Pricing Barrier Options in Foreign Exchange Market

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I dedicate my thesis to my lovely wife, who support me in completing this Master course. Without her support, this study would be impossible.
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Abstract

This thesis studies the applicability of SABR model to FX markets, and change the model to adapt to the specifics in FX. It further develop a pricing method which could be used to valuate first generation exotic FX options. The results are compared with standard Black-Scholes and other common replication methods.
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Chapter 1

Introduction

1.1 The Foreign Exchange Market

The foreign exchange (FX) market is a worldwide decentralized over-the-counter financial market for the trading of currencies. It is estimated 4 trillion change hand in this market every day. The FX option market is also the largest and one of the most liquid option market in the world. Currently, the various traded products range from simple vanilla options to various generation of exotics products. First generation exotic refers to barrier options and one-touch options. Second-generation exotics involves options with a fixing-date structure, whereas third-generation exotics are hybrid products where FX mixed with other asset classes. One of the most noticeable example is Power Reverse Dual Currency (PRDC). Of all the above, the simple barrier option account for large share of the traded volume. This makes it imperative for any pricing system to provide a fast and accurate mark-to-market for this family of products. Although using the Black-Scholes model [4], it is possible to derive analytical prices for barrier options, this model is unfortunately based on constant volatility throughout the life of options. This is clearly wrong as these quantities change continuously, reflecting the traders’ view on the future of the market. Today the Black-Scholes theoretical value is used only as a reference quotation, to ensure that the involved counterparties are speaking of the same option.

The aim of this thesis is – (a) to study the applicability of SABR model to FX option market, (b) extend the model in a way to fit the particularities in FX market, (c) develop a speedy pricing method that could be used in pricing high-volume vanilla and first-generation exotic options.
1.1.1 Peculiarity of FX Option Market

One of the most notable peculiarity of FX derivative market is option volatility are quoted with respect to delta rather than strike like the case in any other asset class. Practically this means that, if the FX spot rate moves, with all other things being equal, the curve of implied volatility vs. delta will remain unchanged, while the curve of implied volatility vs. strike will shift. Some argue this brings more efficiency in the FX derivatives markets. Readers may want to refer to [7] for a discussion on the appropriateness of the delta-sticky hypothesis. On the other hand, it makes it necessary to precisely agree upon the meaning of delta. In general, delta represents the derivative of the price of an option with respect to the spot. In FX markets, the delta used to quote volatilities depends on the maturity and the currency pair at hand. An FX spot for a currency pair FOR/DOM quoted as $X_t$ implies that 1 unit of FOR equals $X_t$ units of DOM. Currency pairs, mainly those with USD as domestic currency DOM, such as EURUSD or GBPUSD, use the Black-Scholes spot delta, the derivative of the price with respect to the spot:

\[
\Delta_{\text{Call}} = D_f(T)N(d_+) \\
\Delta_{\text{Put}} = -D_f(T)N(-d_+). 
\] (1.1)

However, for those currency pair with USD quoted as foreign currency FOR, the following premium included delta convention is used:

\[
\Delta_{\text{Call}} = \frac{K}{X} D(T)N(d_-) \\
\Delta_{\text{Put}} = -\frac{K}{X} D(T)N(-d_-). 
\] (1.2)

The quantities in equation 1.1 and 1.2 are expressed in FOR, which is by convention the risk asset from the domestic point of view. Taking the example of USDJPY, setting up the corresponding Delta hedge will make one’s position insensitive to small FX spot movements if one is measuring risks in a USD (foreign) risk-neutral world.

Regarding to the term structure, the common currency pairs, G11 currencies, use a spot delta convention defined in 1.1 and 1.2 for short maturities ending up to 1 year, whereas any maturities beyond 1 year where interest rates are less likely to be constants as assumed in Black-Scholes, the forward delta is used instead; and their equations are

\[
\Delta_{\text{Call}}^{(F)} = N(d_+) \\
\Delta_{\text{Put}}^{(F)} = -N(-d_+). 
\] (1.3)
for USD quoted as domestic currency; and

\[
\Delta_{\text{Call}}^{(F)} = \frac{K}{X D_f(T)} \frac{D(T)}{D_f(T)} N(d_-)
\]

\[
\Delta_{\text{Put}}^{(F)} = -\frac{K}{X D_f(T)} \frac{D(T)}{D_f(T)} N(-d_-),
\]

for USD quoted as foreign currency. These conventions are very particular to the FX market and needs to be fully understood because volatility term-structures are quoted as a function of delta \(\sigma(\Delta)\). We need to apply the appropriate formula to extract the strike information out of the delta before proceeding with any calculation.

Apart from the delta convention, volatility in FX option markets display profound smile shape across various delta rather than flat straight line. In consideration of this, FX market uses various combination of options at different deltas, namely strangle, risk-reverse, and butterfly, to describe the smile surface. Fortunately, the volatility data\(^1\) I gathered is already expressed in the terms of deltas, which are 10% and 25% puts, and 10%, 25%, 50% calls; thus I would not go into the details of these market conventions.

### 1.2 Smile Risk in Fixed-income Market

The SABR model is one of the few stochastic volatility models widely accepted in the financial institutions for modelling volatility surface nowadays. Historically, it was emerged from fixed-income market, and was designed primarily to model vanilla fixed-income instruments, such as caps, floors, and swaptions. The virtue of being the simplest stochastic volatility model among the others and the existence of approximation analytical formula relating between model parameters and market observable volatilities makes the SABR model gain popularity rapidly. In order to appreciate the design of the SABR model, we first briefly describe other models that had been used before the time of SABR model.

#### 1.2.1 Black model and Implied Volatility

Under the Black model \(^5\), the forward price \(F\) is geometric Brownian motion

\[
dF = \sigma_B F dW.
\]

\(^1\)All market data quoted in this paper and used in my study was obtained from an internal financial data repository in courtesy of JP Morgan.
The result of the time-zero value of call and put from this model is well-known.

\[ C(0) = D(T) \{ F_0 N(d_+) - KN(d_-) \} \]  
\[ P(0) = C + D(T)(K - F_0) \],

where

\[ d_\pm = \frac{\ln \frac{F_0}{K} \pm \frac{1}{2}\sigma^2_B T}{\sigma_B \sqrt{T}} \].

(1.6)

(1.7)

(1.8)

All parameters in Black’s formula could be easily observed, except for the volatility \( \sigma_B \). Since the Black’s option prices are increasing functions of \( \sigma_B \), the volatility \( \sigma_B \) implied by the market price of an option is unique. In markets where volatility is a smile or frown, one-to-one correspondence is obviously not true. Indeed, the implied volatility needed to match market prices nearly always varies with both the strike \( K \) and the time-to-exercise \( T \). Changing the volatility \( \sigma_B \) means a different model is used for each combination of \( K \) and \( T \). Apparently, this would cause problems when managing a large books of options, where trading desks normally group options of a certain type of asset but with various strikes and maturities together for effective hedging.

Consider an example, where we have options of a given asset struck at both 90 and 100 maturing in 1 month. \( \sigma_B \) for the 90 and 100 are 10% and 22% respectively. To calculate the vega risk by bumping \( \sigma_B \), it would be hard for someone to decide whether to bump both of them by a certain percentage, say 10%, or by an absolute value, say adding 1% to both of them. To hedge the net exposure of the book only by consolidating delta and vega risks of all options on a given asset is a common practice because it will bring down the transaction costs significantly. However, it is not certain how this could be achieved with Black’s model as it is not the same model across strikes and maturities.

1.2.2 Local Volatility Model

To tackle these problems with Black model, Dupire [6] proposed local volatility model, whereby the dynamic of a forward price follows

\[ dF = \sigma_L(t, F)FdW \],

(1.9)

in a forward measure. Dupire argued that instead of theorising about the unknown local volatility function \( \sigma_L(t, F) \), one should obtain it directly from calibrating to the market prices of liquid options. In practice, with option markets offering prices only at a discrete set of exercise dates \( T_i \), \( \sigma_L(t, F) \) is assumed to be piecewise constant.
between $T_{i-1}$ and $T_i$. Unfortunately, the local volatility model predicts the wrong dynamics of the implied volatility curve, which leads to inaccurate and often unstable hedges. To illustrate the problem, let’s suppose that today’s implied volatility is a perfect smile

$$\sigma_B(K) = \alpha + \beta [K - F_0]^2,$$  \hspace{1cm} (1.10)

around today’s forward price $F_0$. The relationship between Black volatility and local volatility function is obtained through singular perturbation method [1]

$$\sigma_B(K, F) = \sigma_L \left( \frac{1}{2}[F + K] \right) \left\{ 1 + \frac{1}{24} \frac{\sigma''_L \left( \frac{1}{2}[F + K] \right)}{\sigma_L \left( \frac{1}{2}[F + K] \right)} (F - K)^2 + \cdots \right\}$$  \hspace{1cm} (1.11)

Ignoring the second term and onwards in the expansion series together with assumption 1.10, the local volatility is

$$\sigma_L(F) = \alpha + 3\beta (F - F_0)^2 + \cdots.$$  

As the forward price $F$ evolves away from $F_0$ because of normal market fluctuations, equation 1.11 predicts that the implied volatility is

$$\sigma_B(K, F) = \alpha + \beta \left[ K - \left( \frac{3}{2}F_0 - \frac{1}{2}F \right) \right]^2 + \frac{3}{4} \beta (F - F_0)^2 + \cdots.$$  \hspace{1cm} (1.12)

The implied volatility curve not only moves in the opposite direction as the underlying, but also shifts upward regardless of whether $F$ increases or decreases because of the square terms in 1.12.

The incorrect prediction of curve movement has another implication. Hedges calculated from the local volatility model are also wrong. To demonstrate this, suppose $BS(F, K, \sigma_B, T)$ be Black’s formula for call option, say. Under the local volatility model, the value of a call option is given Black’s formula

$$C_{BS} = BS(F, K, \sigma_B(K, F), T)$$  \hspace{1cm} (1.13)

with volatility $\sigma_B(K, F)$ given by 1.11. Differentiating with respect to $F$ gives delta

$$\Delta \equiv \frac{\partial C_{BS}}{\partial F} = \frac{\partial BS}{\partial F} + \frac{\partial BS}{\partial \sigma_B} \frac{\partial \sigma_B}{\partial F}.$$  \hspace{1cm} (1.14)

The first term is the same $\Delta$ risk as what Black’s model will give. The second term is the local volatility model’s correction to the $\Delta$ risk, which consists of the Black vega risk multiplied by the change in $\sigma_B$ due to changes in the underlying forward price $F$. As illustrated above, this correction term is of opposite sign to the true value.
1.2.3 The SABR model

The failure of the local volatility model means that single-factor Markovian model is not capable of managing smile risk. Instead of making the model non-Markovian, or basing it on non-Brownian motion, such as Levy process, Patrick et al. [1] chose to develop a two-factor model. Observing that most markets experience both relatively quiescent and relatively chaotic periods, one comes to the conclusion that volatility is not constant nor deterministic. The SABR model attempts to capture the dynamics of a single forward rate $F$. Depending on the context, this forward rate could be a LIBOR forward, a forward swap rate, the forward yield on a bond, etc. The SABR model, which can be thought of as an extension of the CEV model, has the dynamics

$$dF = \alpha F^{\beta} dW_1$$

$$d\alpha = \nu \alpha dW_2,$$

where $0 \leq \beta \leq 1$, and $\nu$ is a constant volatility of volatility. In general, the two Wiener processes $W_1$ and $W_2$ are correlated,

$$dW_1 dW_2 = \rho dt,$$

with a constant correlation coefficient $\rho$. The $\alpha, \beta, \rho, \nu$ are collectively called SABR parameters.

Evidence has been given in [1] that the SABR model can be accurately fit to most of the implied volatility curves observed in the marketplace for any single exercise date $T$. More importantly, it also predicts the correct dynamics of the implied volatility curves. This makes the SABR model an effective mean to manage the smile risk in markets where each asset only has a single exercise date; these markets include the swaption and caplet or floorlet markets.

As mentioned above, the phenomenal success of SABR model is largely because an analytical approximation formula exists for European options, which makes calibration and repricing large portfolio of vanilla options extremely fast. Hagan et al [1] used singular perturbation techniques to derive the formula, which is too complicated and lengthy to repeat, and so I just quote the result here. For price of European options given by Black’s formula 1.6 – 1.8, and the implied volatility $\sigma_B(L, K)$ is given by

$$\sigma_B(F, K) = \frac{\alpha}{(FK)^{(1-\beta)/2}} \left\{ 1 + \frac{(1-\beta)^2}{24} \ln^2 \left( \frac{F}{K} \right) + \frac{(1-\beta)^4}{1920} \ln^4 \left( \frac{F}{K} \right) + \cdots \right\} \cdot \left( \frac{z}{x(z)} \right) \cdot \left\{ 1 + \frac{(1-\beta)^2}{24} \frac{\alpha^2}{(FK)^{1-\beta}} + \frac{1}{4} \frac{\rho \beta \nu \alpha}{(FK)^{(1-\beta)/2}} + \frac{2 - 3 \rho^2}{24} \nu^2 \right\} T + \cdots \right\},$$

(1.18)
where
\[ z = \frac{\nu}{\alpha} (FK)^{(1-\beta)/2} \ln \left( \frac{F}{K} \right), \quad (1.19) \]

and
\[ x(z) = \ln \left\{ \frac{\sqrt{1 - 2\rho z + z^2 + z - \rho}}{1 - \rho} \right\}. \quad (1.20) \]

This is the core result of Hagan et al’s paper [1], and is the formula I used to implement the SABR model for fitting market FX smiles.

### 1.3 Applying SABR to FX Market

With the SABR model defined based on generic forward rate \( F \), one can easily jump into conclusion that the SABR model is also applicable to FX. It is not, however. Consider the same setup as defined in 1.15 to 1.17. Since \( F \) is representing a FX forward, the reciprocal of \( F \) is still a valid FX rate. Let
\[ f(x) = \frac{1}{x} \quad Y = f(F) \quad (1.21) \]

The dynamic of \( Y \) becomes
\[ dY = df(F) = f'(F)dF + f''(F)d[W_1, W_1] = \frac{-1}{F^2} \alpha F^{\beta} dW_1 + \frac{1}{F^3} \alpha^2 F^{2\beta} dt \]
\[ = \frac{-\alpha}{F^2} F^{\beta-2} dW_1 + \alpha^2 F^{2\beta-3} dt. \quad (1.22) \]

Although \( \beta \) can theoretically be ranged from 0 to 1 inclusively in the original SABR model, it can be hardly justified that \( \beta \) is any value except 1 based on equation 1.22. With \( \beta = 1 \), equation 1.22 reduces to
\[ dY = -\alpha F^{-1} dW_1 + \alpha^2 F^{-1} dt \]
\[ = Y \{ \alpha^2 dt - \alpha dW_1 \}, \]

which is still a geometric Brownian motion with a drift term. In case \( \beta \neq 1 \), the dynamic of \( Y \) will not follow geometric Brownian motion. Since the risky currency and the domestic currency in \( F \) and \( Y \) are both arbitrary, there is no sound foundation to justify one currency following one type of stochastic process while the other following another totally different. Judging from this perspective, not even the normal model, where \( \beta = 0 \), is theoretically sound.
In other word, we need a modified version of the SABR model on FX forward rate

\[ dF = \alpha F dW_1 \quad (1.23) \]
\[ d\alpha = \nu dW_2, \quad (1.24) \]
\[ dW_1 dW_2 = \rho dt. \quad (1.25) \]

1.4 Implementation

Implementing fitting to SABR model seems to be trivial at first glance; in a nutshell, one only need to code up the equations (1.18) - (1.20) and feed them into one of the quadratic programming functions in Matlab to search for the SABR parameters that minimise the sum of square of the errors between the model-given values and market values. Nevertheless, there are a couple of details needed to handle with care. First of all, notice that since there are actually some theoretical boundary conditions for each SABR parameters,

\[ 0 \leq \alpha \leq \infty \]
\[ 0 \leq \beta \leq 1 \]
\[ -1 \leq \rho \leq 1 \]
\[ 0 \leq \nu \leq \infty, \]

unconstrained search algorithm such as fminsearch is not suitable. fmincon is constrained quadratic algorithm that is more appropriate to this situation. In practice, volatility or volvol of 1000% is considered to be extremely high. I make these as the upper boundaries for \( \alpha \) and \( \nu \) in my implementation, instead of \( \infty \).

Secondly, if one just implement straightforwardly (1.18) - (1.20), fmincon in most cases would return NaN or Inf. To see the cause of the problem, consider the case where the searching algorithm makes \( z \) very close to zero, then \( x(z) \) in (1.20) will tend to zero as well. The term \( \frac{x}{x(z)} \) in (1.18) will become \( \frac{0}{0} \), an indeterminant.

To resolve this problem, one can either make use of L'Hospital’s rule to seek for an equivalent form, or look for power series approximation of \( \frac{x}{x(z)} \). I choose the latter approach. To get a power series approximation, first of all, notice that, by rearranging (1.20), we get

\[ (1 - \rho)e^x = \sqrt{1 - 2\rho z + z^2} + (z - \rho) \quad (1.26) \]
On the other hand, we can rewrite (1.20)
\[
x = -\ln \left\{ \frac{1 - \rho}{\sqrt{1 - 2\rho z + z^2} + (z - \rho)} \right\}
\]
\[
= -\ln \left\{ \frac{(1 - \rho)[\sqrt{1 - 2\rho z + z^2} - (z - \rho)]}{\sqrt{1 - 2\rho z + z^2} + (z - \rho)[\sqrt{1 - 2\rho z + z^2} - (z - \rho)]} \right\}
\]
\[
= -\ln \left\{ \frac{(1 - \rho)[\sqrt{1 - 2\rho z + z^2} - (z - \rho)]}{1 - 2\rho z + z^2 - (z^2 - 2\rho z + \rho^2)} \right\}
\]
\[
= -\ln \left\{ \frac{(1 - \rho)[\sqrt{1 - 2\rho z + z^2} - (z - \rho)]}{1 - \rho^2} \right\}
\]
\[
= -\ln \left\{ \frac{\sqrt{1 - 2\rho z + z^2} - (z - \rho)}{1 + \rho} \right\}
\]
\[
(1 + \rho)e^{-x} = \sqrt{1 - 2\rho z + z^2} - (z - \rho)
\] (1.27)

Subtracting (1.27) from (1.26), we have
\[
2(z - \rho) = (1 - \rho)e^x - (1 + \rho)e^{-x}
\]
\[
z = \frac{e^x - e^{-x}}{2} - \rho \frac{e^x + e^{-x}}{2} + \rho
\]
\[
= \sinh(x) - \rho \cosh(x) + \rho
\]
\[
= x + \frac{x^3}{3!} + \cdots - \rho \frac{x^2}{2!} - \cdots + \rho
\]
\[
= x - \rho \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots ,
\] (1.28)

which is a power series of \( z \) in terms of \( x \). Because \( x \) is actually a function of \( z \), we eventually want to express \( x \) in terms of \( z \). The power series (1.28) can be easily inverted by letting
\[
x = a_1 z + a_2 z^2 + a_3 z^3 + \cdots ,
\]
where \( a_i \) are real coefficients for \( i = 1, 2, 3 \). Substituting into (1.28), we have
\[
z = a_1 z + a_2 z^2 + a_3 z^3 - \frac{\rho}{2!} (a_1 z + a_2 z^2 + a_3 z^3)^2
\]
\[
\quad + \frac{1}{6} (a_1 z + a_2 z^2 + a_3 z^3)^3 + \cdots .
\]

By expanding each term on the right and grouping terms in power of \( z \), we have
\[
a_1 = 1, \quad a_2 = \frac{\rho}{2}, \quad a_3 = \frac{\rho^2}{2} - \frac{1}{6} .
\]
Hence,

\[ x = z + \frac{\rho}{2} z^2 + \frac{1}{6} (3\rho^2 - 1) z^3 + \cdots \]

\[ \frac{z}{x(z)} \approx \left( 1 + \frac{\rho}{2} z + \frac{1}{6} (3\rho^2 - 1) z^2 \right)^{-1}, \]

from which we can see the fraction approaches 1 when \( z \) approaches zero.

Initial guess is also important to stable fitting. Picking wrong starting values could result in failures from the searching algorithm. \( \beta \) is normally fixed during the process of calibration, either by some aesthetic or other a priori considerations, in our case is 1. The initial \( \alpha \) and \( \nu \) are both chosen to be the ATM volatility in the smile. \( \rho \) is heuristically chosen to be zero, just in the middle between -1 and 1. See the \texttt{sabr*.m} files for the actual implementation.

### 1.5 Calibration Results

Figure 1.1 to 1.3\(^2\) shows smile curves of some major currency pairs for all maturities liquidly traded in the market. It has been often said in finance literature that FX option market normally exhibit smile curve whereas equity option market tends to have skew instead. It might have some degree of truth, but certainly not always. While EURUSD and CHFUSD shows obvious smile shape, GBPUSD has a more skew curve. This could be due to the market’s bearish view on GBPUSD exchange rate in recent years since the financial crisis fallout, and thus the curves are very skew towards the downside.

The solid lines in the figures represents the volatility observed in the market, whereas the triangular markers are the actual market data points, indicating the quality of the fitted results of SABR calibrations. All of these smiles are calibrated to SABR model with \( \beta = 1 \). Both the exponent \( \beta \) and correlation \( \rho \) in fact cause a downward sloping skew in \( \sigma_B(K,F) \) as the strike \( K \) varies. Thanks to this fact, fixing \( \beta = 1 \) does not deteriorate the quality of fitting at all.

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\(^2\text{Readers are referred to the spreadsheet }\texttt{fxvolanalysis.xlsm} \text{ for the actual data. Please also read the appendix B for more detailed description on supplementary files.}\)
Figure 1.1: Calibration result of EURUSD smile on 31 August 2009

Figure 1.2: Calibration result of CHFUSD smile on 30 July 2010
Figure 1.3: Calibration result of GBPUSD smile on 25 August 2010

Figure 1.4: Calibration result of JPYUSD smile on 25 August 2010
Chapter 2

Extension of SABR model and Pricing Equation

In the previous chapter, we have shown that the SABR is a viable stochastic volatility model describing FX smiles. Based upon this conclusion, we proceed in deriving a pricing method that can be used in evaluating vanilla options and barrier options. To this end, a partial differential equation will be obtained directly from the model setup. Since the equation does not have any close-form solution, we need to resort to numerical methods – finite difference method to find solution for options. Because the equation involves 3 state variables, namely a spatial variable, volatility, and time, the 3-dimensional PDE is first converted to a simpler form using splitting scheme.

2.1 The SABR Model for FX market

The SABR model is modified to model spots, instead of forwards in the original model. The reason for this is some exotic options, such as barrier options which we will mainly consider in this paper, are concerned with the spots rather than the forwards.

\[
\begin{align*}
    dX_t &= X [(r - r_f) dt + \alpha_t dW_1] \\
    d\alpha_t &= \nu \alpha_t dW_2 \\
    dW_1 dW_2 &= \rho dt,
\end{align*}
\]

where \( W_1 \) and \( W_2 \) are Brownian motions with correlation \( \rho \) on a risk-neutral measure. Let the value of the target option, either vanilla or exotic, be \( V(t, x, y) \). Using iterated conditioning, the discounted option value is a martingale, and thus, by its differential
is

\[ d \left( e^{-rt} V(t, X, \alpha) \right) = \]

\[ e^{-rt} \left\{ -rV dt + \dot{V} dt + V_x dX + V_y d\alpha + \frac{1}{2} V_{xx} dX dX + V_{xy} dX d\alpha + \frac{1}{2} V_{yy} d\alpha d\alpha \right\} \]

Substituting equations 2.2 to 2.3, the right hand side becomes

\[ e^{-rt} \left\{ -rV dt + \dot{V} dt + V_x X [(r - r_f) dt + \alpha dW] + V_y \nu \alpha dW_2 + \nu \alpha V_y dW_2 + \frac{1}{2} V_{xx} \alpha^2 X^2 dt + \nu^2 \alpha \rho V_{xy} dt + \frac{1}{2} \nu^2 \alpha^2 V_{yy} dt \right\} \]

By equating the \( dt \) term to zero, and reversing to time scale for converting terminal boundary condition to initial boundary condition, we have a forward equation

\[-\dot{V} + \frac{1}{2} \alpha^2 \nu V_{xx} + \rho \nu \alpha^2 X V_{xy} + \frac{1}{2} \alpha^2 \nu^2 V_{yy} + (r - r_f) XV_x = rV \quad (2.4)\]

### 2.2 Splitting scheme for multi-factor PDE

Equation 2.4 involves two state variables and one time variable. Close-form solution to this equation is not achievable. Instead, we look for numerical solution through multi-dimensional finite difference scheme. Alternate Direction Implicit (ADI) and splitting scheme are two main methods to solve multi-dimensional PDE. Both ADI and splitting scheme approximate the solution of an initial boundary value problem by partitioning an n-dimensional PDE to a set of one-dimension PDE. However, ADI is not good at approximating mixed derivatives, which we do have in (2.4); we need to pursue splitting scheme for this particular model. The idea of splitting scheme can be described by the following generic second-order PDE. We define operators:

\[ L_S U \equiv A \frac{\partial^2 U}{\partial S^2} + B \frac{\partial U}{\partial S} + C U \quad (2.5) \]

\[ L_v U \equiv D \frac{\partial^2 U}{\partial v^2} + E \frac{\partial U}{\partial v} \quad (2.6) \]

\[ F \equiv \rho \sigma \nu S. \quad (2.7) \]

Then, the splitting scheme divides the following PDE

\[ \frac{\partial U}{\partial t} + L_S U + L_v U + F \frac{\partial^2 U}{\partial S \partial v} = 0 \quad (2.8) \]
into the two equations:

\[-\frac{\partial U}{\partial t} + L_S U + F \frac{\partial^2 U}{\partial S \partial \nu} = 0\]
\[-\frac{\partial U}{\partial t} + L_\nu U = 0.\]

Let’s define discrete difference operators on some dummy variables \(x, y, t\).

\[
\Delta_t^n U_{ij}^n = \frac{U_{ij}^{n+1} - U_{ij}^n}{h_t}
\]
\[
\Delta_x^n U_{ij}^n = \frac{U_{i+1,j}^n - U_{ij}^n}{h_x}
\]
\[
\Delta_y^n U_{ij}^n = \frac{U_{ij}^n - U_{i-1,j}^n}{h_y}
\]
\[
\Delta_{x}^2 U_{ij}^n = \frac{U_{i+1,j}^n - 2U_{ij}^n + U_{i-1,j}^n}{h_x^2}
\]
\[
\Delta_{xy} U_{ij}^n = \frac{U_{i+1,j+1}^n - U_{i+1,j-1}^n - U_{i-1,j+1}^n + U_{i-1,j-1}^n}{4h_x h_y},
\]

and with similar definitions on the other dimensions, such as \(\Delta_y u_{ij}^n\). The operators defined in 2.5 and 2.6 are replaced by the discrete equivalents. We then have

\[
\tilde{L}_S U_{ij}^n = A_{ij} \Delta_x^2 U_{ij}^n + B_{ij} \Delta_S U_{ij}^n + C_{ij} U_{ij}^n
\]
\[
\tilde{L}_\nu U_{ij}^n = D_{ij} \Delta_x^2 U_{ij}^n + E_{ij} \Delta_S U_{ij}^n.
\]

The splitting scheme is formulated as the following. The first leg calculates a solution at a fictitious level \(n + \frac{1}{2}\) given the solution at level \(n\):

\[
-\frac{U_{ij}^{n+\frac{1}{2}} - U_{ij}^n}{h_X} + \tilde{L}_S U_{ij}^{n+\frac{1}{2}} + \frac{1}{2} F_{ij}^n \Delta_S U_{ij}^n = 0,
\]

with \(n \geq 0, 1 \leq i \leq I, 1 \leq j \leq J\). And, the second leg takes the solution from level \(n + \frac{1}{2}\) to level \(n + 1\):

\[
-\frac{U_{ij}^{n+1} - U_{ij}^{n+\frac{1}{2}}}{h_\nu} + \tilde{L}_\nu U_{ij}^{n+1} + \frac{1}{2} F_{ij}^n \Delta_S U_{ij}^{n+\frac{1}{2}} = 0.
\]

The splitting scheme is mainly implicit. Indeed, both the time step and the spatial derivatives, including first-order and second-order terms, are implicit. The cross derivatives term is explicit and proceeds by half in each of the equation 2.9 and 2.10.
2.3 Splitting the FX SABR equation

With the formulation of the splitting scheme, we now in a position to devise the discretisation form of PDE (2.4) for the FX SABR model. Adopting our usual notations and substituting the actual parameters from (2.4), we have

\[
- \frac{V_{i,j}^{n+1/2} - V_{i,j}^n}{h_t} + \frac{1}{2} \alpha_i^2 X_j^2 \left( \frac{V_{i,j+1}^{n+1/2} - 2V_{i,j}^{n+1/2} + V_{i,j-1}^{n+1/2}}{h_X^2} \right) - \frac{1}{2} \rho \nu \alpha_i^2 X_j \left( \frac{V_{i,j+1}^n - V_{i,j}^n - V_{i,j+1}^n - V_{i,j-1}^n}{4h_X h_\alpha} \right) + (r - r_f) X_j \left( \frac{V_{i,j+1}^{n+1/2} - V_{i,j-1}^{n+1/2}}{2h_X} \right) - rV_{i,j}^{n+1/2} = 0 \quad (2.11)
\]

and,

\[
- \frac{V_{i,j}^{n+1} - V_{i,j}^{n+1/2}}{h_t} + \frac{1}{2} \nu^2 \alpha_i^2 \left( \frac{V_{i,j+1}^{n+1} - 2V_{i,j}^{n+1} + V_{i,j-1}^{n+1}}{h_\alpha^2} \right) - \frac{1}{2} \rho \nu \alpha_i^2 X_j \left( \frac{V_{i,j+1}^n - V_{i,j+1}^n - V_{i,j+1}^n - V_{i,j-1}^n}{4h_X h_\alpha} \right) = 0. \quad (2.12)
\]

Rearranging the terms, these two equations become

\[
\left( \frac{\alpha_i^2 S_j^2}{2h_X^2} + \frac{(r - r_f)X_j}{2h_X} \right) V_{i,j+1}^{n+1/2} - \left( \frac{1}{h_t} + \frac{\alpha_i^2 S_j^2}{h_X^2} + r \right) V_{i,j}^{n+1/2} + \left( \frac{\alpha_i^2 S_j^2}{2h_\alpha^2} - \frac{(r - r_f)X_j}{2h_X} \right) V_{i,j-1}^{n+1/2} = -\frac{\rho \nu \alpha_i^2 S_j}{8h_X h_\alpha} (V_{i+1,j+1}^n - V_{i+1,j-1}^n - V_{i-1,j+1}^n + V_{i-1,j-1}^n) - \frac{1}{h_t} V_{i,j}^n \quad (2.13)
\]

\[
\frac{\nu^2 \alpha_i^2}{2h_\alpha^2} V_{i+1,j}^{n+1} - \left( \frac{1}{h_t} + \frac{\nu^2 \alpha_i^2}{h_\alpha^2} \right) V_{i,j+1}^{n+1} + \frac{\nu^2 \alpha_i^2}{2h_\alpha^2} V_{i-1,j}^{n+1} = -\frac{\rho \nu \alpha_i^2 S_j}{8h_X h_\alpha} (V_{i+1,j+1}^{n+1/2} - V_{i+1,j-1}^{n+1/2} - V_{i-1,j+1}^{n+1/2} + V_{i-1,j-1}^{n+1/2}) - \frac{1}{h_t} V_{i,j}^{n+1/2} \quad (2.14)
\]

for \( i = 1, \ldots, I \), and \( j = 1, \ldots, J \).

Putting these equations into matrix form, we have a tridiagonal matrix which could be easily solved by \( LU \) decomposition. Denoting the matrix with the following
notation,

\[
A \equiv \begin{bmatrix}
-\frac{1}{h_t} - \frac{\alpha^2 S^1}{h_X} - r & \frac{\alpha^2 S^1}{2h_X} + (r-r_f)S_1 & 0 & 0 \\
\frac{\alpha^2 S^2}{2h_X} & -\frac{1}{h_t} - \frac{\alpha^2 S^2}{h_X} - r & \frac{\alpha^2 S^2}{2h_X} + (r-r_f)S_2 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \vdots & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
B \equiv \begin{bmatrix}
\left(\frac{\alpha^2 S^1}{2h_X} - \frac{(r-r_f)S_i}{2h_X}\right)V_{i0} \\
0 \\
0 \\
\vdots \\
0 \\
\left(\frac{\alpha^2 S^2}{2h_X} + \frac{(r-r_f)S_i}{2h_X}\right)V_{i,J+1}
\end{bmatrix}
\]

\[
C \equiv \begin{bmatrix}
S_1 (V_{i+1,2} - V_{i+1,0} - V_{i-1,2} + V_{i-1,0}) \\
\vdots \\
\vdots \\
S_J (V_{i+1,J+1} - V_{i+1,J-1} - V_{i-1,J+1} + V_{i-1,J-1})
\end{bmatrix}
\]

\[
V \equiv \begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
\vdots \\
V_{i,J-1} \\
V_{i,J}
\end{bmatrix}
\]

we end up with a recursion equation of the first leg

\[
A V^{(n+\frac{1}{2})} + B^{(n+\frac{1}{2})} = -\frac{\rho \nu \alpha_t^2}{8h_X h_\alpha} C^{(n)} - \frac{1}{h_t} V^{(n)},
\]

(2.15)

for \(i = 1, \cdots, I\). Similarly, denoting

\[
\hat{A} \equiv \begin{bmatrix}
-\frac{1}{h_t} - \frac{\nu^2 \alpha_t^2}{h_\alpha} - \frac{\nu^2 \alpha_t^2}{2h_\alpha} & \frac{\nu^2 \alpha_t^2}{2h_\alpha} & 0 & 0 \\
\frac{\nu^2 \alpha_t^2}{2h_\alpha} & -\frac{1}{h_t} - \frac{\nu^2 \alpha_t^2}{h_\alpha} & \frac{\nu^2 \alpha_t^2}{2h_\alpha} & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
we have another recursion equation for the second leg of the splitting scheme

\[
\hat{A}\hat{V}^{(n+1)} + \hat{B}^{(n+1)} = -\frac{\rho \nu^2 \alpha^2}{2h_x h_\alpha} \hat{C}^{(n+\frac{1}{2})} - \frac{1}{h_t} \hat{V}^{(n+\frac{1}{2})}.
\] (2.16)

### 2.4 Boundary Conditions

In this paper, we are mainly concerned with standard options and barrier options, and thus only the boundary conditions relevant to these options are to be considered in this section.

The boundary conditions for calls and puts can be defined in a number of ways. Readers can refer to [3]. The boundary conditions of call options used in our implementations are:

\[
V(0, X, \alpha) = (X - K)^+
\] (2.17)

\[
V(t, 0, \alpha) = 0
\] (2.18)

\[
V(t, \infty, \alpha) = X e^{-r_f(T-t)} - Ke^{-r(T-t)}
\] (2.19)

\[
\frac{\partial V}{\partial \alpha} \bigg|_{\alpha=0, \alpha=\infty} = 0.
\] (2.20)
The first two conditions are fairly obvious. The condition 2.19 requires the call option to behave as if a forward in case the option is very in-the-money as the underlying asset becomes very large. The fourth one is condition of no flux imposed on the volatility space. Discretising all of the above conditions, which are shown below, should be fairly straightforward. One is worth mentioning, though, is the 2.20, where one-side discretisation is used instead of central-difference for simplicity.

\[
\begin{align*}
V_{ij}^0 &= (X_j - K)^+ \\
V_{i0}^n &= 0 \\
V_{ij+1}^n &= e^{-r_ft_n} X_{j+1} - e^{-r_t} K \\
V_{0j}^n &= V_{ij}^n \\
V_{i+1j}^n &= V_{ij}^n
\end{align*}
\] 

where \( i = 0, \ldots, I + 1 \), and \( j = 0, \ldots, J + 1 \). The condition of no flux applies to all types of the options we consider in this paper. Therefore, the conditions for put options are

\[
\begin{align*}
V(0, X, \alpha) &= (K - X)^+ \\
V(t, 0, \alpha) &= K - X \\
V(t, \infty, \alpha) &= 0.
\end{align*}
\] 

The boundary conditions for barrier options are often small variations, if there is any, from what we have seen so far. For example, the discretised boundary constraints for a down-and-out call are exactly the same as a call given the \( i \) axis \((j = 0)\) of the grid is aligned to the barrier \( B \). Thus, we are not going into the detail of boundary conditions of barrier options.

### 2.5 von Neumann Stability Analysis

One of the methods to check the stability of a finite difference scheme is von Neumann analysis. This analysis is based on the Fourier decomposition of numerical error. To illustrate, consider the data points as Fourier series which is formulated in terms of complex exponential forms

\[
u_i^n = \sum_{k=0}^{m} A_k e^{i\beta k h_x}, \quad \text{for } i = 0, 1, \ldots, m,
\]

where \( I = \sqrt{-1} \). As this analysis applies to linear partial differential equations, we can, by the additivity property, investigate the propagation of only one value \( e^{i\beta k h_x} \),
instead of the whole summation series. Also, since $A_k$ is constants for all $k$, it can be neglected in the analysis. We can, therefore, examine the behaviour of this term as time increases. To this end, we let

$$u^n_i = e^{I\beta x} e^{\alpha t} = e^{I\beta ih_x} e^{\alpha k\Delta t} = \gamma^k e^{I\beta ih_x}$$

(2.29)

where $\alpha$ is some constant and $\gamma \equiv e^{\alpha \Delta t}$ is called the amplification factor. A finite difference scheme is said to be stable, in the sense of Lax-Richtmyer, if the absolute value of the exact solution of the scheme remains bounded for all $k$ as $h_x \to 0$ and $\Delta t \to 0$. From 2.29, we see that a sufficient condition is

$$|\gamma| \leq 1.$$

(2.30)

Strictly speaking von Neumann stability analysis applies only to problems with \textit{constant coefficients}. However, because we are only concerned with the stability of the finite difference scheme in question and the real test of it is to with high-frequency data components. In this regard, even though the coefficients in our pricing PDE 2.4 are time-dependent, they would not be changing as fast as those high-frequency components and can be considered as constants relatively; thus it is still appropriate to apply von Neumann analysis to our problem.

Now, in order to work out the amplification factor of the splitting scheme, we first simplify the equations by substituting the followings:

$$\lambda = \frac{\alpha^2 X^2}{h^2_X}, \quad \eta = \frac{(r - r_f)X_j}{2h_X}, \quad \xi = \frac{\rho \nu_0 \gamma^2 S_j}{8h_X h_\alpha}, \quad \epsilon = \frac{1}{h_t}, \quad \theta = \frac{\nu_0 \alpha^2}{h^2_\alpha}.$$

Equation 2.13 turns to

$$\left(\frac{\lambda}{2} + \eta\right) V_{i,j+1}^{n+\frac{1}{2}} - (\epsilon + \lambda + r) V_{ij}^{n+\frac{1}{2}} + \left(\frac{\lambda}{2} - \eta\right) V_{ij-1}^{n+\frac{1}{2}} = -\xi(V_{i+1,j+1}^n - V_{i+1,j-1}^n - V_{i-1,j+1}^n + V_{i-1,j-1}^n) - \epsilon V_{ij}^n.$$

To apply von Neumann analysis, assuming $V_{ij}^n = \gamma^n e^{Ikih_x} e^{Iljh_x}$, the left hand side would become

$$\left(\frac{\lambda}{2} + \eta\right) \gamma^{n+\frac{1}{2}} e^{Ikih_x} e^{Il(j+1)h_x} - (\epsilon + \lambda + r) \gamma^{n+\frac{1}{2}} e^{Ikih_x} e^{Iljh_x} + \left(\frac{\lambda}{2} - \eta\right) \gamma^{n+\frac{1}{2}} e^{Ikih_x} e^{Il(j-1)h_x}$$

$$= \gamma^{n+\frac{1}{2}} e^{Ikih_x} e^{Iljh_x} \left[\left(\frac{\lambda}{2} + \eta\right) e^{Ilh_x} - (\epsilon + \lambda + r) + \left(\frac{\lambda}{2} - \eta\right) e^{-Ilh_x}\right]$$

$$= V_{ij}^{n+\frac{1}{2}} \left[\frac{\lambda}{2}(e^{Ilh_x} + e^{-Ilh_x}) + \eta(e^{Ilh_x} - e^{-Ilh_x}) - (\epsilon + \lambda + r)\right]$$

$$= V_{ij}^{n+\frac{1}{2}} [\lambda \cos lh_X - \epsilon - \lambda - r] + 2I\eta \sin lh_X]$$
Right hand side becomes
\[-\xi e^{Ikh_\alpha}e^{IlhX}\left(e^{Ikh_\alpha}e^{IlhX} - e^{-Ikh_\alpha}e^{-IlhX} - e^{-Ikh_\alpha}e^{IlhX} + e^{Ikh_\alpha}e^{-IlhX}\right) - \epsilon V^n_{ij} \]
\[= -\xi V^n_{ij}\left[e^{Ikh_\alpha}e^{IlhX} - e^{-Ikh_\alpha}\left(e^{IlhX} - e^{-IlhX}\right)\right] - \epsilon V^n_{ij} \]
\[= -2I\xi V^n_{ij} \sin lh X\left(e^{Ikh_\alpha} - e^{-Ikh_\alpha}\right) - \epsilon V^n_{ij} \]
\[= (4\xi \sin lh X \sin k\alpha - \epsilon) V^n_{ij} \]

Thus,
\[
\frac{V^{n+\frac{1}{2}}_{ij}}{V^n_{ij}} = \frac{4\xi \sin lh X \sin k\alpha - \epsilon}{(\lambda \cos lh X - \epsilon - \lambda - \rho) + 2\eta \sin lh X} \quad (2.31)
\]

Similarly, left hand side of equation 2.14 turns to
\[
\left[\frac{\theta}{2}e^{Ikh_\alpha} - (\epsilon + \theta) + \frac{\theta}{2}e^{-Ikh_\alpha}\right] V^{n+1}_{ij} = (\theta \cos k\alpha - \epsilon - \theta) V^{n+1}_{ij} \]
\[= -\left[\theta(1 - \cos k\alpha) + \epsilon\right] V^{n+1}_{ij} \]
\[= -\left[\epsilon + 2\theta \sin^2 \frac{k\alpha}{2}\right] V^{n+1}_{ij} \]

The right hand side of equation 2.14 is same as above. Therefore,
\[
\frac{V^{n+1}_{ij}}{V^n_{ij}} = -\frac{4\xi \sin lh X \sin k\alpha - \epsilon}{\epsilon + 2\theta \sin^2 \frac{k\alpha}{2}} \quad (2.32)
\]

Combining both half-step equations together, we have
\[
\frac{V^{n+1}_{ij}}{V^n_{ij}} = -\frac{(4\xi \sin lh X \sin k\alpha - \epsilon)^2}{(\epsilon + 2\theta \sin^2 \frac{k\alpha}{2})[(\lambda \cos lh X - \epsilon - \lambda - \rho) + 2\eta \sin lh X]} \]

Hence, the overall amplification factor is
\[
|\gamma| = \frac{(4\xi \sin lh X \sin k\alpha - \epsilon)^2}{(\epsilon + 2\theta \sin^2 \frac{k\alpha}{2}) \sqrt{(\lambda \cos lh X - \epsilon - \lambda - \rho)^2 + (2\eta \sin lh X)^2}} \quad (2.33)
\]

Unfortunately, we are unable to derive a simple general relation linking between all the parameters in the finite-difference scheme from (2.33). Nevertheless, we can show that although the splitting scheme is not unconditionally stable, it is stable under the normal parameters I used for simulations.

As it will be discussed in section 2.7, the grid sizes in spot and volatility axes are determined by number of point per standard derivation. Thus, we can say
\[
h_X \equiv \frac{\alpha X \sqrt{\tau}}{N_X} \quad h_\alpha \equiv \frac{\nu\alpha \sqrt{\tau}}{N_\alpha} ,
\]
where \(N\) are the number of grid points in various axes, and \(\tau\) is the expiration. Substituting them back, we have

\[
\lambda = \frac{N_X^2}{\tau}, \quad \eta = \frac{(r - r_f)N_X}{2\sqrt{\tau}}, \quad \xi = \frac{\rho N_X N_\alpha}{8\tau}, \quad \epsilon = N_t, \quad \theta = \frac{N_\alpha^2}{\tau}.
\]

Letting \(x = lh\) and \(y = kh_\alpha\), equation 2.33 thus becomes

\[
|\gamma| = \left( \frac{\xi^2 N_\alpha N_X \sin x \sin y - \tau N_t}{\tau N_t + 2N_X^2 \sin^2 \frac{y}{2}} \right)^2 \left( \frac{\xi^2 N_\alpha N_X \sin x \sin y - \tau N_t}{\tau N_t + 2N_X^2 \sin^2 \frac{y}{2}} \right)^2 \cdot \frac{N_X^2 \cos x - \tau N_t - N_X^2 - r\tau}{N_X^2 \cos x - \tau N_t - N_X^2 - r\tau} \cdot \frac{(r - r_f)^2}{2} - N_X^2 \sin^2 x
\]

\[
< \left( \frac{\xi^2 N_\alpha N_X \sin x \sin y - \tau N_t}{\tau N_t + 2N_X^2 \sin^2 \frac{y}{2}} \right)^2 \left( \frac{\xi^2 N_\alpha N_X \sin x \sin y - \tau N_t}{\tau N_t + 2N_X^2 \sin^2 \frac{y}{2}} \right)^2 \cdot \frac{N_X^2 (1 - \cos x) + \tau N_t}{N_X^2 (1 - \cos x) + \tau N_t}.
\]

Equation (2.34) repeats itself every \(2\pi\), and thus it is not necessary to consider \(x, y\) further than that.

\[\text{2.6 Order of Accuracy}\]

To complete the analysis on the splitting scheme, we also need to obtain the order of accuracy, which can be done fairly straightforwardly. Given the discretisation formulation we have used to substitute the derivative terms in the derivation above

\[
\begin{align*}
\frac{\partial U}{\partial t} &= \Delta_t^+ U_{ij}^n + O(h_t) \\
\frac{\partial U}{\partial x} &= \Delta_x^2 U_{ij}^n + O(h_x^2) \\
\frac{\partial^2 U}{\partial x \partial y} &= \Delta_{xy} U_{ij}^n + O(h_x^2) + O(h_y^2),
\end{align*}
\]

we can conclude that the error of our scheme is

\[\epsilon \equiv O(h_t) + O(h_x^2) + O(h_y^2).\]
Figure 2.1: Plot of $|\gamma|$ for $0 < x < 2\pi$ and $0 < y < 2\pi$, with $\rho = -1$, $N_X = 50$, $N_\alpha = 50$, $N_t = 365$, and $\tau = 1$.

2.7 Implementation and Test Results

The finite difference scheme is implemented in Matlab. Source files are named `sabrfd-pde*.m`, with each of them dedicates to different type of options. The grid size vector $N$ is passed in as function parameters. It is defined to be the number of grid points used per day for the time variable, or per standard derivation for the other two state variables, namely the asset price and volatility. This is to ensure a consistent level of accuracy no matter how long the expiration of the simulated option is or how far the distribution of the state variables would spread out. Otherwise, if the number of total grid point are fixed, one would need to cautiously adjust the number of grid points accordingly when the maturity goes longer, or the domains covered by the state variables become larger.

For the asset price and volatility, there are cases these variables can go to infinity theoretically. For example, for a down-and-out call, the domain of the asset price is unbounded on the upside. In these cases, the program would limit the grid to six standard derivations from the mid point, normally the current price of the asset or the ATM volatility.

Table 2.1 shows some of the simulations results to demonstrate the correctness of our devised FDM. The first column is a call price in some fictitious SABR parameters,
whereas the other two calls are priced based on SABR parameters calibrated from real market data of EURUSD. $\Delta_{BS}$ is the percentage difference between Black-Scholes and FX SABR. In the fictitious setting, I deliberately set $\alpha$ equals to the Black-Scholes volatility, and $\rho$ and $\nu$ to zero. With these values, the FX SABR model should be equivalent to Black-Scholes. Indeed, the result shows that our FDM reproduces Black-Scholes value with only some small numerical error. As mentioned, the other two columns show call values in more realistic situations. Nevertheless, the percentage differences $\Delta_{BS}$ are not far from the Black-Scholes results either.

Figure 2.3, 2.4, and 2.5 depict the convergence of the FDM with $h_x$, $h_y$, and $h_t$ respectively. Simulations were done on the options listed in the second column of table 2.1. It started with the coarsest grid size and successively cutting the size into half in each subsequent iteration. Each of the graphs has an trend line added to it; the goodness of fitting value $R^2$ equals to 1, confirming the error decreases quadratically with $h_x$ and $h_y$, and linearly with $h_t$, as derived in section 2.6.

Table 2.2 shows the performance of the splitting scheme in a modern computer. It requires around 20 points per standard derivation to get a result in good accuracy, which takes a bit more than 10 seconds. I believe it could be much faster if the code are parallelised.
<table>
<thead>
<tr>
<th>Option Type</th>
<th>Call</th>
<th>Call</th>
<th>Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>Currency</td>
<td>Fictitious</td>
<td>EURUSD</td>
<td>EURUSD</td>
</tr>
<tr>
<td>Spot</td>
<td>1.4320</td>
<td>1.4320</td>
<td>1.4154</td>
</tr>
<tr>
<td>Strike</td>
<td>1.3604</td>
<td>1.3604</td>
<td>1.3446</td>
</tr>
<tr>
<td>Maturity (M)</td>
<td>12</td>
<td>12</td>
<td>24</td>
</tr>
<tr>
<td>Domestic (%)</td>
<td>1.33</td>
<td>1.33</td>
<td>2.39</td>
</tr>
<tr>
<td>Foreign (%)</td>
<td>1.30</td>
<td>1.30</td>
<td>1.72</td>
</tr>
<tr>
<td>$\sigma_B$ (%)</td>
<td>12.69</td>
<td>12.69</td>
<td>12.15</td>
</tr>
<tr>
<td>$\alpha$ (%)</td>
<td>12.69</td>
<td>11.97</td>
<td>11.46</td>
</tr>
<tr>
<td>$\rho$ (%)</td>
<td>0.00</td>
<td>-0.03</td>
<td>-2.60</td>
</tr>
<tr>
<td>$\nu$ (%)</td>
<td>0.00</td>
<td>72.32</td>
<td>50.85</td>
</tr>
<tr>
<td>BS (%)</td>
<td>11.10</td>
<td>11.10</td>
<td>13.98</td>
</tr>
<tr>
<td>FX SABR (%)</td>
<td>11.09</td>
<td>11.01</td>
<td>13.87</td>
</tr>
<tr>
<td>$\Delta_{BS}$ (%)</td>
<td>0.00</td>
<td>0.77</td>
<td>0.79</td>
</tr>
</tbody>
</table>

Table 2.1: Comparison of vanilla call value by Black-Scholes and FX SABR

<table>
<thead>
<tr>
<th>$N_{X,\alpha}$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$ (s)</td>
<td>5.19</td>
<td>11.92</td>
<td>25.26</td>
<td>40.09</td>
</tr>
</tbody>
</table>

Table 2.2: Performance of the splitting scheme with $N_t = 1$ per day

Figure 2.3: Convergence in $h_x$. Grid size vs. option value.
Figure 2.4: Convergence in $h_y$

Figure 2.5: Convergence in $h_t$
Chapter 3

Pricing Barrier Options

In the previous chapter, a finite difference scheme has been devised that allow us to price barrier options numerically under stochastic volatility. In this chapter, we first reprise some other common methods used in pricing barrier options, and then we gauge the theoretical prices produced by the FX SABR model by comparing with these methods under various settings.

We are going to review three well-established pricing methods applicable to barrier options, namely Black-Scholes model, Carr’s static hedging, and Derman’s replication. The latter two are both classified to static replication, which decompose a target option into a portfolio of standard options. The market value of the portfolio provides a practical estimate for the fair value of a target option. This value may reflect the true cost of the option more realistically than the usual theoretical value, especially in the presence of volatility smiles, and other market conditions that violate the assumptions behind Black Scholes. This is the very reason why we compare the FX SABR model to these methods.

3.1 Black Scholes

No development of a new pricing method is considered to be complete without comparing against this de-facto standard of option pricing. Pricing barrier options with Black-Scholes rely on its reflection property, which summaries as follows. Suppose $V(X,t)$ is a solution of the Black-Scholes equation

$$
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 V}{\partial X^2} + (r - r_f)X \frac{\partial V}{\partial X} - rX = 0.
$$
Reflection property states that for any constant $B$, which is normally set to the level of the barrier in this context,

$$W(X,t) = X^{2\alpha} V\left(\frac{B^2}{X}, t\right),$$

with $\alpha = \frac{1}{2} - \frac{(r-r_f)}{\sigma^2}$, is also a solution to the same Black-Scholes equation. Hence, by linearity property of Black-Scholes,

$$V(X,t) + cW(X,t)$$
is also a solution to the Black-Scholes equation for any constant $c$.

This result is of great importance because we can now make use of it to synthesise solutions for various types of barrier options. For example, consider a down-and-out call option with barrier $B < K$. The solution to it is

$$DOC(X,t) = C(X,t) - \left(X\frac{B}{X}\right)^{2\alpha} C\left(\frac{B^2}{X}, t\right), \quad (3.1)$$

with $C(X,t)$ being the vanilla call option having exactly the same features of the corresponding barrier option except without the barrier $B$. Intuitively, equation 3.1 constitutes two parts – a vanilla call option, which reproduces the same payoff as the down-and-out option if the barrier is never hit during its lifetime. The second part is the option that cancels out the vanilla call whenever $X$ hits the barrier $B$, i.e.

$$\left(X\frac{B}{X}\right)^{2\alpha} C\left(\frac{B^2}{X}, t\right) = (\frac{B}{B})^{2\alpha} C\left(\frac{B^2}{B}, t\right) = C(X,t).$$

Close-form formulae for other types of barrier options can be derived similarly together with down-out put-call parity. The result for up-and-out call is included here for the sake of comparison later on.

$$UOC(X,t; K, B) = C(X,t; K) - \left(X\frac{B}{X}\right)^{2\alpha} C\left(\frac{B^2}{X}, t; K\right)$$

$$- \left[C(X,t; B) - \left(X\frac{B}{B}\right)^{2\alpha} C\left(\frac{B^2}{B}, t; B\right)\right]$$

$$- (B - K) \left[BC(X,t; B) - \left(X\frac{B}{B}\right)^{2\alpha} BC\left(\frac{B^2}{B}, t; B\right)\right], \quad (3.2)$$

where $BC$ is a binary call.

Black-Scholes pricing is doomed to failure, however, in the context of volatility smile as the reflection property holds only if the equations for the vanilla option and the one struck at barrier are exactly the same. This rigid criterion is apparently violated when the implied volatility is a smile, or simply not constant.
3.2 Carr’s Static Hedging

Another commonly used method in valuing barrier options is static hedging by Carr et al. [8]. Static hedging aims to provide exotic option writers a way to construct a portfolio constituting only standard vanilla options that perfectly replicate the payoff in the entire lifetime of these options. Although barrier options can also be valued with dynamic hedging argument much like the one used in Black-Scholes, it offers little help to writer of these options in hedging. Static hedging is particularly attractive to barrier option writers because dynamic hedging barrier options is very costly due to their high gamma nature, whereas positions in the static-hedging portfolio is invariant to volatilities and interest rates.

Peter Carr assumes the markets under consideration are frictionless, meaning no cost in buying and selling various instruments and trades can be executed instantaneously. Derivation of static hedging portfolio requires an important property – put-call symmetry, which holds under some restrictions. In particular, they assume that the underlying price process is a diffusion, with zero drift under any risk-neutral measure, and where the volatility coefficient satisfies a certain symmetry conditions.

The assumption of zero risk-neutral drift is innocuous for options written on forward or futures price. Barrier options, our focus in this paper, however, are in general written on spot prices as it is the spot price hitting the barrier triggers subsequent actions, not forward. Thus, zero-drift assumption, or equivalently foreign interest rate equals to domestic interest rate, cannot be always true.

Another crucial criterion for put-call symmetry (PCS) to hold is that volatility smile must satisfy the following symmetry condition

\[ \sigma(cX, t) = \sigma \left( \frac{X}{c}, t \right), \]

for some positive constant \( c \). Thus, the volatility is assumed to be the same for any two levels whose geometric mean is the current \( X \). Black-Scholes with deterministic volatility satisfies this condition as \( \sigma(X, t) = \sigma(t) \). This condition is not as restrictive in the first glance. Notice that if \( Y = \ln(X/K) \), and letting \( \nu(Y, t) = \sigma(X, t) \), the equivalent condition is:

\[ \nu(y, t) = \nu(-y, t) \]

Thus, the symmetry condition is also satisfied in models with a symmetric smile in the log of \( K/F \). The graph of volatility against \( K/F \) will be skew towards the lower side from the forward, i.e. with higher put volatility than call volatility for strikes equidistant from the forward. Moreover, the symmetry condition also allows
for volatility frowns or more complex patterns such as sine waves (although it is rather unusual).

Given frictionless markets, no arbitrage, zero carrying cost, and the volatility symmetry condition, Put-Call Symmetry states the following relationship:

\[ C(X, t; K)K^{1/2} = P(X, t; B)B^{1/2}, \]

where \( \sqrt{KB} = X \). The proof can be found [8].

Similar to pricing barrier options in Black-Scholes using reflection property, we can concentrate on pricing knock-out call only, and all the other types of barrier options can be worked out through in-out put-call parity relations.

### 3.2.1 Down-and-Out Calls

A down-and-out call (DOC) struck at \( K \) with barrier \( B < K \) is by definition same as a vanilla call if the barrier has never been hit by its expiration; otherwise, it becomes worthless. To hedge this option, both the terminal payoff and the payoff along the barrier have to be matched. The first step in constructing a hedge is to match the terminal payoff by purchasing a standard call \( C(X, t; K) \). Now consider when the underlying hits the barrier \( X = B \). When this happens, DOC becomes zero whereas our first hedge, a vanilla call, still have positive value. Thus it is necessary to sell off another instrument to cancel out the standard call value at the barrier. By PCS, when \( X = B \), the vanilla call option

\[ C(X, t; K) = \frac{K}{B} P \left( X, t; \frac{B^2}{K} \right). \]

Hence, the complete replicating portfolio for a down-and-out call is to purchase one standard call struck at \( K \) and sell \( KB^{-1} \) standard puts struck at \( B^2K^{-1} \).

\[ DOC(X, t; K, B) = C(X, t; K) - \frac{K}{B} P \left( X, t; \frac{B^2}{K} \right), \quad B < K. \tag{3.3} \]

If the barrier is hit before expiration, the replicating portfolio can be liquidated with PCS guaranteeing the proceeds from selling the call will be offset exactly by the cost of buying back the puts. In case the barrier is not hit, the long call gives the exact terminal payoff as the DOC, and the put will expire worthless, as \( B^2K^{-1} < B \) when \( B < K \).

It should be apparent to the readers that the equation 3.3 obtained from replicating portfolio resembles some similarity 3.1 from Black-Scholes reflection. Indeed,
they in fact agree each other if brought under the same assumption. The appendix A proves that 3.1 equals to 3.3 when the foreign interest rate is same as the domestic interest rate, and volatility is constant as assumed in Black-Scholes.

3.2.2 Up-and-Out Calls

Contrast to DOC that always has the barrier set below its strike, an up-and-out call (UOC) has a knockout barrier set above the current underlying price and strike level ($B > K$). The replicating portfolio turns out to be

$$
UOC(X, t; K, B) = C(X, t; K) - UIP(X, t; K, B) - (B - K)UIB(X, t; B),
$$

(3.4)

where the up-and-in bond $UIB(X, t; B)$ pay $1$ at expiration if the barrier $B$ has been hit before then, and $UIP(X, t; K, B)$ is an up-and-out put. Needless to say, the standard call in the portfolio is to match the terminal payoff in case the barrier has never been hit in its lifetime. Conversely, at the first passage time to the barrier $B$, the UIC and UIB knock in. Because the underlying price is at $B$, the assumption implies that the replicating portfolio can be liquidated at zero cost. The UIP cancels the time value of the vanilla call, while the $(B - K)$ UIB offsets the call’s instinct value. However, equation 3.5 is not yet a practical result as it composes of UIC and UIB, which may rarely trade in markets. We need to replace these uncommon instruments with more standard ones.

Considering the UIC, when underlying hits $B$, the option kicks in and its value immediately equals to a $P(X, t; K)$. PCS implies that its value equals to $\frac{K}{B}C\left(X, t; \frac{B^2}{K}\right)$. Because $\frac{B^2}{K} > B$ when $B > K$, this call option will expire worthless if barrier has never been hit before expiration, and thus matching the payoff of UIC at any time.

As for the UIB, it is proved in the appendix of [8] that

$$
UIB(X, t; B) = 2BC(X, t; B) + \frac{1}{B}C(X, t; B), \quad B > X,
$$

where $BC$ stands for a binary call. Putting all these together, equation 3.5 can be rewritten as

$$
UOC(X, t; K, B) = C(X, t; K) - \frac{K}{B}C\left(X, t; \frac{B^2}{K}\right)
- (B - K)\left[2BC(X, t; B) + \frac{1}{B}C(X, t; B)\right].
$$

(3.5)

Furthermore, binary call can actually be synthesised using an infinite number of standard calls as in (3.6). In practice, however, we could use a couple of calls to
approximate it. Hence, we succeed ending up with a portfolio perfectly replicating an up-and-out call using liquidly-traded standard options in the market. And, to most traders, this is the most realistic pricing method as it also represent the cost of the hedge at the same time.

\[
BC(X, t; K) = \lim_{n \to \infty} n \left[ C(X, t; K) - C(X, t; K - \frac{1}{n}) \right].
\]

(3.6)

3.2.3 Non-Zero Carrying Costs

As mentioned in the previous section, for all of the above derivation of static replication to hold, we need to assume zero carrying cost of the underlying on which the options are written. This assumption is realistic if option is written on forward or futures. This is, however, inappropriate as in commonly-traded barrier options, it is the spot of the underlying touching the barrier. In particular, in the context of FX barrier options, zero carrying costs of spot FX means the foreign interest rate is exactly equal to the domestic interest rate, which is often not the case.

Bowie and Carr relax the assumption of zero drift in [10]. Instead of obtaining a single value for barrier options, they developed a tight bounds. When interest rates differ, the forward relates to the spot as \( F = X e^{(r - r_f)T} \) because of interest-rate parity. Define the initial forward barrier

\[
\hat{B} = Be^{(r - r_f)T}.
\]

When carrying cost are positive, i.e. \( r > r_f \), then the forward is above the spot, \( F > X \), and similarly, the initial forward barrier is above the spot, \( \hat{B} > B \). In this case, when the barrier of a down-and-out call is below its strike, equation 3.3 becomes

\[
C(X, t; K) - \frac{K}{B}P(X, t; \frac{\hat{B}^2}{K}) \leq DOC(X, t; K, B) \leq C(X, t; K) - \frac{K}{B}P(X, t; \frac{B^2}{K}),
\]

(3.7)

as the higher the strike the more valuable is the put. In case the carrying cost is negative \( r < r_f \), the upper and lower bound will be swapping their places in 3.7.

3.3 Derman’s Static Replication

Derman et al. [2] developed another strategy (DEK) to find static replication for exotic options. In contrast to Carr’s method, Derman does not assume any special
conditions about the market, namely symmetry of volatility, zero drift, and put-call symmetry. Nevertheless, their approach requires an infinite number of options to perfectly replicate an exotic option. This can never be achieved in practice, and thus only an approximation in nature.

Without loss of generality, consider to find a hedging portfolio for a single-barrier option of price $H(t)$, the DEK method does the following:

- A standard option with the same strike $K$ and maturity $T$ as the barrier option. This would be the principle option for matching the terminal payoff of the barrier option. Its price is assumed to be $S$. In a general context, it could be a call or put or even a portfolio of calls and puts in case of complex terminal payoff. It could be even nothing, for example, in case down-and-in option as this kind of barrier option expire worthless if barrier has not been hit.

- A portfolio $\Phi = (\phi_1, \ldots, \phi_n)$ of standard options struck at the barrier with expiration $T = (T_1, \ldots, T_n)$ where $T_{i-1} < T_i$. The type of option depends on the relative level of barrier $B$ and the strike $K$. They should be puts if $B < K$, and calls otherwise. Assume the prices of the options are $P^T = (P_1, \ldots, P_n)$. This portfolio should be balanced such that it would be sold off together with the vanilla option with net value zero in case the barrier is touched.

The total value of the portfolio at any time $t < T$ is $\Gamma(t) = \Phi P(t) + S(t)$. To obtain a perfect replicating portfolio, it is required

$$\Gamma(t) = 0,$$

if the barrier is hit at time $t$. This means the entire portfolio can be sold off with net amount zero. Furthermore,

$$\Gamma(T) = H(T),$$

in case the barrier has not been touched before expiry. All the options $(\phi_1, \ldots, \phi_n)$ will expire worthless and the principle option will match the barrier option’s terminal payoff. In order to find the correct weights $\Phi$ for each of the options, we follow the procedure below:

- If we position ourselves at time $T_n$, our portfolio will be $(0, 0, \ldots, \phi_n)$. Thus,

$$\Gamma(T_n) = \Phi P(T_n) + S(T_n) = \phi_n P_n(T_n) + S(T_n) = 0.$$
Hence,

\[ \phi_n = -\frac{S(T_n)}{P_n(T_n)}. \]

- Having determined the number of contracts that needs to be taken in the n option, we now position ourselves at \( T_{n-1} \), where our portfolio weight vector is \((0, 0, \ldots, \phi_{n-1}, \phi_n)\) and the only unknown in the equation \( \Gamma(T_{n-1}) = 0 \) is \( \phi_{n-1} \). A similar analysis gives

\[ \phi_{n-1} = -\frac{S(T_n) + \phi_n P_n(T_{n-1})}{P_{n-1}(T_{n-1})}. \]

- The subsequent weights can be determined similarly with the following recursive formula:

\[ \phi_i = -\frac{S(T_i) + \sum_{j=i+1}^{n} \phi_j P_j(T_i)}{P_i(T_i)}. \]

To illustrate the construction of a replicating portfolio, let us consider a real example of EURUSD, dated on 31 August 2009. Suppose we need to price an at-the-money down-and-out call option struck at 1.432 with the barrier 10% below the spot, which is 1.289, expiring in 1 year. Figure 3.1 depicts the market volatility smiles at that moment. According to the scheme, the portfolio will be composed of a 1Y standard call struck at 1.432, the strike of down-and-out call, and puts struck at 1.289 expiring at as many terms as the market can offer. In this particular example, they would be 7D, 1M, 2M, 3M, ... etc. Table 3.1 summaries the weightings on each options to form the static hedging portfolio of the down-and-out call. Figure 3.2 shows that the graph of the replication resembles the down-and-out call option very well. Figure 3.3 depicts the result of the replication and the relative error comparing to theorical Black-Scholes value respectively. Notice there are spikes of errors along the barrier throughout the life of the options. Notice the error is more serious than it would be if Black-Scholes is compared with DEK replication in constant volatility. After all, replicating under volatility smiles is bounded to be a different model from Black-Scholes with a single constant volatility. Nevertheless, the maximum relative error in this case is no more than 0.15%.

The market value of the portfolio provides a practical estimate for the fair value of a target option. The value may actually reflect the true cost of the option more realistically than the theoretical value, especially in the presence of volatility smiles.
<table>
<thead>
<tr>
<th>Option Type</th>
<th>Strike</th>
<th>Expiration (Months)</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Put</td>
<td>1.289</td>
<td>0.2301</td>
<td>-0.0040</td>
</tr>
<tr>
<td>Put</td>
<td>1.289</td>
<td>1</td>
<td>-0.0141</td>
</tr>
<tr>
<td>Put</td>
<td>1.289</td>
<td>2</td>
<td>-0.0222</td>
</tr>
<tr>
<td>Put</td>
<td>1.289</td>
<td>3</td>
<td>-0.0319</td>
</tr>
<tr>
<td>Put</td>
<td>1.289</td>
<td>6</td>
<td>-0.1000</td>
</tr>
<tr>
<td>Put</td>
<td>1.289</td>
<td>9</td>
<td>-0.2052</td>
</tr>
<tr>
<td>Put</td>
<td>1.289</td>
<td>12</td>
<td>-0.2737</td>
</tr>
<tr>
<td>Call</td>
<td>1.432</td>
<td>12</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table 3.1: Replicating Portfolio to a EURUSD down-and-out option

which is not perfectly symmetric that CEG method assumes, or other market conditions that violate the assumptions behind Black-Scholes. The barrier option value is now a direct derivation from the values of other options liquidly traded on the market where, such as in the example we consider, exhibits profound volatility smiles that are neither symmetric nor a single constant volatility can represent well.

While both CEG and DEK probably have clear advantage over Black-Scholes model, they themselves however have no obvious merit over the other. That is because, on the one hand, CEG assumes frictionless market and perfect volatility symmetry, which needless to say are fantasy. On the other hand, DEK requires infinity number of options to have perfect hedge, which is also unrealistic as markets only offer options traded on finite number of fixed maturities. Which of these methods provide more accurate estimate of the true value of barrier options, or exotic options in general, will largely depend on the market conditions.

### 3.4 Comparisons to the FX SABR model

In this section, I am going to apply the replication methods described in the above sections to real market data, and then comparing the results against the FX SABR model that I have developed in the chapter 2 in order to have an idea how the new model performs. Before we proceed pricing barrier options with finite difference, we first need to answer one more fundamental question.

Notice that the volatility surfaces gathered from the market are for FX forwards. We, on the other hand, are trying to price options that concerns FX spots hitting barriers. For this reason, the spatial variable in the finite difference scheme is FX spot, instead of forward. However, we need to supply calibrated SABR parameters, namely $\alpha, \nu, \rho$, to the FD equation, and those are supposed to come from calibrating
Consider the original SABR model,

\[ dF = \alpha F^\beta dW_1 \]
\[ d\alpha = \nu dW_2 \]
\[ dW_1dW_2 = \rho dt. \]

But, since \( X = Fe^{(r_f - r)(T-t)} \), therefore

\[
\begin{align*}
    dX &= e^{(r_f - r)(T-t)}dF - (r_f - r)Fe^{(r_f - r)(T-t)}dt \\
    &= \alpha e^{(r_f - r)(T-t)}F^\beta dW_1 - (r_f - r)Fe^{(r_f - r)(T-t)}dt \\
    &= \alpha e^{(r_f - r)(T-t)(1-\beta)} \left( e^{(r_f - r)(T-t)}F \right)^\beta dW_1 - (r_f - r)Fe^{(r_f - r)(T-t)}dt \\
    &= (r - r_f)Xdt + \alpha Xe^{(r_f - r)(T-t)(1-\beta)}X^\beta dW_1 \quad (3.8)
\end{align*}
\]

Hence, in general, the stochastic volatility model for forwards is not equivalent the corresponding one for spots. Luckily, when \( \beta \equiv 1 \), equation 3.8 becomes

\[ dX = (r - r_f)Xdt + \alpha XD_1, \]
which is exactly the same as the model setup we used in deriving the finite difference scheme; thus, it is appropriate to use calibrated SABR parameters from the forwards in the spot equation.

Table 3.2 depicts values of barrier options produced by the methods discussed above under various market conditions, such as domestic and foreign interest rates, maturities, and smiles, etc. Although I only show down-and-out and up-and-out call options; other type of single-barrier options can easily obtained with down-and-up parity and put-call parity. This is the main result of this paper. Prices produced by the FX SABR model tends to agree more with the estimates by static replications, rather than simple Black-Scholes model.

Implementation of CEG for DOC and UOC are based on equations 3.3 and 3.5 respectively. The UOC is inherently less accurate than DOC because of the approximation stated in 3.6 involved in computing the digital option using formula 3.5. Thus, looking across the table, discrepancy between FX SABR Model and CEG is generally bigger for UOC than DOC, with some lucky exceptions though.

Among the prices of DOC, the price of 1Y EURUSD DOC in column 1 by SABR PDE differs from CEG by only 0.8%. Given these methods have very different theoretical base, this result is fairly surprising. However, this is far from a coincidence. In the SBAR model, $\rho$ is controlling the skew of smile. If $\rho$ is zero, volatility would

\footnote{JPYUSD omitted because no JPY interest-rate data available for the corresponding period.}
Figure 3.3: Relative error between replication and theoretical Black-Scholes value

a symmetric smile shape. Indeed, the correlation $\rho$ for 1Y EURUSD smile is only -0.03%, and at the same time, the interest rate difference of EURUSD is as tiny as 0.03%. Both of these conditions are very close to the fundamental assumptions required by Carr’s replication. In fact, it is proved [9] in appendix A that SABR model with $\rho = 0$ and no cost of carry implies PCS, which is the foundation for CEG replication to hold. In contrast, when these conditions are violated like in GBPUSD, the difference $\Delta_{CEG}$ increases. CEG$_F$ represents the values priced based on forward barrier instead of spot barrier as discussed in section 3.2.3. In general, results from the FX SABR model hover around the range of CEG and CEG$_F$.

Reading across the table 3.2 for the discrepancies against DEK, one would notice that the DEK in general produces a price closer to the FX SABR model with longer maturity date. This can be explained by the fact that accuracy of DEK depends on the number of options expiring on or before the target option maturity. For example, taking a barrier option expiring in 2 years, DEK replication need to use standard options from 7D, 1M and up to 2Y, whereas replicating a 1 year barrier can use only up to the 1Y standard option. It is true in our case studies that the market volatilities at barrier levels are mostly not available for shorter date options, such 7D, 1M, etc. We resort to pricing these short-dated options with volatilities estimated from extrapolating smile curves. Although the weightings of short dated options are relatively small, their impact will be more significant to target options with short maturity, and hence the approximation by DEK are less accurate. In other words, the longer the maturity of the target option is, the better the accuracy produced.
by the DEK replication is. This also explains why the accuracy of CHFUSD are so poor; as you can see from the graph 1.2, the barriers for CHFUSD up-and-out call are around 1.24 to 1.28 level, where most of the volatility smiles on shorter maturities do not cover.

Finally, there are cases where market data prevent a certain type of options to be priced. For example, as shown in figure 1.3, the volatility smile of GBPUSD is so skew towards the downside that pricing up-and-out barrier option is unlikely to be accurate as it is difficult to estimate the market view on the volatilities above at-the-money. Therefore, only DOC are shown for GBPUSD.
Chapter 4
Conclusions

This paper investigates the possibility to apply SABR model on FX smiles, and subsequently modify the original SABR slightly to adapt to the particularities of FX market. It then proceed to use the new formulation to price vanilla and first-generation (barrier options) exotic options using finite-difference scheme. The results are compared against the Black-Scholes and other popular static replication methods.

SABR calibration algorithm is implemented in Matlab. It does produce good fitting to the market data collected for various currency pair. The SABR, which is originally designed to model forward rates, are modified to model spot FX rate, which is necessary in pricing barrier options, where barrier are to be hit by spots but not forwards. The model is then transform to a partial differential equation that in turn is solved numerically in finite difference splitting-scheme. This scheme is relatively quick compared to Monte-Carlo simulation, and will be suitable to price high-volume options liquidly traded on FX markets, as those are the focus of this paper. Smiles surfaces need to calibrate only once in the start of a trading day, and the same parameters could be reused to price various contracts.

Having compared with Derman’s and Carr’s static replication, the pricing produced by the FX SABR model can be concluded to be reasonable. Discrepancies between them are expected as they are all derived from different basis. Nevertheless, there are cases where their difference comes very close, such as where small correlation and interest-rate difference in foreign and domestic currency make Carr’s replication and FX SABR very close. This could also be proved in rigorous mathematics. Also, Derman’s and FX SABR would not differ much if market volatility smiles from short to long tenor cover the strike as well as the barrier level, as in the case of DOC in GBPUSD.

The FX SABR model is viable way to price vanilla and simple FX barrier options with more realistic results than Black-Scholes model. Future study can be done on
the hedging result based on the FX SABR model, and further investigation on risk management of a book of barrier options. This would complete the model, and will be of great use in practice.
Appendix A

Proof of Theorems

A.1

Theorem A.1.1. Equivalence of Black-Scholes and Static replication.
Under zero cost of carry, the pricing formula of an down-and-out barrier option resulting from Black-Scholes reflection and Carr’s static replication are equivalent.

Proof. Given the Black-Scholes call option formula

\[
C(X, t; K) = X e^{-r_f(T-t)} N(d_+) - K e^{-r_f(T-t)} N(d_-)
\]

\[
d_+ \equiv \frac{\ln X - (r - r_f + \frac{\sigma^2}{2}(T - t))}{\sigma \sqrt{T - t}}
\]

\[
d_- \equiv d_+ - \sigma \sqrt{T - t}.
\]

Then, under the assumption that \( r = r_f \),

\[
C\left(\frac{B^2}{X}, t; K\right) = \frac{B^2}{X} e^{-r_f(T-t)} N(d_+) - K e^{-r_f(T-t)} N(d_-),
\]

with

\[
d_+ = \frac{\ln \frac{B^2 X}{K} + (r - r_f + \frac{\sigma^2}{2}(T - t))}{\sigma \sqrt{T - t}}
\]

\[
= - \frac{\ln \frac{X}{B^2/K} - (r - r_f + \frac{\sigma^2}{2}(T - t))}{\sigma \sqrt{T - t}}
\]

\[
= - \frac{\ln \frac{X}{B^2/K} - \frac{\sigma^2}{2}(T - t)}{\sigma \sqrt{T - t}}
\]

\[
= -d_-
\]

\[
= -(d'_+ - \sigma \sqrt{T - t}).
\]
On the other hand, for a put option struck at $\frac{B^2}{K}$,

$$P \left( X, t; \frac{B^2}{K} \right) = \frac{B^2}{K} e^{-r(T-t)} N \left( -d'_- \right) - X e^{-r_f(T-t)} N \left( -d'_+ \right)$$

Hence,

$$\frac{X}{B} C \left( \frac{B^2}{X}, t; K \right) = \frac{X}{B} \left[ \frac{B^2}{X} e^{-r_f(T-t)} N(d_+) - K e^{-r(T-t)} N(d_-) \right]$$

$$= B e^{-r_f(T-t)} N(d_+) - \frac{X K}{B} e^{-r(T-t)} N(d_-)$$

$$= B e^{-r_f(T-t)} N(-d'_-) - \frac{X K}{B} e^{-r(T-t)} N(d'_+)$$

$$= \frac{K}{B} P \left( X, t; \frac{B^2}{K} \right).$$

Hence, both formulae are identical and the result follows. $\square$
A.2

**Theorem A.2.1. Stochastic Volatility and PCS.** Suppose that \( X_t \) is a martingale, \( Q_t \equiv \ln \frac{X_t}{X_0} \), and the two-dimensional process \((Q_t, V_t)\) satisfies

\[
\begin{align*}
\frac{dQ_t}{dt} &= -\frac{1}{2} f^2(Q_t, V_t, t) dt + f(Q_t, V_t, t) dW_1 \\
\frac{dV_t}{dt} &= g(Q_t, V_t, t) dt + h(Q_t, V_t, t) dW_2,
\end{align*}
\]

where \((W_1, W_2)\) is a standard Brownian motion and the functions \(f(\cdot), g(\cdot), h(\cdot)\) are even in \(q\) and imply weak uniqueness for equation. Then PCS holds.

**Proof.** We have

\[
\begin{align*}
\frac{d(-Q_t)}{dt} &= \frac{1}{2} f^2(Q_t, V_t, t) dt - f(Q_t, V_t, t) dW_1 \\
&= -\frac{1}{2} (Q_t, V_t, t) dt - f(Q_t, V_t, t) dW_1^Q \\
&= -\frac{1}{2} (-Q_t, V_t, t) dt + f(-Q_t, V_t, t) d\tilde{W}_1^Q, 
\end{align*}
\]

because Girsanov's theorem implies that \((W_1^Q, W_2)\) is a Brownian motion under \(Q\), where

\[
W_1^Q \equiv W_1 - \int_0^t f(Q_s, V_s, s) ds, \tag{A.3}
\]

and hence so is

\[
\left(\tilde{W}_1^Q, W_2\right) \equiv (-W_1^Q, W_2). \tag{A.4}
\]

Moreover, by the evenness of \(g\) and \(h\) in \(x\),

\[
\frac{dV_t}{dt} = g(-Q_t, V_t, t) dt + h(-Q_t, V_t, t) dW_2. \tag{A.5}
\]

By equations A.1, A.2, and A.5, both \((Q, V)\) under measure \(P\) and \((-Q, V)\) under measure \(Q\) solve the SDEs A.1. By weak uniqueness, \(Q_T\) under \(P\) has the same distribution as \(-Q_T\) under \(Q\).

**Corollary A.2.2. FX SABR Model.** The FX SABR model implies Put-Call Symmetry if the cost of carry is zero and the correlations \(\rho\) between the two Brownian motions \((W_1, W_2)\) is zero.

**Proof.** With the interest rate differential \(r - r_f\) and the correlation \(\rho\) equal to zero, the FX SABR model becomes

\[
\begin{align*}
\frac{dX_t}{dt} &= \alpha X_t dW_1 \\
\frac{d\alpha}{dt} &= \nu dW_2,
\end{align*}
\]

\(\alpha\) being a deterministic function of \(t\).
where \((W_1, W_2)\) are independent Brownian motions under measure \(\mathbb{P}\). Thus, \(X_t\) is a martingale and the log moneyness \(Q_t \equiv \ln \left( \frac{X_t}{X_0} \right)\) satisfies equations A.1, and \(\alpha, \nu\) are even with respect to \(Q_t\). Hence, the result follows.
Appendix B

List of Supplementary Files and their Brief Descriptions

1. sabrpdefdvanillacall.m, sabrpdefdupandoutcall.m, sabrpdefddownandoutcall.m
   FX SABR model implemented in the splitting scheme. This is for pricing vanilla call options, up-and-out, and down-and-out calls respectively.

2. cegupandoutcallreplication.m, cegdownandoutcallreplication.m
   CEG method to price up-and-out, and down-and-out calls respectively.

3. dekbarrieroptionreplicationwithsmile.m
   DEK method to price barrier options.

4. blackscholes.m, blackscholesdownandoutcall.m, blackscholesupandoutcall.m
   Price vanilla and barrier call options with Black-Scholes reflection.

5. sabr_sigma.m
   The implementation of volatility approximation formula (1.18) - (1.20).

6. sabr_fit.m
   Use of constrained quadratic algorithm to find the optimal solution of a fitting.

7. sabr_fit_err.m
   The SABR objective function for the minimisation algorithm.

8. fxvol_analysis.xlsm
   This spreadsheet contains raw data collected from JPMorgan system. *-VOL pages are original volatility data for different currency pairs, whereas *-IR pages are original interest rate data. Interest rates beyond 1 year were not available and were extrapolated using cubic spline. Graphs of calibration results and tables of simulation outcomes are also created in this spreadsheet.
Bibliography


