\tilde{X}_s \geq \chi_T(X_T) \ P \ a.s. \right\} 
\]

(3.19)

Let us consider the càdlàg modification of the essential supremum process, and denote by \( X_t^* \) the upper-hedging price process

\[
X_t^* = \text{ess. sup}_{Q \in \mathcal{A}} \mathbb{E}^Q [\chi(T, X_T) | \mathcal{F}_t] 
\]

(3.20)

It is assumed that the expected claim pay-off is bounded from above,

\[
X_0^* \triangleq \sup_{P^* \in \mathcal{M}} \mathbb{E}^{P^*} [\chi_T(T, X_T) | \mathcal{F}_t] < \infty
\]

(3.21)

From Karatzas and Shreve (2004, Proposition 6.5, p.213), we know that the corresponding value process \( \xi_t X_t^* \) is a supermartingale under any equivalent martingale measure in \( \mathcal{M} \). For any stopping time \( \tau \in [0, T] \), define \( \tilde{X}(\tau) \) as the deflated essential supremum of the claim price From Doob’s optional sampling theorem, we have that

\[
\hat{\xi}_\tau X_t^* \geq \mathbb{E} \left[ \hat{\xi}_\tau \chi(T, X_T) | \mathcal{F}_\tau \right] 
\]

(3.22)

Dividing by \( \hat{\xi}_\tau \) and taking essential supremum on both sides of the inequality, it follows that

\[
X_t^* \geq \tilde{X}_\tau 
\]

(3.23)

Since \( X_t^* \geq \tilde{X}_\tau \) almost surely if \( \tau \) takes only finitely many values, we may construct a sequence \((\tau_n)_{n=1}^\infty\) so that \( \tau_n \) converges to \( \tau \) almost surely when \( n \to \infty \). It follows from Fatou’s lemma and from the right-continuity of \( V^* \) that

\[
\int_A \hat{\xi}_\tau X_t^* dP \leq \lim_{n \to \infty} \int_A \hat{\xi}_{\tau_n} X_t^* dP = \lim_{n \to \infty} \int_A \hat{\xi}_{\tau_n} \tilde{X}_\tau dP \leq \int_A \hat{\xi}_\tau X_t^* dP 
\]

(3.24)

Hence, the opposite of inequality (3.23) is also true and we must have equality

\[
X_t^* = \tilde{X}_\tau 
\]

(3.25)

As shown in Karatzas and Shreve (1991, pp. 24-27), each stopped supermartingale has a unique Doob-Meyer decomposition and this leads to the optional decomposition from Kramkov (1996) of the non-stopped supermartingale \( V^* \)

\[
\hat{\xi}_t X_t^* = X_0^* + \int_0^t \pi_s dX_s - L_t 
\]

(3.26)
Here \( L_t \) is an increasing adapted process, \( \pi \) a progressively measurable admissible strategy. As a corollary of the optional decomposition, the value \( V_t^* \) can be characterised as the least initial capital needed to cover the claim \( \chi \)

\[
V_t^* = \text{ess.inf} \left\{ V_t : V_t + \int_t^T \pi_s dX_s \geq \chi(T, X_T) | \mathcal{F}_t \right\}
\]

In the spirit of Møller (1998), we consider the pure financial value \( V^f_t \) of the portfolio of claims, and identify unique superhedging strategy from the optional decomposition (3.26)

\[
V_t^f = \mathbb{E}^Q[B^{-1}\chi(T, X_T) | \mathcal{F}_t^f] = \mathbb{E}^Q[B^{-1}\chi(T, X_T)] + \sum_{i=1}^{t} \alpha_i \Delta S_i e^{ri}
\]

**Proposition 3.3.1 (Upper hedging strategy)** The unique admissible super-hedging strategy is given by

\[
\begin{cases}
\pi_t = (n_x - N_{t-1})\alpha_t \\
v_t = n_x V_0^f + \sum_{i=1}^{t} \pi_i \Delta S_i - \pi_t S_t
\end{cases}
\]

The super-replication price corresponds to a total risk aversion and is in this sense not suitable for a competitive insurance firm. The large initial capital required to carry out this strategy results in structural overhedging and generates prohibiting claim price and is there. From the factor \((n_x - N_{t-1})\) in the equation for \( \pi_t \), it appears that the superhedging price of a living benefit assumes that all clients are alive at maturity. Föllmer and Leukert (1999) tackled this issue by including some tolerance to shortfall risk in the form of a quantile hedging model which is little more than a dynamic version of the value-at-risk principle. Employing a combination of optional decomposition and Neyman-Pearson lemma, Artzner et al. (1999) introduced an axiomatic approach to pricing contingent claim with respect to coherent risk measures. It is shown that the value-at-risk principle fails to comply to the sub-additivity condition, essential for the risk measure to be viable.\(^3\). Föllmer and Leukert (2000) used the optional decomposition on a modified claim and reduced the initial capital required.

\(^3\)A proof of the existence of a solution with coherent measure can be found in Schied (2004)
3.4 Convex duality

Whereas the methodologies presented so far combine a complete financial market with an acturial market, the utility based dual method described below models an incomplete financial market directly. The claim contingent on $X$ is hedged with an imperfectly correlated asset with price process $Y_t = (Y_t)_{t \in [0,T]}$. Utility based strategies were first introduced by Hodges and Neuberger (1989) in the context of optimising hedge performance in the presence of transaction costs. The issuer’s risk preferences is characterised by a utility function $U(X_T)$, and the primal problem is that of maximising the utility among all admissible trading strategies.

$$\max_{\pi \in A} \mathbb{E}[U(X_T)]$$

In order to avoid doubling strategies, following Schachermayer (1999), we restrict the set $A$ containing all admissible asset allocation plans is bounded from below

$$A = \{ \pi \in \mathcal{L}^2(P_X) : X_t \geq a_\pi, \forall t \in [0,T], a_\pi \in \mathbb{R} \} \quad (3.30)$$

The constraint that arbitrage opportunities should be ruled out is enforced by dualising the problem. We are now left with the problem of maximising the Lagrangian $\mathcal{L}$ with respect to $X_T$ and $\eta$

$$\max_{\pi \in A, \eta \geq 0} \mathcal{L}(X_T, \eta) = \max_{\pi \in A, \eta \geq 0} \mathbb{E}[U(X_T)] + \eta \left[ x - \mathbb{E} \left( \frac{dQ}{dP} \right) X_T \right] \quad (3.31)$$

Defining the convex conjugate of $U(X_T)$ as $V(\eta) = \mathbb{E}[U(x) - x\eta]$, the dual problem reads

$$\max_{\pi \in A, \eta \geq 0} \mathcal{L}(X_T, \eta) = \max_{\pi \in A, \eta \geq 0} \eta x + \mathbb{E} V \left( \frac{dQ}{dP} \eta \right) \quad (3.32)$$

Hugonnier et al. (2002) solved the dual problem using utility based price in a general semimartingale setting. Kramkov and Schachermayer (1999) established necessary and sufficient conditions on the utility functions for the theory to be valid. The existence of an optimal process when the wealth is constrained to take values in a non-empty closed convex set is due to Cvitanić and Karatzas (1992).

If we specify our study to the case of exponential utility $U(x) = -\exp\{-\alpha x\}$, a fairly simple calculation shows that the convex conjugate $V(\eta)$ is given by

$$V(\eta) = \frac{\eta}{\alpha} \log \left( \frac{\eta}{\alpha} - 1 \right) \quad (3.33)$$
If we purchase (or sell) a claim on the non-traded asset $Y$, the primal problem becomes $\max_\pi EU[X_T + h(Y_T)]$ and the dual problem is to minimise

$$
\min_{Q \in \mathcal{M}} \left\{ V(\eta) + \frac{\eta}{\alpha} \mathbb{E} \left( \frac{dQ}{d\mathbb{P}} \log \frac{dQ}{d\mathbb{P}} \right) + \eta \mathbb{E} \left( \frac{dQ}{d\mathbb{P}} h(Y_T) \right) \right\}
$$

(3.34)

If we specify our case to exponential utility of the form $u(x) = e^{-\delta x}$ it is observed that the optimum is reached when the relative entropy $\mathcal{H}(Q, \mathbb{P}) = \mathbb{E}^Q \left( \log \frac{dQ}{d\mathbb{P}} \right)$ is minimised. Davis (1997, 1999) found explicit hedging results under the assumption that $Y$ is governed by a GBM, with a Brownian motion correlated to $W$ with coefficient $\rho$. The argument used is the marginal utility based price defined as the limit of the utility indifference price when $n \to 0$. In this setting, classical utility maximisation problem from Merton (1969)

$$
\sup_\pi [U(X_T + n\xi(t, T, X_T)) | \mathcal{F}_t] = -\frac{1}{\gamma} e^{-\gamma X_t} g(T-t, \log S_t)
$$

(3.35)

Using the fact that the primal value is a supermartingale, Henderson (2002) shows that gives rise to a non-linear PDE for $g$. The method used by Zariphopoulou (2001) and Musiela and Zariphopoulou (2004) consists of using the power distortion $g(\tau, y) = e^{b\tau} G^b(\tau, y + \beta \tau)$ and setting the constants $b, \alpha$ and $\beta$ so that $G$ solves a linear partial differential equation.

**Proposition 3.4.1 (Utility based hedge)** The optimal hedging strategy that minimises the relative entropy $\mathcal{H}(Q, \mathbb{P})$ is given by

$$
\pi^* = \mu g + \frac{\sigma d\eta}{\sigma^2} g^Y \rho
$$

(3.36)

The utility indifference argument can be used to find the claim price as

$$
p^e = n\mathbb{E}^M h(X_T) - \frac{\gamma}{2} n^2 (1 - p^2) \left[ \mathbb{E}^M \chi^2(X_T) - \mathbb{E}^2 \chi(X_T) \right] + O(n^3)
$$

(3.37)

A noteworthy connection of this methodology with one of the classical actuarial pricing principle\(^5\) was shown by Monoyios (2007), who relates the minimal entropy measure to the minimal martingale measure from Föllmer and Schweizer (1991) by an Esscher transform.

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\(^4\)See for example Rouge et al. (2000)
\(^5\)Esscher transforms have been applied by Gerber and Shiu (1994) for determining a candidate pricing measure. Simple pricing formulas were found using a power utility function of the form $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, for a positive parameter $\gamma \neq 0$. 57
\[
\frac{dQ^e}{dQ^M} = \frac{\exp \left( -\frac{1}{2} \left( 1 - \rho^2 \right) \right) \int_0^T \lambda^2 du}{\mathbb{E}^{Q^M} \exp \left( -\frac{1}{2} \left( 1 - \rho^2 \right) \right) \int_0^T \lambda^2 du}
\]  

(3.38)

Utility based hedging strategies are extremely sensitive to the value of the Sharpe ratio. Monoyios (2008) considers the cases where \( \lambda \) is constant and that is a martingale under the physical measure \( \mathbb{P} \). One advantage of setting \( \lambda_t \) independent of \( \pi_t \), is that the minimum entropy measure coincides with the minimal martingale measure. Due to its lack of intuitive risk measure and to the absence of a linear pricing rule, utility based hedging has not been widely adopted by practitioners so far. Another limitation of this technique for the specific market in question is that the distortion transformation doesn’t yield the desired linear equation in the presence of several sources of randomness.
Chapter 4

Hedge effectiveness results

The numerical analysis below displays the results of the different approaches to hedging in incomplete market described in chapter 3. Cumulative profit and losses arising from the hedge assuming a complete market is shown each time in comparison in order to isolate the hedging error due to discrete trading in a continuous model. We consider the hedging results over a period of 5 years of a policy with an account value of 100,000 $ maturing in 20 years including a principal protection (possibly paid as annuity using the appropriate change of measure) and a ratchet death benefit. The outcome of Monte-Carlo simulation of the respective strategies P&L is given below using 10,000 pseudo-random scenarios. Mortality improvement and interest rates are governed by a CIR process, and equity returns by a GBM. Other parameters are assumed constant. As may be expected, the quadratic hedge reduces the P&L volatility but limits gains and losses equally. The superhedge is stable but consistently expensive. Utility-based hedge allows more P&L volatility than other methods but favours trading gains efficiently, provided that average returns of equity investments are accurate. As may be expected from an exponential utility (See chapter 2), the left tail is long and thick.

<table>
<thead>
<tr>
<th></th>
<th>Naive</th>
<th>Quadratic</th>
<th>Superhedge</th>
<th>Utility based</th>
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<tbody>
<tr>
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<td>-11689.31</td>
<td>-3993.09</td>
<td>-165.77</td>
<td>-1816.94</td>
</tr>
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<td>1980.06</td>
<td>-130.85</td>
<td>10827.52</td>
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<td>5020.85</td>
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<td>1606.80</td>
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<td>3804.03</td>
</tr>
<tr>
<td>10%</td>
<td>-3864.65</td>
<td>2165.19</td>
<td>-156.70</td>
<td>2657.39</td>
</tr>
</tbody>
</table>

Table 4.1: P&L distribution statistics in $ for different optimality criteria
Figure 4.1: Naive hedge P&L distribution
Figure 4.2: Quadratic hedge P&L distribution
Figure 4.3: Superhedge P&L distribution
Figure 4.4: Utility based hedge P&L distribution
Chapter 5

Conclusions and future research directions

The problem of pricing and hedging in incomplete markets remains largely unresolved. The flattering results obtained using the dual approach can be largely explained by the fact that we took the expected equity return as given. This is however a critical parameter that is hardly observable. While superhedging proves too costly, shortfall minimising hedge on the other hand may deserve a fair shot. In this study, we considered the most important aspects of a general framework for pricing and hedging insurance derivatives accurately. The study is obviously far from being exhaustive and relevant refinements could be made to address the specific risks associated with a particular portfolio. The large diversity of products offered, with multiple elective benefits and gimmicks makes it hard to address all aspects with a single specific model. One important source of risk that hasn’t been covered explicitly is the significant amount of credit risk resulting from investments in fixed income securities of different grades. In our analysis, credit risk is modelled as part of the numerous unhedgeable risks. However, a stochastic model with specific dynamics for credit spreads could lead to a more precise representation of risk. Furthermore, the risk might be partly hedged by taking positions in credit derivatives. The price to pay for this gain of accuracy would be an additional difficult incomplete market model and probably also a loss of analytical tractability for the stochastic optimal control problems from our second chapter. The originality of this dissertation lies in the treatment of dynamic policy-holder lapsation. Introducing the concept of implied utility, we refined predictions of the game option exercise. We also found a useful application of implied utility for structuring products, determining optimal investments ratios that are consistent with periodical withdrawals patterns and desired level of principal protection. While the relevant literature put the focus mostly on financial options or mortality, predicting...
dynamic lapse is important to approximate the true financial value of the policy since it adds to the negative convexity of the guarantees. Moreover, it makes the claim more difficult to hedge. The implied utility model was kept as tractable as possible in order to allow for some ease of calibration. Important improvement may be achieved by refining the assumed utility function and the funds value process. Possibly, even a similar tractable model might perform better if parameters are set carefully for specific groups divided by categories of age, gender, geographic region and possibly other relevant grouping criteria. A limitation to this procedure is that it requires that all subgroup contains a sufficiently large number of policies for the law of large numbers to be applied. Concerning the lagged filtration presented in the first chapter in the context of modelling the insured person’s attitude towards exercising the game option, this concept could be extended and applied to valuation from the issuer’s perspective. Delays in access to in-force policies data cause insurance companies to trade without updated information on the latest status of the liabilities. The dynamic asset allocation could also be improved, particularly for contracts that do not allow for automatic rebalancing according to undetermined ratios. In that case a model with stochastic volatility and drift would seem more appropriate to determine the investment strategy deterministically at time of issue of the contract. A more important limitation of the closed from solution for dynamic asset allocation is that the optimisation relies of simplified assumptions on continuous trading, which do not allow for taking transaction costs or budget constraints into account.
Appendix A

Proofs and results from Chapter 1

A.1 Relation between mortality decrements

The mortality decrement for a period of one year is given in terms of annual mortality rate at time \( t \) \( M_{Ann}(t) \) by \( \{1 - M_{Ann}(t)\} \). Let us consider a partition of the one year time space \( T_0, T_1, ..., T_n = 1 \text{ year} \), and define the mortality rate \( M_{\delta_i} \) expressed in some year fraction \( \delta_T(i) = (T_{i+1} - T_i) \). Compounding of the decrement over one year must yield the same result independently of the compounding period chosen. Hence the following relation must hold

\[
\prod_{i=0}^{n-1} \{1 - M_{\delta_T}(i)\} = 1 - M_{Ann}(t) \quad (A.1)
\]

Assuming that \( \delta_T(i + 1) = \delta_T(i), \forall i \in [0, 1, ..., n] \), and that mortality is uniformly distributed throughout the year, the relation between \( \delta \)-periodical mortality and annual mortality follows

\[
M_{\delta_T} = \{1 - [1 - M_{Ann}(t)]\}^n \quad (A.2)
\]

In the limit as the partition becomes infinitesimally small, we obtain the instantaneous hazard rate defined above

\[
\mu(x, t) = \lim_{\delta T \to 0} \delta T \sum_{i=1}^{N} \{1 - (1 - M_{\delta_T}(i))\}^{\delta T} \quad (A.3)
\]
A.2 Proof of Proposition 1.36

The proof of Proposition 1.36 follows from a standard application of the optional stopping time theorem, that may be found in Klebaner (1999). Define the scale function \( S(\zeta) \) such that it is a martingale. By Ito’s lemma we have

\[
S(\mu) = \frac{dS(\mu)}{d\mu}d\mu + \frac{1}{2} \frac{d^2S(\mu)}{d\mu^2}(\mu)d\mu^2
\]  

(A.4)

Since \( S \) is to be a martingale, the systematic drift must vanish, hence the scale function is given by

\[
S(\mu) = \int_c^x \left( \frac{c}{\mu} \right)^{2\theta} \exp \left\{ \frac{2\kappa(\mu - c)}{\sigma^2} d\mu \right\}
\]  

(A.5)

An application of the optional stopping time theorem yields the probability \( H \) of hitting \( \infty \) before zero as

\[
H(\mu) = \frac{S(\mu) - S(0)}{S(0) - S(\infty)}
\]  

(A.6)

H equals infinity if the exponent of \( c/\mu \) in (A.5) is positive. As required, the condition (1.36) prevents \( \mu \) to reach zero.
Bibliography


Schachermayer, W., 1999. Optimal investment in lifeinsurance when wealth may become negative. preprint, Vienna University of Technology.


