Credit Barrier and Dynamic Correlation Techniques for Pricing Collateralized Debt Obligations of European Small and Medium-sized Enterprises

Louis Loizou
Mathematical Finance Group
Mathematical Institute
University of Oxford

and

Dresdner Kleinwort Benson

louis.loizou@christ-church.oxon.org

Supervised by

Professor Claudio Albanese
Department of Mathematics
Imperial College
University of London

Working Paper:
Credit Risk Evaluation Designed for Institutional Targeting in finance
Risks in Small Business Lending
Venice, 25-26 September 2006

http://www.greta.it/credit/credit2006/credit2006.htm
Acknowledgements

I would like to thank Professor Claudio Albanese, Imperial College, London, for supervising this work. I would also like to thank Herve-Pierre Flammier, Managing Director of Structured Finance, Standard & Poor’s, for providing me the SME credit ratings data of Standard & Poor’s.

Disclaimer

The opinions herein expressed are the author and not necessarily endorsed in any way by Dresdner Kleinwort Benson, University of Oxford or any other individual.
Abstract

The distinct nature of SMEs makes the structuring and pricing of SME CDO products particularly difficult. First of all, SMEs are publicly unrated causing information asymmetries and moral hazard complexities within the SME CDO market.

I propose that the issue of how to price SME CDOs should receive special attention by the quantitative analysts, in the same way that the SMEs are treated differently relatively to large enterprises under the new Capital Accord. For instance, there is a smaller risk weight factor for economic capital when the bank's exposure is coming from an SME than a large firm.

Hence, this research paper is devoted exclusively to the pricing of SME CDOs, aiming at contributing to the studies of the SME debt securitization from both a quantitative and qualitative perspective. In specific, the SME CDO pricing framework of this research paper has been developed in such a way as to offer an intuitive and flexible way of pricing those instruments. It presents a genuine extension to the area of SME structured finance of the quantitative techniques in Albanese, Chen, Dalessandro and Vidler (2006).

This research paper develops a framework for pricing synthetic tranches of collateralized debt obligations (CDOs) backed by loans to small and medium-sized enterprises (SMEs). By using functional calculus the pricing of SME CDOs is done in the light of a constructive, functional analytic and intuitive approach, while it overcomes the Monte Carlo simulation noise of copula implementation schemes. In addition, the functional calculus framework is rich and flexible enough to accommodate both jumps and dynamic correlation structures within the credit quality process of the SME obligors and hence the CDO pricing kernel.

Furthermore, the mathematical components enter the pricing framework in an intuitive way. Firstly, the construction of the Credit Barrier Model (CBM) satisfies both the fundamental theorem of finance and the theorem of measure changes. Secondly, the pricing framework is justified from an economic perspective, by taking on board special issues arising in the SME structured finance. For instance, the complex correlation interrelationships of the SMEs with large enterprises and financial observables and the fact that the asset side of an SME CDO consists of hundreds names.

The calibration is performed under both the statistical (real-world) and risk-neutral (pricing) measure. Under the statistical measure the model is calibrated by using historical credit ratings data for SMEs at the aggregate level. Then, the CBM and the dynamic correlation techniques are combined to price an actual SME synthetic CDO from the German securitization market. The pricing and the calculation of the various hedge ratios of the different tranches with or without trading management constraints (unwind constraints) has been successful.
List of Tables

Table [1]. The tridiagonal elements of the Markov generator
Table [2]. The elements of the subordinated Markov operator
Table [3]. European SMEs PDs by rating class and term to maturity
Table [4]. One-year Transition Matrix for European CDOs (1987-2004)
Table [5]. CLN spreads and recovery rates by rating class
Table [6]. Composition of names and capital structure by rating class
Table [7]. The CDO tranche prices, no unwind barrier, March 1st 2005
Table [8]. The CDO tranche prices, unwind barrier of 10.5%, March 1st 2005
Table [9]. The CDO tranche hedge ratios, unwind barrier of 10.5%, March 1st 2005

List of Figures

Figure [1]. A credit default swap transaction
Figure [2]. The global credit derivatives market trend
Figure [3]. Market shares of credit derivative products, 2003
Figure [4]. A general CDO securitization process
Figure [5]. European SME CDO issuance trends
Figure [6]. The European SME CDO issuance by pool location, 2003
Figure [7]. Credit default assumptions for European SMEs and LargeCaps
Figure [8]. The dynamic correlation structure
Figure [9]. The local drift function under the statistical measure
Figure [10]. The local volatility function under the statistical measure
Figure [11]. Calibration of the CBM on SME 1, 3 and 5-year PDs
Figure [12]. Calibration of the CBM on the transition matrix
Figure [13]. The structure of the PROMISE Mobility 2005-1 SME CLO
Figure [14]. The risk-neutral drift function
Figure [15]. The term to maturity of the CLN spreads by rating class
Figure [16]. The term structure of implied versus actual recovery rates
Figure [17]. The distance to default by rating class
Figure [18]. Specification of the local beta function, March 1st 2005
Figure [19]. Specifications of the weights and percentile levels
Figure [20]. The contagion skew
Figure [21]. The PandL distribution of the CLO, unwind barrier 0%
Figure [22]. The PandL distribution of the CLO, unwind barrier 10.5%
Figure [23]. The PandL distribution of the CLO, unwind barrier 0% and time-horizons 1-10 years
## Contents

1 Introduction 1
  1.1 Introductory Remark ........................................... 1
  1.2 Credit Derivatives ............................................ 2
  1.3 The CDO Market of European SMEs ............................. 3
    1.3.1 Securitization Structures ................................. 3
    1.3.2 Market Trends ........................................... 5
    1.3.3 Motivation of Issuance ................................... 6
    1.3.4 Pricing Complexities ..................................... 7
  1.4 Credit Risk Modelling and Pricing ................................ 7
    1.4.1 Structural Models ....................................... 8
    1.4.2 Reduced-form Models .................................... 8
    1.4.3 Credit Barrier Models ................................... 9

2 Credit Barrier Models 11
  2.1 Markov Generator ............................................. 11
    2.1.1 Specification ........................................... 11
    2.1.2 Diagonalisation ......................................... 15
  2.2 Pricing Kernel ................................................. 16
    2.2.1 Derivation ............................................. 16
    2.2.2 Jump Subordination ..................................... 17

3 Dynamic Correlation 21
  3.1 Intuition and Construction .................................... 21
    3.1.1 Intuition ............................................... 21
    3.1.2 Construction ........................................... 23
  3.2 The Conditional Pricing Kernel ............................... 24
    3.2.1 Derivation ............................................. 24
    3.2.2 Contagion Skew ......................................... 26
4 Markov Estimation and Statistical Calibration 28
4.1 Estimation of the Markov Generator . . . . . . . . . . . . . . 28
4.1.1 Lattice Representation . . . . . . . . . . . . . . . . . . . . 28
4.1.2 Numerical Estimation . . . . . . . . . . . . . . . . . . . . . 29
4.2 Calibration under the Statistical Measure . . . . . . . . . . . . 30
4.2.1 Estimation of Parameters . . . . . . . . . . . . . . . . . . . 30
4.2.2 Calibration on Default Probabilities . . . . . . . . . . . . 30
4.2.3 Calibration on Transition Probabilities . . . . . . . . . . . 31

5 Risk-Neutral Calibration and Pricing 34
5.1 The PROMISE Mobility 2005-1 SME CLO . . . . . . . . . . . . 34
5.1.1 Background to the Transaction . . . . . . . . . . . . . . . 34
5.1.2 Structural Data . . . . . . . . . . . . . . . . . . . . . . . . . 35
5.2 Risk-Neutral Calibration . . . . . . . . . . . . . . . . . . . . . 36
5.2.1 Model Specification . . . . . . . . . . . . . . . . . . . . . . . 36
5.2.2 Calibration on CLN spread curves . . . . . . . . . . . . . . 37
5.3 Pricing of the PROMISE CLO Tranches . . . . . . . . . . . . . 38
5.3.1 Dynamic Correlation Structure . . . . . . . . . . . . . . . 38
5.3.2 The Profit and Loss Distribution . . . . . . . . . . . . . . 39
5.3.3 Tranche Prices and Hedge Ratios . . . . . . . . . . . . . . 39

6 Conclusion 41
6.1 Concluding Remarks . . . . . . . . . . . . . . . . . . . . . . . . . 41
6.2 Further Work . . . . . . . . . . . . . . . . . . . . . . . . . . . . 42

A Mathematical Proofs 44
A.1 Derivation of the drift and volatility of the Markov process . . 44
A.2 A heuristic proof of the fundamental theorem of finance (2.1.1)
with functional calculus . . . . . . . . . . . . . . . . . . . . . . . . 46
A.3 Proof of the measure change theorem (2.1.2) . . . . . . . . . . 47
A.4 Proof of the pricing kernel’s properties (2.2.1)-(2.2.3) . . . . . 47

B Tables and Figures 49

C Bibliography 66
Abstract

This thesis develops a framework for pricing synthetic tranches of collateralized debt obligations (CDOs) backed by loans to small and medium-sized enterprises (SMEs). By using functional calculus the pricing of SME CDOs is done in the light of a constructive, functional analytic and intuitive approach, while it overcomes the Monte Carlo simulation noise of copula implementation schemes. In addition, the functional calculus framework is rich and flexible enough to accommodate both jumps and dynamic correlation structures within the credit quality process of the SME obligors and hence the CDO pricing kernel.

Furthermore, the mathematical components enter the pricing framework in an intuitive way. Firstly, the construction of the Credit Barrier Model (CBM) satisfies both the fundamental theorem of finance and the theorem of measure changes. Secondly, the pricing framework is justified from an economic perspective, by taking on board special issues arising in the SME structured finance. For instance, the complex correlation interrelationships of the SMEs with large enterprises and financial observables and the fact that the asset side of an SME CDO consists of hundreds names.

The calibration is performed under both the statistical (real-world) and risk-neutral (pricing) measure. Under the statistical measure the model is calibrated by using historical credit ratings data for SMEs at the aggregate level. Then, the CBM and the dynamic correlation techniques are combined to price an actual SME synthetic CDO from the German securitization market. The pricing and the calculation of the various hedge ratios of the different tranches with or without trading management constraints (unwind constraints) has been successful.
Chapter 1

Introduction

1.1 Introductory Remark

This thesis develops an intuitive framework for pricing the tranches of synthetic collateralized debt obligations (CDOs) backed by a reference pool of loans to small and medium-sized enterprises (SMEs). SME synthetic CDOs are tradable illiquid credit derivative instruments, which slice and transfer the credit risk exposures of the SME reference portfolio through tranche-linked notes (TLNs) to market investors. The Basel Committee of the Bank for International Settlements (BIS) defines SMEs as enterprises of having no more than EUR 50 million of turnover; either of the latest accounting year’s figure or the average of the past three (Basel, 2001). The European Commission (EC) adds to the definition that SMEs should have assets of less than or equal to EUR 27 million and no more than 25% of their capital is to be owned by non-SME entities (EC, 2003).

According to the Chairman of the Basel Committee "SMEs are important for the growth and stability of the European economy" (McDonough, 2002). Indeed, the EC’s SME Observatory (2003) measures that SMEs account for almost 40% of the private sector output in the EU and employ two thirds of the workers in the European Economic Area (EEA). Additionally, there are 20 million SMEs providing employment for 120 million individuals within the EEA. Especially in Spain and Germany SMEs constitute 98% and 99% respectively of their registered companies. SMEs has an important role to play within the banking sector, since 70% of the credit portfolios of European regional banks are devoted to SMEs (EC, 2003).

However, the distinct nature of SMEs makes the structuring and pricing of SME CDO products particularly difficult. First of all, SMEs are publicly unrated causing information asymmetries and moral hazard complexities within
the SME CDO market. Therefore, I propose that the issue of how to price SME CDOs should receive special attention by the quantitative analysts, in the same way that the SMEs are treated differently relatively to large enterprises under the new Capital Accord. For instance, there is a smaller risk weight factor for economic capital when the bank’s exposure is coming from an SME than a large firm.

Hence, this thesis is devoted exclusively to the pricing of SME CDOs, aiming at contributing to the studies of the SME debt securitization from both a quantitative and qualitative perspective. In specific, the SME CDO pricing framework of this thesis has been developed in such a way as to offer an intuitive and flexible way of pricing those instruments. It presents a genuine extension to the area of SME structured finance of the quantitative techniques in Albanese, Chen, Dalessandro and Vidler (2006).

This paper is organised as follows. The rest of section 1 discusses briefly the credit derivatives and CDO securitization market for European SMEs, while it reviews the literature on credit risk modelling. Sections 2 and 3 present the construction of the pricing framework by elaborating on the credit barrier models (CBMs) and dynamic correlation techniques (DCTs) respectively. Then, section 4 calibrates the CBM on SME credit ratings historical data. After that, section 5 shows how to apply the risk-neutral pricing framework on the PROMISE Mobility 2005-1, which is a SME CDO transaction from the German securitization market. Finally, section 6 concludes and points out further work on this topic. The accompanied CD includes the dataset and full estimation results.

1.2 Credit Derivatives

Credit derivatives present a remarkable development in the market of credit risk-transfer instruments. They, unlike traditional credit insurance products, are over-the-counter (OTC) tradable financial contracts whose payoffs are contingent to changes in the credit quality of some underlying asset(s), which are usually named as reference entities. The working principle of credit derivatives is common from the simplest vanilla products, like credit default swaps (CDSs), to the most advanced ones, like CDOs. A CDS, as figure [1] illustrates, is a negotiable bilateral contract with a specific predetermined maturity in which the credit protection buyer (buyer counterparty) pays a periodic premium (swap rate) on a predetermined notional amount in exchange for a contingent guaranteed terminator payment from the credit protection seller (seller counterparty) to cover losses following a specified credit event (e.g. bankruptcy, moratorium) on a specific asset (reference asset) before
maturity (ISDA, 2005).

Figure [1]. A credit default swap transaction

The credit derivatives market is small compared to interest rate and foreign exchange derivatives which constitute around USD 177 trillion and USD 32 trillion respectively and are the largest global OTC derivative markets (BIS, 2004). Nevertheless, the fact that global financial markets have much larger exposures to credit risk than to interest rate or currency risk indicates a high growth potential for the credit derivatives market. Indeed, according to the British Bankers’ Association (BBA) and the International Swaps and Derivatives Association (ISDA) the global credit derivatives market in terms of the outstanding principal has grown dramatically over the past decade (BBA, 2002; ISDA, 2005). As figure [2] shows, it grew from a USD 0.4 trillion outstanding notional value in 1996 to an estimated USD 1.2 trillion at the end of 2001, whereas by June 2005 has been soared to USD 12.4 trillion (BBA, 2002; ISDA, 2005).

Figure [2]. The global credit derivatives market trend, 1996-2005

The credit derivatives product market is not only fast growing, but also highly innovative and segmented. As figure [3] depicts, the CDO product market is dominated by single-name CDSs with a share of 51%; multi-name synthetic CDOs and credit indices of 16% and 11% respectively, while the rest 22% of the market consists of hybrids, like default basket products.

Figure [3]. Market shares of credit derivative products, 2003

1.3 The CDO Market of European SMEs

1.3.1 Securitization Structures

CDOs products, which are also considered as being asset-backed securities (ABSs), diversify and transfer the risk exposure of a pool of assets; if the assets are loans then these are collateralised loan obligations (CLOs). In general, there are two types of CDO securitization structures: cash CDOs and synthetic CDOs.

Cash CDOs typically involve the conversion of cash flow from a pool of assets of varying maturity and quality into negotiable capital market papers (tranches), which slice the credit risk of the reference pool into different risk levels, are issued by either the originator of the assets or a special-purpose
vehicle (SPV) which is a single-asset finance company (Jobst, 2005). The tranches are ranked as equity, mezzanine, senior and super-senior on a high to low risk-exposure scale from the investor’s point of view. The lower bound of the risk level of a tranche is often referred to as an attachment point and the upper bound a detachment point. The tranches are rated by one or more rating agencies, underwritten by the sponsoring bank and sold to institutional investors. A cash CDO is backed by true assets, such as bonds or loans. Its payoffs, either coupons or principals, come from the actual cash flows of the assets in the pool.

Synthetic CDOs are not backed by cash flows of assets. Instead, they are linked to their reference entities by credit derivatives, such as CDSs or CLNs. The payoffs of most synthetic CDOs are only affected by credit events and are not related to the actual cash flows of the pool (Hirata and Shimizu, 2004). Most synthetic CDOs in the market do not have a so-called cash flow waterfall structure (Satyajit, 2005). They can be customized so that the payment structure of a tranche does not depend on the payment structures of other tranches. Such CDOs are thus sometimes called single tranche CDOs and they are very popular. The most commonly used credit derivatives in synthetic CDOs are credit default swaps (CDSs on CDO tranches) and tranche-linked notes (CDO notes) to CDO tranches (Jacobson et al, 2004). Synthetic CDOs consisting only of CDSs are unfunded CDOs, whereas synthetic CDOS issuing only CLNs to CDO tranches are funded or partially funded.

Given a CDO structure, each tranche has a notional amount and a coupon. Such a structure behaves the same way as a bond does, until the accumulated loss of the reference pool touches the attachment point of the tranche. For example, a 5-10% tranche has an attachment point of 5% and a detachment point of 10%. When the accumulated loss of the reference pool is no more than 5% of the total initial notional of the pool, the tranche will not be affected. However, when the loss has exceeded 5%, any further loss will be deducted from the tranche’s notional principal until the 10% detachment point is reached (Bluhm, 2003). In addition, the structure of a CDO may be imposing trading management constraints, which are expressed as percentages of unwind barriers. A general cash securitization process is illustrated in figure [4];

Figure [4]. A general CDO securitization process

The buyer (investor) of a CDO note pays a certain amount, the note’s principal, upfront and receives periodic coupon payments from the CDO that are based on the principal and a fixed coupon rate. If the tranche does not
suffer any loss during the life of the note, the investor will receive a full repayment of the amount invested at the beginning (Dietsch and Petey, 2004). However, if there are defaults in the reference pool and the pool’s accumulated loss exceeds the attachment point of the linked tranche, the principal will be reduced and thus the future coupon and the principal repayment will be reduced.

1.3.2 Market Trends

An important, recent and fast-growing niche segment of the European CDO market is the securitization of SME portfolios through CDO structures. Figure [5] depicts the issuance of SME CDO volumes in terms of notional credit risk transfer amounts in Europe (KfW Bankengruppe, 2005; Commerzbank Securities, 2004). The CDO notional amount reached a peak in 2001 at EUR 16.5 million from almost EUR 10 million in 1999. In 2004, the market exceeded EUR 18 million. So, in 5 years time the European SME CDO market has doubled in size. Of course, it is still a small fraction of the total CDO issuance in Europe. I estimate it to be about 15-20% of the whole CDO European market. However, after the implementation of the new Capital Accord through the Financial Services Action Plan (FSAP) the market would continue to grow even more. The Basel Committee, through the New Accord, permits banks on the internal ratings-based (IRB) approach to adjust downward the capital requirements on exposures to SMEs. In addition, it permits aggregate exposures to a single business borrower of up to EUR 1 million to be treated as retail exposures and hence enjoying a risk weight reduction of 75% (Basel, 2001).

Moreover, the steadily increase in the proportion of government agency sponsored securitization schemes, especially in Germany and Spain, led to the increase in the issuance patterns and volumes of SME CDOs (KfW Bankengruppe, 2005). As figure [6] displays, the German SME CDO market is the largest in Europe (57%) followed by the Spanish market (24%) and then the UK one (8%). The German market is largely synthetic whereas the Spanish one is true-sale. Despite that Germany is the leading market for SME securitisations in Europe, is still in its infancy; it is estimated that only 2-3% of the SME loans in German banks’ balance sheets had been securitised by the end of 2004 (Ranné, 2005). The most important reason, as the next two sections explain, is the complexity of the SME securitization and the lack of knowledge and expertise of structuring those products.
1.3.3 Motivation of Issuance

Research by the European Investment Bank (EIB) indicates that European regional banks devote to SME lending 70% of their credit portfolio (Wagenvoort, 2003). Essentially, the motivation of issuance for regional commercial banks is twofold:

- Firstly, CDO securitization is an efficient risk management tool; it enables regional banks to defuse risk concentrations of SME loans without disrupting client relations.

- Secondly, it allows commercial banks to exploit the regional lending potential by originating additional SME loans and using more favourable term conditions.

Besides, the EIB found that SME lending is more profitable than lending to large enterprises by higher interest rates of 90-160 basis points (bps). Therefore, European state development banks, such as Kreditanstalt fuer Wiederaufbau (KfW) of Germany and Instituto de Credito Oficial (ICO) of Spain, usually act as the SPV in CDO deals. SPV’s motivation to engage in the securitization process is again twofold:

- Firstly, to assist in the growth of SMEs by ensuring easier access and lower cost to loan financing for the SME lenders.

- Secondly, to provide regional banks with cost-effective regulatory capital relief, since SME capital exposures are treated more favourably than exposures to large enterprises (Basel, 2001).

From the SMEs point of view, the CLO issuance induces better access to the lending markets. In addition, the implementation of SME securitization platforms offer to investors a regular and relatively standardised but distinct source of investment. However, CDO protection sellers do not consider SME securitization products as an established and reliable asset class in spite of the fact that the probability of default (PD, in Basel’s notation) for SMEs is lower than the PD for large enterprises. This relationship is considered in figure [7] and is based on research conducted by Standard & Poor’s (2003a; 2004c) and Moody’s (2003). For example, investors have become comfortable with the PROMISE SME securitization platform in Germany and more willing to assume risk at competitive rates, but still the percentage of SME
securitization is small (Ranne, 2005). This is partially due to the absence of transparency restraining the existence of a deeper and broader secondary market for SME CDOs (Ranne, 2005).

Figure [7]. Credit default assumptions for European SMEs and LargeCaps

1.3.4 Pricing Complexities

There are certain inherited complexities, which make the structuring and pricing of CDOs very difficult. Specifically, CDO instruments are less liquid than CDSs and their prices are unquoted, meaning that the pricing in CDO markets is opaque. In addition, CDOs contain multiple names so is very difficult to price and risk manage. SME CDOs imposes additional structuring and valuation difficulties than CDOs in general (Effenberger, 2003). Therefore, in spite of the large exposures to SME credit risk and the usefulness of those securitization methods, regional banks make little or no use of SME CDOs.

The information asymmetry problem, that is, the uneven distribution of information between the regional banks and SME borrowers causes adverse selection and moral hazard issues in closing CDO deals. Particularly, SMEs unlike large enterprises, are neither rated nor subject to extensive disclosure requirements in order to have a public measure of the PD. For instance, SMEs tend to borrow via bank loans rather than bonds, thus valuation by the market is not possible. Hence, the protection seller is not able to ascertain fully the quality of the underlying SME assets and demands from the protection buyer an additional distrust premium to make up for this adverse selection risk. Additionally, once a SME CDO contract is closed, it reduces the protection buyer’s incentive to monitor the reference SME borrowers in ensuring credit servicing (moral hazard).

Therefore, in this research study I consider only the pricing of SME synthetic CDOs mostly due to data limitations (e.g. credit ratings for individual SME obligors) and additional complexities induced by true-sale structures, like the transferring of the legal ownership of the SME assets, which makes the pricing of true-sale instruments rather opaque.

1.4 Credit Risk Modelling and Pricing

This section presents the quantitative developments in credit risk models, with reference to SMEs. In turn, sections 1.4.1-1.4.3 discusses the structural, reduced-form and credit barrier models respectively.
1.4.1 Structural Models

In structural models the firm defaults if its total assets, which are modelled by a geometric Brownian motion process with jumps, is below the barrier point of its outstanding debt (Merton, 1974). Black and Cox (1976) extended this framework by modelling the borrower’s balance sheet directly. Other prominent work in this category includes Kim et al (1993), Nielsen et al (1993), Leland (1995), Longstaff and Schwartz (1995), Zhou (1996) and CreditMetrics credit risk model (Kealhofer, 1998). Structural models are very intuitive with clear financial interpretation. However, their estimation is subjective due to the necessity of extracting a default barrier from accounting statements with complex liability structures. Moreover, structural models do not take into account current market data, like debt spreads. Hence, their pricing applicability on CDOs is limited, because CDOs are usually hedged by using CDSs (Hull et al, 2005).

The modelling of credit risk for SME individual obligors is conducted by employing structural models. In particular, I note the SME CDO tracker of FitchRatings (2005) and Moody’s (2003) RiskCalc, which are both calibrated on SME historical default data based on credit scoring statistical techniques.

1.4.2 Reduced-form Models

Reduced-form models assume that an exogenous random variable, which is modelled as the first arrival of a point process, drives firm’s default while the PD is nonzero over any time interval. These models have been considered by Literman and Ibel (1991), Jarrow and Turnbul (1995), Madan and Unal (1996), Lando (1998), Hull and White (2001), Bielecki and Rutkowski (2002) and Schonbucher (2003). They are suitable for pricing single-name credit derivatives, as they are calibrated on observed market data, such as bond spreads.

The pricing of CDOs is conducted by using correlated stochastic rates for the different names based on copula models (Duffie, 1998). Copula models present the market standard for pricing CDOs. Copula is a method of combining given marginal distributions of PD for individual obligors to a common multivariate distribution, which is usually a Gaussian or Student-t and reflects the dependencies between single default times. Their numerical implementation is relatively straightforward, as copulas are estimated by employing MCS techniques; ideally by variance reduction methods to mitigate the problems of large simulation noise and slow convergence (Li, 2000; Laurent and Gregory, 2003). However, copulas might generate both unrealistically low correlations and inconsistent calibrations on CDS spreads, when
the number of reference names is large (Duffie and Singleton, 1999; Bielecki and Rutkowski, 2000).

On the SMEs side of modelling credit risk, I note the development of the Credit Risk Tracker, which calibrated the CLO model on credit default correlation assumptions in order to model the creditworthiness of SMEs (Standard and Poor’s, 2004).

1.4.3 Credit Barrier Models

CBMs estimate the distance to default of an obligor, which is a measure of the obligor’s leverage relative to the volatility of its asset value. The literature on CBMs, which is more recent than structural and reduced-form models, includes Hyer et al (1999), Gordy and Heitfield (2001), Douady and Jeanblanc (2002) Hull and White (2001), Avellaneda and Zu (2001), Albanese et al (2003), Albanese and Chen (2005a), Albanese and Chen (2005b), Albanese and Chen (2005c). In Hull and White (2001) and Avellaneda and Zu (2001), the underlying stochastic process is a time-changed diffusion process, which triggers default by hitting a time-dependent barrier. In Albanese and Chen (2005a) the credit quality of individual obligors is estimated with respect to aggregate credit ratings and CDS spreads data, whereas transitions are associated to barrier crossings. Moreover, jumps are added by subordinating the diffusion process on the variance-gamma process (Bochner, 1955; Madan and Senata, 1990; Madan, Carr and Chang, 1993) instead of following Merton’s (1974) jump model. CBMs combine advantages of both the structural and reduced-form models, as they are based on observables reflecting balance sheet information (credit ratings) and market data (CDS spreads).

In this study, I consider the CBMs framework, which is based on functional calculus and lattices, as discussed in Albanese, Chen, Dalessandro and Vidler (2006). These are useful techniques that have been largely remained unused due to the lack of attention to the quantitative problems arising in social sciences by mathematical physicists (Albanese, 2005d). Some of the disadvantages of CBMs and DTCs is the lack of development of multi-factor dynamic correlation structures and the absence of a regression function which is re-estimated at each node of the time-dependent path of the credit quality process.

The CBM functional calculus framework can be used to find prices and hedge ratios for CDOs with more than hundred names, by establishing a dynamic correlation among the distance to default processes of the individual obligors and macroeconomic financial observables (Albanese and Chen, 2005). Furthermore, CBMs carry out a greater spectrum of information by taking into account the trading management constraints for CDOs (Albanese,
2005d). In addition, they are analytically simple and numerically tractable, as the estimation is based on linear algebra algorithms. There is no need to use MCS techniques when evaluating the CDO profit and loss distribution. The DTCs for CDO pricing offer an alternative than the copulas way in modelling correlation structures. DTCs are estimated by regression analysis based on the movement of financial indices.

There is no published academic paper utilizing the CBMs and DCTs frameworks in constructing a pricing model for SME CDOs. This is a rich analytical framework; I believe that by using functional calculus the pricing of SME CDOs is done in the light of a constructive, functional analytic and intuitive approach, while it overcomes the MCS noise and the non-existence of SME ratings.
Chapter 2

Credit Barrier Models

Section 2.1 demonstrates the translation of the SME obligor’s credit quality process to a Markov generator, whereas section 2.2 shows the derivation and subordination of the pricing kernel of the SME credit quality process over the transformed lattice representation.

2.1 Markov Generator

2.1.1 Specification

The analysis starts by denoting the credit quality process of an individual SME obligor over a time span \( t \) with the variable \( \xi_t \). This process represents a relative measure of the SME obligor’s leverage to the volatility of its assets’ value over time.

Then, I translate the credit quality process \( \xi_t \) to a Markov chain process \( X_t \) over the lattice \( \Omega \) within the interval \([0, 1]\). The discretization step is given by \( h \) and the last lattice point by \( hN \), where \( h \not= 0 \) and \( N \) is a positive integer. Hence, the lattice may be specified in the following terms

\[
\Omega = \{0, h, \ldots, hN\} \subset [0, 1]
\]  

To achieve the lattice specification in (2.1), I consider a discrete Markov operator \( \ell \) with individual elements \( \ell(x, y) \), describing the probability density function of moving from a point \( x \) to a point \( y \) on the credit quality scale for times \( t \) to \( t' \). In specific, the discrete Markov operator is characterized by the properties (2.1.1)-(2.1.4).

**Property 2.1.1** \( \ell \) is a finite tridiagonal matrix of size \( N \times N \).

**Property 2.1.2** \( \ell \) is positive definite and \( \ell(x, y) \geq 0 \) if \( x \neq y \).
Property 2.1.3 \( \ell \) is probabilistic conservative so \( \sum \ell(x,y) = 0 \ \forall x, y. \)

Property 2.1.4 The diagonal elements of \( \ell \) sum to the negative of all others \( \ell(x,x) = -\sum_{y \neq x} \ell(x,y). \)

Then, by using the forward finite-difference and Laplace operators respectively

\[
\nabla^h f(x) = \frac{f(x + h) - f(x)}{h} \\
\Delta^h f(x) = \frac{f(x + h) + f(x - h) - 2f(x)}{h^2}
\]

the infinitesimal Markov operator is specified as follows

\[
\ell_h = a(x) \Delta^h + \left[ b(x) - a(x) \right] \nabla^h
\]

where,

\[
a(x) = \frac{\sigma(x)^2}{2} \\
b(x) = \mu(x)
\]

is the volatility and drift of (2.4) respectively.

In order to ensure a continuous time-space limit for the lattice \( \Omega \), the functions \( a(x) \), and \( b(x) \) are derived from a state-dependent drift function \( \mu(x_t) \) and local volatility \( \sigma(x_t) \) respectively. The derivation of (2.5) and (2.6), which is shown in appendix A.1, is implemented by introducing a non-linear transformation of the process \( x_t \)

\[
F(x_t,t)
\]

The introduction of (2.7) makes it possible to define the Markov generator \( \ell \) in accordance with the fundamental theorem of finance (FTF). The proof of theorem 2.1 can be found in appendix A.2.

**Theorem 2.1.1** The Fundamental Theorem of Finance (FTF). Consider a Markov generator \( \ell \) with properties (2.1.1)-(2.1.4) and let \( F(x_t,t) \) be an adapted non-linear transformation of the process \( x_t \), as it is shown in (2.7). The transformed process is called \( \ell \)-martingale if and only if the propagator of \( \ell \) is

\[
\frac{\partial F}{\partial t} + \ell F = 0
\]

Consider a family of adapted asset price processes \( u_i(x,y) \), where \( i = 0, \ldots, N \). This family is locally complete at the point \( (x,y) \) if the set of differences

\[
\zeta_i(y) = \left( u_i(y, W(x,y)) - u(x,y) \right)
\]
for \( i = 1, \ldots, N \) is complete in the vector space \( \vee \) generated by the eigenfunctions \( \eta \) of \( \ell \). Then, there exists a measure change \( G(x, y, x', y', t) \) such that the discounted asset price processes \( u_i(x, y) \) are \( \ell_G \)-martingales. Moreover, if the family of of assets \( u_i(x, y), i = 0, \ldots, N \) is locally complete at the point \((x, y)\), then the function \( G(x, y, x', y', t) \) is uniquely specified at that point.

Basically, the FTF says that martingale processes are the ones of most direct relevance to the pricing theory (Delbaen and Schachermeyer, 1994). In the functional calculus framework, the arbitrage freedom argument is achieved by applying the following translational invariant conditions on \( \ell \):

**Condition 2.1.1 Drift condition**

\[
E_t \left[ f(x_t, t) \right] = \mu(hx) = \sum_y \ell(x, y) \left( F(y) - F(x) \right).
\]

**Condition 2.1.2 Volatility condition**

\[
\text{Var} \left[ f(x_t, t) \right] = \sigma(hx)^2 = \sum_y \ell(x, y) \left( F(y) - F(x) \right)^2.
\]

**Condition 2.1.3 Probability conservation condition**

\[
\sum_y \ell(x, y) = 0.
\]

Furthermore, the conditions (2.1.1)-(2.1.3) correspond to each sub-diagonal of \( \ell \). So, by assuming that \( a_1, \ldots, a_N \) and \( b_1, \ldots, b_N \) are elements of the matrix \( \ell \) then

\[
\ell = \begin{pmatrix}
-b_0 & b_0 & 0 & 0 & \cdots & 0 \\
-a_1 + b_1 & b_1 & 0 & \cdots & 0 \\
0 & a_2 & -(a_2 + b_2) & b_2 & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & a_{N-1} & -(a_{N-1} + b_{N-1}) & b_{N-1} \\
0 & 0 & \cdots & 0 & a_N & -a_N
\end{pmatrix} \tag{2.10}
\]

Hence, the conditions (2.1.1)-(2.1.3) are incorporated within the Markov operator by solving them as a simultaneous linear system, consisting of the equations (2.11)-(2.13), as it is shown below

\[
\sum_y \ell(x, y) \left( F(y) - F(x) \right) = \mu(hx) \tag{2.11}
\]

\[
\sum_y \ell(x, y) = \sigma(hx)^2 \tag{2.12}
\]
\[
\sum_y \ell(x, y) = 0 \tag{2.13}
\]

The solutions to the system (2.11)-(2.13) are the elements \(a_1, \ldots, a_N\) and \(b_1, \ldots, b_N\) of the tridiagonal matrix \(\ell\), as it is specified in (2.10).

The conditions (2.1.1)-(2.1.3) hold under both the statistical \((P)\) and risk-neutral \((Q)\) measures. In the functional calculus framework, the change of measure from \(P\) to \(Q\) is defined in terms of the theorem (2.1.2). A sketchy proof of this theorem is shown in appendix A.3.

**Theorem 2.1.2** Let \(\ell\) be a Markov generator with properties (2.1.1)-(2.1.4) and let \(G\) be a numeraire. The numeraire is a positive function such that

\[
\ell_G(t) = \frac{\partial G(t)}{\partial t} + \frac{1}{G(t)} \ell(t) G(t) \tag{2.14}
\]

The numeraire \(G(t)\) defines a numeraire-changed generator by the following equation

\[
\ell_G(t) = \frac{\partial G(t)}{\partial t} + \frac{1}{G(t)} \ell(t) G(t) \tag{2.15}
\]

Then, the propagator \(F_G(t, t')\) for the measure-changed process \(\ell_G(t)\) is given by

\[
F_G(t, t') = \frac{1}{G(t')} F(t, t') G(t) \tag{2.16}
\]

In essence, the change of measure in theorem (2.1.2) is the Radon-Nikodym derivative with respect to \(P\), which is specified in general terms as

\[
\frac{dQ_t}{dP_t} \tag{2.17}
\]

The above change of measure from \(P\) to \(Q\) is performed through a risk-neutralizing drift, which is describe by a suitable matrix \(A_h(x, y)\) such that

\[
\ell_Q(x, y) = \ell_P(x, y) A_h(x, y) - I(x, y) \sum \ell_P(x, y) A_h(x, y) \tag{2.18}
\]

where, \(I(x, y)\) is the \(N \times N\) identity matrix. In (2.9) the function \(A_h(x, y)\) imposes that the conditions (2.1.1)-(2.1.3) are robust with respect to \(h\) and hence the FTF is satisfied.
2.1.2 Diagonalisation

To manipulate the Markov generator \( \ell \) by means of functional calculus, it is necessary to diagonalize it. The properties of \( \ell \), as described in (2.1.1)-(2.1.4), guarantees that \( \ell \) is diagonalizable and admits a complete set of \( N \) eigenvalues \( \lambda_1, \ldots, \lambda_N \).

The diagonalization of \( \ell \) proceeds into thee steps. Firstly, I find the \( N \) linearly independent eigenvectors \( \eta_1, \ldots, \eta_N \) corresponding to the eigenvalues \( \lambda_1, \ldots, \lambda_N \) respectively; secondly, I form the matrix \( U \) which has as its columns the eigenvectors \( \eta_1, \ldots, \eta_N \); thirdly, the specified matrix \( \ell = U \Lambda U^{-1} \) will be diagonal with \( \lambda_1, \ldots, \lambda_N \) as its successive entries. Hence, by elementary linear algebra

\[
\ell \eta = \lambda \eta
\]  

(2.19)

Then, denoting with \( \text{det} \) the determinant of a square matrix, the eigenvalues of \( \ell \) are the solutions to the characteristic equation

\[
\text{det}(\ell - \lambda \eta) = 0
\]  

(2.20)

By solving the above equation, the Markov operator \( \ell \) is expressed as a diagonal matrix such that

\[
\ell = U \Lambda V
\]  

(2.21)

where \( V = U^{-1} \).

Given that \( \ell \) is diagonalizable, then it is possible to make use of an important property of functional calculus. However, this is done by applying an arbitrary function \( F \) to \( \ell \). Hence, the Taylor’s expansion of \( F(\ell) \) is given by

\[
F(\ell) = a_0 + a_1 \ell + a_2 \ell^2 + \ldots \text{where} \ a_1, a_2, \ldots \text{are real numbers}
\]  

(2.22)

Therefore, substitution for (2.21) into (2.22) gives the expression

\[
F(\ell) = a_0 + a_1 U \Lambda U^{-1} + a_2 U \Lambda U^{-1} U \Lambda U^{-1} + \ldots \Rightarrow
\]

\[
F(\ell) = U \left[ a_0 + a_1 \Lambda + a_2 \Lambda^2 + \ldots \right] U^{-1}
\]  

(2.23)

Now let the expression \( a_0 + a_1 \Lambda + a_2 \Lambda^2 + \ldots \) which is broken down into

\[
a_0 + a_1 \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \lambda_N
\end{pmatrix} + a_2 \begin{pmatrix}
\lambda_1^2 & 0 & \cdots & 0 \\
0 & \lambda_2^2 & \cdots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \lambda_N^2
\end{pmatrix} + \ldots
\]  

(2.24)
be transformed in terms of a matrix operator $F(\Lambda)$ such that
\begin{equation}
    a_0 + a_1 \Lambda + a_2 \Lambda^2 + \ldots = F(\Lambda) \tag{2.25}
\end{equation}
Hence, (2.23) becomes,
\begin{equation}
    F(\ell) = UF(\Lambda)V \tag{2.26}
\end{equation}
The expression in (2.26) is the cornerstone of functional calculus as the Ito’s lemma is the cornerstone of stochastic calculus, when deriving the pricing kernel (Albanese, 2005). Section 2.2 shows how to apply (2.26) in order to derive the pricing kernel with functional calculus.

2.2 Pricing Kernel

2.2.1 Derivation

Suppose that the arbitrary function $F()$ is specified in terms of the exponential function. Then,
\begin{equation}
    F = \exp(t) \tag{2.27}
\end{equation}
The exponential function is chosen, because of its special mathematical properties and algebraic tractability. These properties are helpful in constructing a meaningful pricing kernel.

Hence, substituting (2.27) into (2.26) results into
\begin{equation}
    \exp(t\ell)(x,y) = U\exp(t\Lambda)(x,y)V \tag{2.28}
\end{equation}
After that, I express the right-hand side (RHS) of (2.28) in terms of summation notation which gives
\begin{equation}
    U\exp(t\Lambda)(x,y)V = \sum_n c_n(x)v_n(y)\exp(t\lambda)(x,y) \tag{2.29}
\end{equation}
Hence, (2.28) becomes
\begin{equation}
    \exp(t\ell)(x,y) = \sum_n \eta_n(x)v_n(y)\exp(t\lambda)(x,y) \tag{2.30}
\end{equation}
Then, let
\begin{equation}
    u(x,t;y,t') = \exp((t'-t)\ell)(x,y) \tag{2.31}
\end{equation}
be the probability kernel that the credit quality process of an individual SME obligor changes from $x_t$ to $y_{t'}$ for times $t$ to $t'$. 

16
So, combining (2.30) and (2.31) the probability kernel can be expressed as follows

\[
u(x, t; y, t') = \sum_n \eta_n(x)v_n(y)e^{(t' - t)\lambda}(x, y)
\]

(2.32)

I claim that the expression in (2.32) is the pricing kernel of the SME credit quality process over the lattice \( \Omega \), since it satisfies the following set of properties (proof in appendix A.4).

**Property 2.2.1** \( u \) is non-negative \( \exp(t\ell(x, y)) \geq 0 \).

**Property 2.2.2** \( u \) is probabilistic conservative \( \sum_y \exp(t\ell(x, y)) = 1, \) if \( x \neq y \).

**Property 2.2.3** \( u \) satisfies the Chapman-Kolmogorov equation

\[
\exp(t\ell(x, y)) = \sum_z \exp(t\ell(x, z)) \cdot \exp((t' - t)\ell(x, z))
\]

2.2.2 Jump Subordination

In order to incorporate jumps within the SME pricing kernel, I add a drift term \( c(x) \) in the Markov operator in (2.4). Hence, it now becomes

\[
\ell_h = a(x) \Delta^h + \left[ b(x) - a(x) \right] \nabla^h_+ + c(x) \nabla^h_+
\]

(2.33)

To reflect asymmetries in the jump intensities, I model separately up and down jumps. This class of jump processes is associated to stochastic time changes, which are given by non-decreasing processes \( T_t \) with independent increments, as defined below.

**Definition 2.2.1** A stochastic time change process on a complete filtration \( \mathcal{F}_T \) is defined as a right-continuous non-decreasing process starting from zero and with values in \( [0, \infty) \), where \( \infty \) is an absorbing point. The lifetime of \( T_t \) in terms of calendar time is defined as \( \eta = \inf\{t \geq 0\} \). In addition, the financial time coordinate, which is denoted as \( s = T_t \) is interpreted as the total number of credit events up to a certain calendar time.

Hence, from the above definition model jumps are considered as being rapid advances in financial activity with respect to the calendar time, meaning that the relative pace at which the financial time proceeds varies stochastically.

There is a special class of time change processes for which the associated stochastic time changes are described by Bochner subordinators. The following definition elaborates on this (Albanese and Kuznetsov, 2005; Bohner, 1995).
Definition 2.2.2 Let $T_t$ be a stochastic time change process, as it is specified in definition 2.2.1. Then, $T_t$ is called a Bochner subordinator if it has independent and homogeneous increments on $[0, \eta)$, that is, $T_{t+s} - T_t$ is independent of the filtration $\mathcal{F}_t$ and has the same distribution as $T_s$. The subordinator $T_t$ can be associated to a so-called renewal measure $\rho_t(dx)$ such that

$$E_0\left[\int_0^\infty f(T_t)\right] = \int_0^\infty f(x)\rho_t(dx)dt,$$

for all continuous functions of compact support $f(x)$.

Thus, by applying definition (2.2.2), $T_t$ is a Bochner subordinator if and only if

$$E_0\left[\exp(-\lambda T_t)\right] = \exp(-\phi(\lambda)t)$$

(2.34)

In this project, I use the variance-gamma process, which is a special case of a Bochner subordinator in order to incorporate the jumps into the pricing kernel in (2.32). The introduction of jumps in credit risk models is necessary given the sharp advances in financial activity through the realization of credit events within the underlying portfolio of SME obligors. In addition, I use the variance-gamma process, because is a standard and well-known jump model in the literature (Madan and Seneta, 1990, Madan et al, 1998).

The distribution of the subordinator $T_t$ when using the variance-gamma process is given by

$$\rho_t(ds) = \left(\frac{\mu}{\nu}\right)^\alpha \Gamma(\alpha)^{-1}s^{\alpha-1}\exp\left(-\frac{\mu}{\nu}s\right)ds$$

(2.35)

where, $\mu$ and $\nu$ is the mean and variance rate respectively, $\alpha = \frac{\nu^2}{\nu}$ and $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}\exp(-t)dt$ is the Gamma function.

Albanese and Kuznetsov (2005) show that the corresponding Bernstein function is given by

$$\phi(\lambda) = \frac{\mu^2}{\nu} \log\left(1 + \lambda \frac{\mu}{\nu}\right)$$

(2.36)

In addition, Philips (1952) shows that if $\ell$ is the Markov generator of the credit quality process $x_t$ and $\phi(\lambda)$ the Bernstein function for the subordinator $T_t$, then the Markov generator $\tilde{\ell}$ of the subordinated credit quality process $x_{T_t}$ is given by

$$\tilde{\ell} = -\phi(-\ell)$$

(2.37)

To produce asymmetric jumps, I specify the $\mu$ and $\nu$ differently for the up (+) and down (−) jumps and compute two Markov generators separately such as

$$\tilde{\ell}_\pm = -\phi_\pm(-\ell) = -U_\pm \phi(-\Lambda_\pm) V_\pm$$

(2.38)
and from (2.36) yields

$$
\phi_{\pm}(\ell) = \frac{\mu^2_{\pm}}{\nu_{\pm}} \log \left(1 + \ell \frac{\mu_{\pm}}{\nu_{\pm}} \right)
$$

(2.39)

**Definition 2.2.3** Let \( \theta_n(x) \), where \( n = 0, 1, \ldots \), be a family of eigenfunctions \( \phi_{\pm}(\ell) \), where \( \phi(\cdot) \) is the Bernstein function and \( \lambda_{n,\pm} \) are the corresponding eigenvalues. In addition, let \( \theta_n(x) \) be a complete orthonormal basis for some measure \( \kappa(dx) \) of the space which spans the subordinated Markov operator \( \tilde{\ell} \).

Then, using definition (2.2.3) the operator \( \tilde{\ell} \) satisfies

$$
\tilde{\ell} \theta_n(x) = -\phi_{\pm}(-\lambda_n)\theta_n(x)
$$

(2.40)

With similar arguments as in section 2.1, the equation

$$
\exp(t\tilde{\ell})\theta_n(x) = \exp(-t\phi_{\pm}(-\lambda_n))\theta_n(x)
$$

(2.41)

holds. Hence, the new generator of the subordinated process with asymmetric jumps \((\pm)\) is obtained by combining the two generators \( \ell_+ \) and \( \ell_- \). This operations results in the following matrix

$$
\tilde{\ell} = \begin{pmatrix}
0 & \ldots & \ldots & \ldots & 0 \\
\tilde{\ell}_-(2, 1) & \tilde{d}(2, 2) & \tilde{\ell}_+(2, 3) & \ldots & \tilde{\ell}_+(2, n) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\tilde{\ell}_-(n - 1, 1) & \tilde{\ell}_-(n - 1, 2) & \ldots & \tilde{d}(n - 1, n - 2) & \tilde{\ell}_+(n - 1, n) \\
0 & \ldots & \ldots & \ldots & 0
\end{pmatrix}
$$

(2.42)

The elements of the upper and lower boundary of (2.42) have been assigned to a zero value. This is to ensure that there is either no probability leakage in the process or reflection at the boundary. Moreover, the diagonal elements of (2.42) are chosen to satisfying the condition of probability conservation, that is,

$$
\tilde{d}(x, x) = -\sum_{y \neq x} \tilde{\ell}(x, y)
$$

(2.43)

In overall, the addition of jumps has not increased the dimensionality of the linear algebra problem, which still remains computationally efficient. At this stage of the construction, I obtain a subordinated generator \( \tilde{\ell} \) for the process of distance to default for the SME obligor, whose dynamics are characterized by a combination of state-dependent local volatility and asymmetric variance-gamma jumps.
Finally, by using similar arguments as in (2.32), the subordinated SME pricing kernel is given by

\[
\tilde{u}(x; t; y, t') = \exp \left( (t' - t) \tilde{\ell}_\pm \right) (x, y) = \sum_n \exp \left( - \phi_{\pm} (t' - t) \lambda_n \right) \theta_n(x) \theta_n(y)
\]

(2.44)
Chapter 3
Dynamic Correlation

Section 3.1 shows how to construct a dynamic correlation structure for SME CDOs and section 3.2 how to incorporate the dynamic correlation structure into the CBM pricing kernel.

3.1 Intuition and Construction

3.1.1 Intuition

Conventionally, the term "correlation" means the relationship between the creditworthiness of the reference names within the CDO underlying portfolio. This is a strong factor which should definitely be taken into account when constructing the CDO pricing kernel. This is because a default by one of the reference names could trigger a downgrade or default of another reference name in the basket causing, in turn, a sharp downgrade of the credit quality of the CDO.

However, this is a rather narrow and insensible way of defining a correlation structure for SME CDOs. For instance, consider two SME reference names on a vertical production chain, where SME-1 sells large proportions of products to SME-2. If SME-2 defaults, assuming low recovery rates, then SME-1 will loose a significant amount of turnover which theoretically may drive its own downgrading or (under the worst case scenario) default. Typically, SME CDOs are well-diversified, since they are based on thousands of different loans of relatively small sizes. Hence, it is unrealistic to argue that, in practice, the default of an SME obligor may cause the downgrading of the CDO. Hence, incorporating the conventional narrow correlation structure within the SME CDO pricing kernel would be inadequate.

Therefore, in this paper, I approach the correlation interdependencies
rather differently. In essence, I consider the modelling of wider correlation interdependencies. The reasons are mainly the facts that, firstly, SMEs are exposed to significant (in a statistical sense) business sector-specific shocks which are mainly driven by the viability of large corporations; and secondly, SMEs are more exposed to macroeconomic conditions than large firms (EC, 2004).

In order to materialize the first point I consider the following example. Assume that a subset of SME CDO reference names are service suppliers of a large corporation. If the large corporation defaults and assuming low recovery rates, then there is empirical evidence that this particular subset of SMEs is likely to be downgraded or even defaulted (Davydenko and Franks, 2004). Hence, the default of a large corporation with close credit relationships with SME names is more likely to induce stronger downgrades of the SME CDO than the default of a single SME reference name itself.

Besides, the second point is also valid, since macroeconometric research indicates that SMEs are more exposed to macroeconomic shocks than large enterprises due to the financial accelerator effect (Bernanke et al., 1996). The financial accelerator is the link between the cost (premium) of external finance and the net worth (capitalization) of the potential borrower. The lower the net worth of the potential borrower, the higher the dependence on bank loans (external finance) is and hence the dependence on the macroeconomic cycle of the banking system. This is indeed a stylized fact which persists for European SMEs, since SMEs by default have low capitalization (Franks and Sussman, 2002).

Therefore, when constructing the correlation structure, it is plausible to make assumption that the default of an SME reference name is not only contingent on the viability of the other SME names (reference names), but also on the financial viability of large corporations and the macroeconomic environment in general. For this reason I propose to follow Albanese, Chen and Dalessandro (2005) and correlate the creditworthiness of the SME reference names to financial observables, which are leading indicators of the macroeconomic cyclicality. This method takes into account the correlation sensitivities of all economic players (SMEs and large entities), including both the narrow (SME CDO names) and wide correlation (business cycle) effect. Hence, incorporating both the internal- and external-correlation sensitivities into a dynamic correlation function is a key factor for the CDO pricing process.

A way to incorporate this complex correlation structure within the SME CDO pricing kernel is to construct a dynamic correlation function. Unfortunately, the copula method does note offer a flexible mathematical framework to construct this (Albanese, Chen and Dalessandro, 2005). Therefore, I con-
sider conditioning the correlation performance of the credit quality of the individual SME obligors on the benchmark iTraxx CDS index for 5- and 10-year term to maturity. The iTraxx CDS index provides the CDS spreads over time of the most liquid 125 names of the European credit market divided into the standard tranches. Hence, it is a financial observable, which it can be thought of as being a proxy for the creditworthiness of the European credit markets and the business cycle (Amato, and Gyntelberg, 2005). The iTraxx index, which was created in 2004, is owned by a group of the largest global investment banks under the name International Index Company. There are important benefits of considering a dynamic correlation function based on the iTraxx CDS index like,

- It promotes standardisation within the SME structured finance industry.
- It introduces greater transparency of the pricing methodology.
- It raises the investors’ trust on the way of pricing SME securitization structures, as they are aware of the credit risk assumptions, particularly for the synthetic CDOs which are more risky than the true-sale ones (Kalckreuth, 2001).

Unfortunately, in this study I consider a one-factor dynamic correlation structure. Despite that CDO multi-factor dynamic correlation structures have not yet been considered within the CBM framework, their implementation is possible and it would be a subject for further work (Albanese, 2005). For instance, in the multi-factor dynamic correlation function, the correlation factors could be the iTraxx sectoral indices. This is important for SMEs, as there are empirical arguments supporting the fact that SMEs in different sectors (e.g. cyclical versus non-cyclical) have different responses to various shock intensities (EIB, 2005).

### 3.1.2 Construction

This section describes the mathematical modelling of the dynamic correlation structure. The procedure is similar as in Albanese, Chen and Dalessandro (2005). The credit quality processes of the individual SME obligors, which are mapped into the continuous time lattice $\Omega$ are conditioned on the iTraxx CDS index. Then, the movements of the iTraxx CDS index are mapped into a binary non-recombing tree forming various probabilistic scenarios over time. The probabilistic scenarios are characterised by a set of complementary
transition probabilities $w$ and $(1 - w)$, $w \in [0, 1]$, remaining constant at each node of the iTraxx CDS index path.

These probabilistic scenarios are combined with a local regression beta function. This method of conditioning allows the avoidance of the noisy MCS methods in the pricing and calculation of the various hedge ratios for SME synthetic CDOs.

Particularly, I make the local beta function to be dependent on the credit quality variable $\xi$. Moreover, I choose for the function $\xi$ an increasing term structure, which captures the stylised fact that higher-rated SME names are more correlated to the economic cycle than lower-rated ones (Henneke and Truck, 2005). Thus, the term structure of the local beta function is specified as taking values from zero to one, that is,

$$\beta(\xi) \in [0, 1]$$

where, the lower limiting case $\beta(\xi) = 0$ implies zero correlation sensitivity between the iTraxx index and an SME name with credit quality process $hx \in [0, 1]$; the middle scenarios $\beta(\xi) \in (0, 1)$ imply partial correlation sensitivity between the iTraxx index and an SME name with credit quality process $hx \in [0, 1]$; and the upper limiting case $\beta(\xi) = 1$ implies full correlation sensitivity between the iTraxx index and an SME name with credit quality process $hx \in [0, 1]$.

Furthermore, I choose a time step of $(t - t') = \Delta t = 1y$. This choice has proven sufficient in providing greater flexibility in tuning the correlation structure. Moreover, the implied index variable takes two values on each period $\Delta t$. An upturn (+) corresponds to a positive movement of the iTraxx index, whereas a downturn (−) corresponds to a negative one.

3.2 The Conditional Pricing Kernel

3.2.1 Derivation

Within this framework, the synthetic tranches of varying seniority for SME CDOs are priced by calibrating the local beta function in (3.1). Hence, for a given scenario along the tree path, the unconditional variance-gamma subordinated pricing kernel in (2.44) is conditioned on the iTraxx financial observables through the local beta function as follows:

$$\tilde{u}_{w, \beta}^{\pm}(t, x; t + \Delta t, y) = (1 - \beta(hx))\tilde{u}_0(t, x; t + \Delta t, y) + \beta(hx)\tilde{u}_{1}^{\pm}(t, x; t + \Delta t, y)$$

(3.2)
In the case where the correlation structure is zero, that is,
\[ \beta(hx) = 0 \] (3.3)
(3.2) casts down to \( \tilde{u}_0(t, x, t+\Delta t, y) \), which is the unconditional pricing kernel in (2.44). In the case where there is full correlation structure, that is,
\[ \beta(hx) = 1 \] (3.4)
the functions \( \tilde{u}_1^+(x, y) \) and \( \tilde{u}_1^-(x, y) \) of the right-hand side of (3.2) takes the form in expression (3.5) and (3.6) respectively,
\[ \tilde{u}_1^+(x, y) = \frac{1}{1-w} \begin{cases} \tilde{u}(x, y) & \text{if } y > m(w, x) \\ \left( w - \sum_{y>m(w, x)} \tilde{u}(x, \zeta(w, x)) \right) & \text{if } y = m(w, x) \\ \tilde{u}(x, y) = 0 & \text{if } y < \zeta(w, x) \end{cases} \] (3.5)
and
\[ \tilde{u}_1^-(x, y) = \frac{1}{w} \begin{cases} 0 & \text{if } y > m(w, x) \\ \tilde{u}(x, m(w, x)) - \tilde{u}_1^+(x, m(w, x)) & \text{if } y = m(w, x) \\ \tilde{u}(x, y) & \text{if } y < \zeta(w, x) \end{cases} \] (3.6)
where,
\[ m(w, x) = \inf \{ m = 0, \ldots, N \mid \sum_{y<m(w, x)} \tilde{u}(x, y) \leq w \} \] (3.7)
Note that for all specifications of (3.1) and (3.2)
\[ \tilde{u}(x, y) = w\tilde{u}_\beta^+(x, y) + (1-w)\tilde{u}_\beta^-(x, y) \] (3.8)
The overall conditioning of the pricing kernel is achieved by forming a weighted summation over all correlation paths in the event tree, as it is depicted in figure [8].

Figure [8]. The dynamic correlation structure

As figure [8] illustrates, on a given path I use \( w_{i|\ell^t}^+ \) and \( w_{i|\ell^t}^- \) for a good (bad) period scenario respectively, corresponding to an upward (downward) movement of the iTraxx CDS index. The weight of a path is the product of a number of factors \( w \) equal to the number of bad periods and a number of factors \( (1-w) \) for each one of the good periods. Thus, marginal default probabilities of the individual SME obligors are kept unchanged, while correlations are induced dynamically into the SME single-name processes.
Moreover, I evaluate the conditional prices \( P_\Gamma \) by using a multi-period function

\[
P_\Gamma = \exp(t_i - t_{i-1})\tilde{\ell}_{\gamma_i} \cdot \ldots \cdot \exp(t_n - t_{n-1})\tilde{\ell}_{\gamma_n}
\]

(3.9)

where, the process \( \Gamma = \{\gamma_1, \ldots, \gamma_n\} \) runs over the sets of conditional paths based on the index’s scenario. Hence, the pricing kernel of the SME CLO tranches is obtained by evaluating

\[
P_{\text{SMECLO Tranche}_j} = \sum_{\Gamma} w^{n-(\Gamma)}(1 - w)^{n+(\Gamma)} P_\Gamma
\]

(3.10)

where \( j = 1, \ldots, 5 \) denotes the tranche seniority ranging from the equity to the senior tranche and assuming that the mezzanine tranche is broken down into three subcategories. The respective attachment-detachment points are taken to be the percentiles 0-3, 3-6, 6-9, 9-12 and 12-22. The super-senior tranche ranges from 22-100%.

In addition, the combination of (3.9) and (3.10) gives the conditional pricing kernel, which is shown below

\[
P_{\text{SMECLO Tranche}_j} = \sum_{\Gamma} w^{n-(\Gamma)}(1 - w)^{n+(\Gamma)} \left[ \exp(t_i - t_{i-1})\tilde{\ell}_{\gamma_i} \cdot \ldots \cdot \exp(t_n - t_{n-1})\tilde{\ell}_{\gamma_n} \right]
\]

(3.11)

The above construction process of the pricing kernel can be generalised. Consider a number \( M > 1 \) of percentile levels \( 0 < w_1 < \ldots < w_M < 1 \) and let \( q_i \in [0, 1], i = 1, \ldots, M \) be a corresponding set of probabilities summing up to unity, that is, \( \sum_i q_i = 1 \). So, the pricing kernel, which is subordinated on the variance-gamma process and is conditioned on the generalised dynamic correlation structures, is given by

\[
\hat{\bar{\mu}}_{w,\beta}^\pm(t, x, t + \Delta t, y) = \sum_{i=1}^M q_i \hat{\bar{\mu}}_{w_i,\beta}^\pm(t, x; t + \Delta t, y)
\]

(3.12)

Furthermore, the formula in (3.11) is extended to the above generalized case as long as the weight \( w \) is replaced by the average weight \( \sum_i q_i w_i \). Hence,

\[
P_{\text{SMECLO Tranche}_j} = \sum_{\Gamma} \sum_i q_i w_i^{n-(\Gamma)}(1 - w_i)^{n+(\Gamma)} \left[ \exp(t_i - t_{i-1})\tilde{\ell}_{\gamma_i} \cdot \ldots \cdot \exp(t_n - t_{n-1})\tilde{\ell}_{\gamma_n} \right]
\]

(3.13)

### 3.2.2 Contagion Skew

The local beta function has a particular specification: It is an increasing function of the credit quality variable. A useful way to assess the impact of the chosen specification of (3.2) on the correlation model (3.12) is the contagion skew.
Definition 3.2.1 The contagion skew $\Psi$ is the ratio of the unconditional to conditional SME default probabilities.

Intuitively, the contagion skew shows the intensity of the probability of default when assuming a correlation of the individual SME obligor with the iTraxx CDS index, as compared to the unconditional probability of default for the same SME obligor.

The contagion skew is constructed by first computing the unconditional default probabilities as a function of credit quality. Next, for each value of credit quality, assuming that a name of that quality defaults within a time-horizon of 5 years, I compute the conditional probability of defaults for all other names. Finally, I average over all values of credit quality and taking the ratio between the conditional and unconditional probabilities. In mathematical terms, the contagion skew is expressed as follows

$$\psi\left(\tilde{u}, \tilde{u}^\pm \right) = \frac{\tilde{u}(t, x; t + \Delta t, 0)}{\tilde{u}_{w,0}^\pm(t, x, t + \Delta t, 0)}$$  \hspace{1cm} (3.14)

where, $\tilde{u}_P$ is defined in (2.44) and $\tilde{u}^\pm$ is defined in (3.8).
Chapter 4

Markov Estimation and Statistical Calibration

Section 4.1 shows how to estimate the Markov generator. This forms the basis for section 4.2, which performs the statistical calibration of the CBM.

4.1 Estimation of the Markov Generator

4.1.1 Lattice Representation

An SME credit rating system denotes current opinion of an SMEs obligor’s overall financial capacity (its creditworthiness) to pay its financial obligations. In this research paper, I use the Standard & Poor’s SME credit ratings, where S&P’s opinion focuses on the SMEs obligor’s capacity and willingness to meet its financial commitments as they come due (Standard and Poor’s, 2003b and 2003c). It does not apply to any specific financial obligation, as it does not take into account the nature of and provisions of the obligation, its standing in bankruptcy or liquidation, statutory preferences, or the legality and enforceability of the obligation.

Consider a credit rating system which consists of $K$ classes. In the case of the Standard and Poor’s extended credit rating system $K = 22$. Hence, the representation of the credit quality variable over the lattice $\Omega$ in (2.1) is discretized as follows

$$\{0, 1, 2, \ldots, 22\} \leftrightarrow \{Default, \ldots, C-, \ldots, CCC+, \ldots, BB, \ldots, AA+, AAA\}$$

Moreover, the nodes of the lattice $\Omega$ are subdivided into $K = 22$ subintervals of adjacent nodes $0 = x_0 < x_1 < \ldots < x_K$ for $i = 1, \ldots, K$ such
that

\[ I_i = [x_{i-1}, \ldots, x_i] \]  \hspace{1cm} (4.2)

Intuitively, the interval \( I_i \) corresponds to the \( i \)th credit rating class. So, if the process of an individual SME obligor is in \( I_i \) at time \( t \), then at this time-point the process has a credit rating of \( i \). To enhance numerical tractability, I specify the barycentre of the population density \( \bar{x} \) in a particular credit class to be the mid-point of \( I_i \). Moreover, the representation in (4.1) maps the SME credit quality process to a homogeneous lattice. Hence, in order to have equal distances between the various credit rating classes the following condition is imposed on (4.2)

\[ \#(x_i - x_{i-1}) = \frac{N}{K} \]  \hspace{1cm} (4.3)

### 4.1.2 Numerical Estimation

The numerical estimation of the Markov generator is carried out on Lapack (Linear Algebra PACKage), which is a publicly available collection of Fortran77 routines. When Lapack is installed locally, it becomes part of the programming libraries of Visual Studio.net. Below, I describe the estimation steps in more detail.

In order to specify the elements of the tridiagonal Markov operator, I solve the linear system (2.11)-(2.13) by utilizing a Gaussian elimination algorithm known as the Thomas Algorithm (Thomas, 1949). The algorithm provides an efficient way of solving tridiagonal linear systems, which arise from finite-difference approximations to partial differential equations (Thomas, 1949). Table [1] presents the estimated Markov generator (2.10) before introducing the jumps. Table [1] contains the tridiagonal elements \( a_x \) and \( b_x \), as they are specified in (2.10).

After that, I proceed with the diagonalization of the estimated Markov operator of table [1]. The last column of table [1] shows the estimated eigenvalues, when diagonalizing the Markov generator (2.10) by applying the diagonalization procedure of section 2.1.2. When solving this numerically, I use the professional numerical diagonalization algorithm \( \text{dgeev} \) from the Lapack library in Visual Studio.net.

The estimated Markov operator admits a complete set of eigenvectors. This is true in general, but in some cases diagonalization might be impossible, as the Markov operator can be reduced at most up to a non-trivial Jordan form with non-zero off-diagonal elements. In practical terms, if non-diagonalizable Markov operators arise, then \( \text{dgeev} \) detects the problem and rectifies it by providing a small perturbation of the CBM's parameters.
Moreover, the estimated Markov operator (2.42) with variance-gamma subordinated elements is presented in table [2]. The subordination has been performed as described in section 2.2.2. Basically, the empirical subordination is conducted by applying (2.37) on each one of the elements of the estimated unsubordinated Matrix generator in (1).

4.2 Calibration under the Statistical Measure

Section 4.2.1 presents the estimated structures of the parameters when calibrating the CBM on transition probabilities (section 4.2.2) and PDs (section 4.2.3).

4.2.1 Estimation of Parameters

Intuitively, the statistical measure is the calibration of the CBM model on historical SME credit ratings data. It is very important for a CDO model to match the historical dimension of the creditworthiness of the SME obligors in order to be a well-informative tool for pricing CDO products.

The drift function $\mu_P(\xi)$, which is graphed in figure [9], becomes larger in absolute terms when the credit quality improves.

Figure [9]. The local drift function

The state-dependent volatility function $\sigma(\xi)$ is shown in figure [10]. As expected, it diminishes as the rating quality improves across the SME rating class.

Figure [10]. The local volatility function

The estimated variance rates of the variance-gamma process (2.35), which are $\nu_+ = 9.5$ and $\nu_- = 6.0$, are relatively large compared to the time-horizon of the historical data. This means that large jump amplitudes are required to carry out the calibration. Given that the jump amplitudes are affected by the state-dependent volatility, the larger jumps occur within the rating categories of lower credit quality.

4.2.2 Calibration on Default Probabilities

An SME obligor which is rated "Selective Default" (SD) or "Default" (D) means that it failed to meet one or more of its (rated or unrated) financial obligations when it became due (Standard and Poor’s, 2002). The SME
PD data has been kindly released by Standard and Poor’s for the research purposes of this thesis. A "D" rating is assigned when Standard and Poor’s believes that the default will be a general default and that the SME obligor will fail to pay all or substantially all of its obligations as they become due. In the lattice representation I treat the "SD" class as one of the rest rating classes, since it is assigned to an SME obligor for which Standard and Poor’s believes that it will continue to meet its payment obligations in a timely manner.

Hence, absorption into the state \( x = 0 \) from the lattice representation (4.1) is interpreted as the occurrence of default. In mathematical terms, the probability starting from the initial rating \( i \) and reaching a default state by time \( t \) is given by

\[
\bar{p}_i(t) = \tilde{u}_P(0, \bar{x}; t, 0)
\]  

(4.4)

where, \( \tilde{u}_P \) is defined in (2.44).

For modelling the SME historical default probabilities I use the discrete credit curves for one-, three- and five-year term to maturities based on cohorts of SME historically observed default frequencies. Given the high importance of the default probabilities in determining credit risk models, I use PDs of more than one-year term to maturity.

The Standard and Poor’s PDs are estimated by using the SME Credit Risk Tracker (Standard and Poor’s, 2004b). The SME Credit Risk Tracker aggregates the PD by rating class from a large pool of SMEs, which is distributed across Europe. In the case where a banking institution utilizes the Internal Ratings-based (IRB) approach in estimating PDs, then Standard and Poor’s maps the internal credit rating scale to the Standard and Poor’s scale by employing actuarial techniques (Standard and Poor’s, 2002). The Standard and Poor’s PDs for European SMEs are shown in table [3].

Table [3]. European SMEs PDs by rating class and term to maturity

Figure [11] presents a comparison of the model equation (4.4) (lines) and historical (dots) default probabilities of table [3]. It is easy to observe that the CBM model has a reasonable fit on historical PD assumptions. However, the longer the term to maturity the less accurate the fit become.

Figure [11]. Calibration of the CBM on SME 1, 3 and 5-year PDs

4.2.3 Calibration on Transition Probabilities

The transition (or migration) probability is the probability that an SME obligor migrates from one credit rating to another by taking either upward
or downward direction. For instance, the credit quality of an SME obligor might be downgraded by the rating agency, when it becomes more susceptible to adverse effects of changes in circumstances and economic conditions than other SME obligors in the same rating class.

When using the lattice representation in (4.1), the transition probability that an obligor with a given initial rating $i$ at time $t = 0$ will have a rating $j$ at a later time $t > 0$ is expressed by

$$
\tilde{p}_{ij} = \sum_{y = a_{j-1}}^{a_{j-1}} \tilde{u}_P \left( 0, \tilde{x}_i; t, y \right)
$$

(4.5)

where, $u_P$ is defined as in (2.44) and $y \in \Omega$ is the credit quality state of the SME reference name after the transition of the credit rating class from $i$ to $j$.

The transition probability in (4.5) can be estimated by matching it with historical averages provided by credit assessment institutions. I use the one-year transition probability matrix for European structured finance transactions for the period 1987-2004 taken from Standard and Poor’s (2003a). The one-year transition matrix is presented in table [4].

Table [4]. One-year Transition Matrix for European CDOs (1987-2004)

As table [4] shows, many of the rating classes are not presented due to the fact that there have not been enough transactions lying within this class and hence Standard and Poor’s does not provide an aggregation over these classes. In addition, I do not use the three- and five-year transition matrices since the data is too noisy for the low probability events and taken over shorter time period than the one-year matrix.

Due to data limitations I use the general matrix for European structure finance since a matrix for European SME structured finance is inexistent. However, this is a sensible choice as the realized losses for the (non-publicly) rated transactions are very low and to date have ranged from 0% to 0.2% of the principal balance on the SME portfolio basis (Standard and Poor’s, 2002).

The stable performance of the SME CDO transactions is due primarily to the reasonably well-diversified underlying loans within each CDO and generally small obligor concentrations. For instance, in some SME CDO securitizations the number of reference loans have excited 25,000, like in the ABN AMRO’s SMILE and Deutche Bank’s PROMISE-Z 2001-1 PLC, whereas the median of all Standard and Poor’s rated SME CDO transactions is 2,500 loans (Standard and Poor’s, 2002b).
The estimation results of calibrating the CBM on the Standard and Poor’s transition matrix are displayed in figure [12].

Figure [12]. Calibration of the CBM on the transition matrix

Figure [12] presents a line-plot of the implied transition probabilities as they are extracted from the CBM in estimating (4.5) and a dot-plot for the one-year historical transition probabilities as shown in table [4]. As it can be seen, the implied transition probabilities are reasonably closed to the historical ones. Hence, the CBM has also good fit on the transition matrix of the aggregate creditworthiness of European CDOs.
Chapter 5

Risk-Neutral Calibration and Pricing

5.1 The PROMISE Mobility 2005-1 SME CLO

Section 5.1 presents a public SME CLO transaction which forms the basis for developing the material in the subsequent sections; section 5.2 calibrates the CBM model under the risk-neutral measure, whereas section 5.3 shows how to price the synthetic tranches and derive the hedge ratios under various assumptions.

5.1.1 Background to the Transaction

The structure of this transaction, under the name Mobility 2005-1, is based on KfW’s SME (Mittelstand, in German language) securitisation platform, called programme for mittelstand loan securitisation (PROMISE). Under the PROMISE platform, KfW acts as intermediary for the synthetic transfer of the credit risk of the SME reference portfolio. The transaction has been rated by both FitchRatings and Moody’s, where both agencies issued a pre-sale report (FitchRatings, 2005b; Moody’s, 2005). The transaction’s date is March 1st 2005.

On the asset side, the initial underlying SME portfolio consists of 365 facilities. These are basically loans, syndicates, revolving credits and guarantees totalling approximately EUR 710m to 265 SME obligors in Germany. The largest loan is EUR 6m and longest maturity is 5 years. The credit-linked notes (CLNs), which are going to be transferred to investors, have been issued by the PROMISE I MOBILITY 2005-1, which is a bankruptcy-remote SPV based in Dublin (Moody’s, 2005).
On the liability side, the classes A+ to F CLNs have been secured by Schuldscheine, initially issued by KfW, and purchased by the issuer from KfW on the closing date (KfW, 2005a). The arranger and underwriter is Deutsche Bank AG, whereas Deloitte & Touche GmbH is acting as the trustee. IKB, which is a regional bank in Germany, is the originator of the CLO. Hence, IKB enters into a CDS with KfW for the reimbursement of losses on its SME underlying portfolio. KfW, in turn, have sought protection against the same risk by issuing the Schuldscheine to be purchased by the issuer with the notes proceeds and by entering into the super-senior CDS swap of EUR 650 m with Deutsche Bank (FitchRatings, 2005b). Figure [13] presents the diagram of the transaction.

Figure [13]. The structure of the PROMISE Mobility 2005-1 SME CLO

5.1.2 Structural Data

Both the investors’ report (Kfw, 2005d) and the rating agencies’ pre-sale reports provide information on a number of structural data of the PROMISE Mobility I-2005 transaction. The term to maturity of the CLN spreads is 5 years, whereas the term structure of recovery rates ranges from 45% to 55%. Table [5] summarizes both the CLN spreads (bps) and recovery rates (%), after aggregating the 365 SME reference names by rating class.

Table [5]. CLN spreads and recovery rates by rating class

Moreover, from the various transaction reports (KfW, 2005d; KfW, 2005c) I obtain information about the capital structure of the CLO. In specific, the initial reference portfolio of the asset side consists of 94% investment graded SMEs, that is, of credit quality BBB and above. This is a rather strong credit quality, even when comparing with CLOs backed by loans to large corporations (Standard and Poor’s 2003; EIB, 2005). Table [6] presents the names’ composition and capital structure by rating class.

Table [6] Composition of names and capital structure by rating class

Moreover, the asset side is highly diversified, across the various business sectors. In particular, the highest market share is 16% for the "industrial and manufacturing" industry and 13% for the "food, beverage and tobacco". The lowest market share comes from the "textiles and furniture" and "transportation" categories with 4% each (FitchRatings, 2005b).

It should be noted that it is a common practice in the rating agencies industry that the pre-sale reports should be as conservative as possible. So,
in reality it might be possible to observe higher recovery rates or even tighten
CLN spreads than the ones which are mentioned in the FitchRatings (2005b)
and Moody’s (2005) pre-sale reports. However, for the purposes of this thesis,
I use the structural data of tables [5] and [6].

Hence, based on the structural data of tables [5] and [6], I develop a the-
oretical spreadsheet to illustrate the applicability of the pricing techniques
of chapters 2 and 3. This spreadsheet, which is included in the accompanied
CD, contains information about the 365 individual SME obligors on the CDS
spread (based on the CLN spreads), recovery rates (based on the term struc-
ture of recovery rates), notional amounts (based on the capital structure)
and rating (based on the rating classification at the aggregate level). Then,
I use this spreadsheet to perform the estimations. The results are presented
in the subsequent sections.

5.2 Risk-Neutral Calibration

Section 5.2.1 presents the intuition behind the model specification for the
risk-neutral estimation. Section 5.2.2 performs the calibration on the CLN
spreads.

5.2.1 Model Specification

Intuitively, the risk-neutral calibration gives the fair pricing measure of the
CDO tranches. The fair price is the lowest spread (price) that the SPV
KfW could sustain when transferring the synthetic slices of credit risk of
the SME portfolio (still under the legal ownership of the originator IKB) to
investors. If investors sell protection (or demand this investment product) at
this risk-neutral price, then this is referred to as being a fair CDS or CLN
swap deal.

It should also be noted that the risk-neutral pricing measure is sensitive
to the model selection (e.g. CBM versus reduced-form model) and param-
eters’ specification. As Walker (2004) and Albanese, Chen and Dalessandro
(2005) points out in a very apposite way, buyers and sellers of CDOs have
a wide range of arbitrage-free prices to choose from and it is the market
that determines, both in principle and in practice, a definite price. Hence,
the risk-neutral pricing might not be (and in most cases is not) the actual
pricing of the transaction.

Besides, actual prices are determined after negotiations with the investors
and the counterparty of the senior CDS. However, it is particularly important
that quantitative analysts calculate a set of risk-neutral prices as a guide-
line for the CDO structuring and sales people when negotiating the most favourable price from either the investors or the SPV’s point of view.

According to the fundamental theorem of finance (2.1.1) and the change of measure theorem (2.1.2) the model under the risk-neutral measure is specified by switching the Markov operator (2.33) from a real-world to a risk-neutral drift $\mu_Q(x)$. The risk-neutral drift measure is chosen to fit the CDS spread curves of the individual SME obligors within the reference portfolio of the PROMISE Mobility 2005-1 CDO transaction. The estimated risk-neutral drift is shown in figure [14].

Figure [14]. The risk-neutral drift function

When performing the calibration I use an inhomogeneous lattice as opposed to the homogeneous lattice of chapter 4. This is due to the fact that the risk-neutral estimation is not performed at the aggregate but the individual SME obligor’s level. In this PROMISE securitization transaction, there are 365 SME assets hence, it was necessary to construct a lattice $\Omega$ with 12,000 lattice points. The estimation is carried on Mathpoint. Mathpoint is an autonomous library on Visual Studio.net which allows for the lattice discretization and the subsequent estimations.

5.2.2 Calibration on CLN spread curves

The SME names share the same process specification, but differ in their credit quality’s starting point, recovery rates’ term structure and spreads. This is equivalent as having a risk-neutralizing drift, where its term structure is adapted to match the observed CDS spread curves of the individual SME obligors.

In order to show how the model is calibrated on CLN spreads, I run various scenarios for the CDS spreads. These scenarios are based on the tranche CLN spreads of table [5]. Hence, I make the implicit assumption that the CLN spreads for term to maturities 4-, 3-, 2- and 1-year are given by the CLN spread when the term to maturity is 5 years and using a downscale factor of 90%, 80%, 65% and 50% respectively. The aggregate CLN spread curves for each of the rating classes are graphed in figure [15].

Figure [15]. The term to maturity of the CLN spreads by rating class

To achieve the matching of the term structure of the CLN spreads in figure [15], I take the liberty of adjusting the term structure of the recovery rate for each one of the 365 SME facilities in the reference portfolio. The objective is to ensure that the term structures of individual SME recovery
rates are as flat as possible and as close as possible to the tentative input value of the actual recovery rates of table [5]. Figure [16] presents the implied recovery cycles for the 365 SME facilities (grouped by rating class) as at the transaction’s initiation date, that is, March 1st 2005.

Figure [16]. The term structure of implied versus actual recovery rates

I observe that the implied term-structures of recover rates are highly correlated across the SME facilities and to the general spread level; unsurprisingly, as recovery levels are known to be linked to the economic cycle (Albanese, Chen and Dalessandro, 2005). This is because the aggregation over rating class is very similar across SME names and that the implied term structure is reasonably flat.

In order to derive the implied term structure of recovery rates I use the implied correlation from the iTraxx CDS index on the March 1st 2005. The correlation matrix which is of size $145 \times 145$ can be seen in the accompanied CD.

The above exercise leads to the estimation of the distance to default for each of the individual SME names: the distance to default becomes smaller when the rating class becomes of lower quality. This is shown in figure [17].

Figure [17]. The distance to default by rating class

5.3 Pricing of the PROMISE CLO Tranches

5.3.1 Dynamic Correlation Structure

The empirical dynamic correlation structure is introduced as per equation (2.12). The structure of the local $\beta$ function when calibrating the pricing model on the transaction’s initiation date is graphed in figure [18]. As the graph shows, the higher the credit quality of a defaulted name is, the large the impact on all other names becomes. The steepness of this curve controls precisely the discrepancy between prices for senior tranches as compared to junior ones.

Figure [18]. Specification of the local beta function, March 1st 2005

The specifications of the weight functions $w_i$ and probability functions $q_i$ are shown in figure [19].

Figure [19]. Specifications of the weights and percentile levels

The contagion skew is graphed in figure [20].

Figure [20]. The contagion skew
5.3.2 The Profit and Loss Distribution

The pricing of the CLO, requires the generation of the probability distribution function (PandL) for the CLO portfolios over the given time horizon (Merino and Nyfeler, 2002). In this case the time period is March 1st 2005. The PandL distribution is derived by estimating the equation (3.13), while making some additional assumptions. Specifically, I assume that there is an upfront fee of 19.25%. In addition, I assume that the unwind barrier (trading management constraint) is either 0% or 5%. Figures [21] and figure [22] present the PandL distributions on the March 1st 2005 with an upfront fee with an upfront fee of 19.25%, an unwind barrier of 0% and 5% respectively.

Figure [21]. The PandL distribution of the CLO, unwind barrier 0%
Figure [22]. The PandL distribution of the CLO, unwind barrier 10.5%

The blue lines correspond to the various attachment and detachment points of the PandL distribution. If the accumulated loss breaches the attachment point, the principal of the tranche will be reduced and so will the interest cash flow. In the worst case, the buyer loses the entire principal and thus all future cash flows.

From the above figures, it is possible to observe that the PandL distribution with no unwind barrier presents a clustering of risk around the detachments and attachment points. This means that there exist a high degree of sensitivity around the tranches of various seniority. Moreover, it is an unstable distribution with five clusters and it is not reduced monotonically as it is expected. However, the PandL distribution presents higher peaks at the lower seniority tranches, when introducing the management constraint of an unwind barrier of 10.5%. In addition, it is more stable than the PandL distribution with no unwind barrier, while it reduces to zero monotonically.

Moreover, the multi-peak PandL distributions tend to the Normal distribution when the time horizon expands and the quality of the CDO deteriorates. This is shown in figure [23], where the PandL distribution is plotted when assuming a time horizon of one to ten years of CDO defaults. The PandL maintains peaks up to year three, but from the year four and onwards the PandL distribution of the CDO tends to the Normal distribution.

Figure [23]. The PandL distribution of the CLO, unwind barrier 0% and time-horizons 1-10 years

5.3.3 Tranche Prices and Hedge Ratios

The model can be calibrated by adjusting the function \( \beta(\xi) \), the thresholds \( w_i \), and the corresponding probabilities \( q_i \) through equations (3.12) and (3.13).
For example, choosing \( q_i = Nw_i^\alpha \) with \( \alpha = 1 \) and the beta function graphed in figure [18], I obtain the tranche prices which are shown in tables [7] and [8].

Moreover, both tables [7] and [8] contain the benchmark CDS spreads for the index. As it can be seen, the CDS spreads for the high seniority tranches are much higher than those of the index when the PandL distribution does not include management constraints. However, when management constraints are incorporated within the PandL distribution of the CLO, then the CDS spreads are closer to the index’s benchmark ones. Furthermore, the maximum loss when the PandL does not impose management constraints is higher than the maximum loss when the PandL has an unwind barrier of 10.5%. Therefore, the incorporation of management constraints within the CLO pricing kernel is an important factor for the risk-neutral pricing of the tranches.

The delta hedge ratios are calculated by re-estimating the PandL distribution after increasing the CDS spreads of the individual SME obligors by one bps. Then, I take the ratio between the CDS spreads of the initial PandL and the re-estimated one. This is a rather simple way of calculating hedge ratios; however, it is a way to illustrate how the model can be used to compute the delta hedge ratios. Table [9] presents the hedge ratios for the various tranches, when using the distribution with the unwind barrier of 10.5%.

I observe that the hedge ratios for the lower seniority tranches are higher than those of higher seniority. In addition, the high values of the hedge ratios indicate the degree of unstableness of the PandL distribution when there is no management constraint.
Chapter 6

Conclusion

Section 6.1 gives the concluding remarks, whereas section 6.2 makes suggestions for additional work on this research paper.

6.1 Concluding Remarks

This paper developed a genuine application of the framework in Albanese and Chen (2005c), Albanese, Chen and Dalessandro (2005) and Albanese, Chen, Dalessandro and Vidler (2006) for pricing synthetic tranches of SME CDOs. This framework is rich and flexible enough to accommodate both jumps and dynamic correlation structures within the credit quality process of the SME obligors and hence the CDO pricing kernel.

In addition, the mathematical components enter the framework in an intuitive way. Firstly, it was shown that the construction of the CBM satisfies both the fundamental theorem of finance and measure changes. Secondly, the framework is justified from an economic perspective and hence it takes on board special issues, arising in the SME structured finance, like the complex correlation interrelationships of the SMEs with large enterprises and financial observables and the fact that the asset side of an SME CDO consists of hundreds of reference names.

Then, the CBM model has been calibrated under both the statistical (real-world) and pricing (risk-neutral) measure. As it has been shown, the CBM has a reasonable fit on historical credit ratings data for SMEs. Hence, the CBM can replicate sharp advances in financial activity.

After that, the CBM and the dynamic correlation techniques have been combined to price an actual SME synthetic CDO from the German securitization market. The framework has been successful in pricing the different tranches with or without trading management constraints, by using the
implied correlation structure of the iTraxx CDS index on the transaction’s initiation date. Hence, this pricing framework offers the potential of marking to market the CDO transaction given the strong liquidisation of the iTraxx CDS index. Moreover, the model is so flexible that allows the calculation of the CDO hedge ratios.

In overall, this SME CDO pricing framework is realistic in the sense that it allows for the pricing of actual SME CDO transactions with hundreds of reference names. It is also possible to allow for even more complex trading and regulatory assumptions than the ones which have been used in this version of the paper. This modelling framework, can be used to price other exotic SME CDO structures. For instance, it can extract prices for SME CDO squared transactions, where each underlying credit risk is itself another SME CDO tranche.

6.2 Further Work

The development of this SME CDO pricing framework is still in its infancy. Therefore, there is potential to improve in order to make it even more realistic, while maintaining the mathematical rigorosity of its components. Below, I suggest a number of improvements and further validation measures that I will incorporate in subsequent versions of this research paper.

- To perform a mark-to-market analysis by calibrating the risk-neutral pricing kernel (with or without unwind barriers) for different points in time. For example, May 1st 2005 and August 1st 2005 so that to obtain the risk-neutral prices after 3 and 6 months. Of course, the reference portfolio on those days might be different than the initial portfolio on the March 1st 2005, or it might even be unnecessary that the CDO transaction should be kept mark-to-market.

- To re-estimate the beta regression coefficients which links the credit quality process to the economic cycle through financial observables, by accommodating for more complex and customized cycle scenarios.

- To consider dynamic correlation structures which are conditional to business-sector specific shocks, by developing a multi-factor dynamic correlation function. This is possible by using the iTraxx CDS sectoral indices. Indeed, research conducted by the European Commission states that the proportion of SMEs is doubled in business sectors which are known to be countercyclical (Observatory, 2003). This means that
considering the size of the reference names of a CDO portfolio unconditional of the business sector can therefore lead to the conclusion that firm size is a strong indicator of CDO correlation even though it is merely a proxy for a business sector.

- To compare the CDS spreads which are derived when using a one-factor Gaussian copula with the ones derived when using a one-factor dynamic correlation function.

- To construct a dynamic correlation function which is based on an index of creditworthiness of the SME reference portfolios on the asset side of the CDO and compare the results when conditioning on the iTraxx CDS index.
Appendix A

Mathematical Proofs

A.1 Derivation of the drift and volatility of the Markov process

Assume that there exist a non-linear transformation $F(x; t)$ as in (2.7) over the filtration $x$. Then, the drift $\mu(x, t)$ is obtained as follows

$$
\mu_F(x, t) = \frac{d}{dt}E_t[F(x', t')] = \frac{d}{dt}E_t\sum_{x'} u(x, t; x', t') F(x', t') \Rightarrow
$$

$$
\mu_F(x, t) = \sum_{x'} \frac{\partial u}{\partial t'} + u(x, t; x', t') \frac{\partial u}{\partial t'} (x, t; x', t') F(x', t') + u(x, t; x', t') \frac{\partial F}{\partial t'} (x', t')
$$

(A.1)

By using the Kolmogorov equation, where $\tau$ is the transport matrix operator, produces

$$
\frac{\partial u}{\partial t'} |_{t' = 0} = \ell' u(x, t; x', t') = \ell u(x, t; x, t')
$$

(A.2)

Then, by substituting (A.2) into (A.1) gives

$$
\mu_F(x, t) = \sum_{x'} \ell u(x, t; x', t') F(x', t') + u(x, t; x', t') \frac{\partial F}{\partial t'} (x', t')
$$

(A.3)

whereas, substituting (A.3) into (A.2) results into

$$
\mu_F(x, t) = \mu(x, t) \nabla_h F(\cdot) + \frac{\sigma(x, t)^2}{2} \Delta^h F(\cdot) + \frac{\partial F}{\partial t}(\cdot)
$$

(A.4)

After that, by taking and solving the following limits (L’Hospital’s rule)

$$
\lim_{h \to 0} \nabla_h F(x, t) = \lim_{h \to 0} \frac{F(x + h, t) - F(x, t)}{h} = \frac{\partial F}{\partial x}(x, t)
$$

(A.5)
\[ \lim_{h \to 0} \nabla_h F(x, t) = \lim_{h \to 0} \frac{F(x + h, t) + F(x - h, t) - 2F(x, t)}{h^2} = \frac{\partial^2 F}{\partial x^2}(x, t) \] (A.6)

the expression in (A.4) becomes

\[ \mu_F(x, t) = \mu(x, t) \frac{\partial F}{\partial x}(x, t) + \frac{\sigma(x, t)^2}{2} \frac{\partial^2 F}{\partial x^2}(x, t) + \frac{\partial F}{\partial t}(x, t) \] (A.7)

The derivation of the volatility function follows similar lines. Basically, the argument is the following

\[ \sigma_F(x, t)^2 = \frac{d}{dt'} |_{t'=t} E_t \left[ \left( F(x', t') - F(x, t) \right) \right]^2 = \]
\[ = \frac{d}{dt'} |_{t'=t} \sum_{x = x' \pm h} u(x, t; x', t') \left( F(x', t') - F(x, t) \right)^2 = \]
\[ = \sum_{x = x' \pm h} \ell(x, t; x', t') \left( F(x', t') - F(x, t) \right)^2 \] (A.8)

Then, by using the Taylor’s expansion for infinitesimal \( h \)

\[ F(x \pm h, t) - F(x, t) = \frac{\partial F}{\partial x}(x, t) h + o(h^2) \] (A.9)

After substituting into (A.8) both the infinitesimal Markov operator (2.4) and (A.9)

\[ \frac{1}{h} \mu(x, t) \left[ f(x + h, t) - F(x, t) \right]^2 + \]
\[ + \frac{1}{2h^2} \sigma(x, t)^2 \left[ \left( F(x + h, t) - F(x, t) \right)^2 + \left( F(x - h, t) - F(x, t) \right)^2 \right] \] (A.10)

Then, by taking the limit \( h \downarrow 0 \) of (A.10)

\[ \lim_{h \to 0} \frac{1}{h} \mu(x, t) \left[ F(x + h, t) - F(x, t) \right]^2 + \]
\[ + \lim_{h \to 0} \frac{1}{2h^2} \sigma(x, t)^2 \left[ \left( F(x + h, t) - F(x, t) \right)^2 + \left( F(x - h, t) - F(x, t) \right)^2 \right] = \]
\[ = 0 + \lim_{h \to 0} \frac{1}{2h^2} \sigma(x, t)^2 \left( \frac{\partial F}{\partial x} \right)^2 h^2 = \]
\[ = \sigma(x, t)^2 \left( \frac{\partial F}{\partial x} \right)^2 \] (A.11)
From (A.7) and (A.11), I obtain precisely the drift and volatility respectively of an infinitesimal change in $F(x; t)$ which is given by the Ito’s lemma, where $B$ is a standard Brownian motion.

$$dF_t = \left[ \frac{\partial F(x; t)}{\partial x} \mu(x; t) + \frac{\sigma(x; t)^2}{2} \frac{\partial^2 F(x; t)}{\partial x^2} + \frac{\partial F(x; t)}{\partial t} \right] dt + \frac{\partial F(x; t)}{\partial x} \sigma(x; t) dB_t$$

(A.12)

The above expression (A.12) is obtainable when the Markov operator is defined in terms of stochastic calculus, that is,

$$dx_t = \mu(x; t) dt + \sigma(x; t) dB$$

(A.13)

Therefore, I have shown that the specification of the volatility and drift in (2.5) and (2.6) respectively is equivalent as specifying the Markov operator in terms of stochastic calculus.

### A.2 A heuristic proof of the fundamental theorem of finance (2.1.1) with functional calculus

Suppose that there exists no arbitrage and fix a time $t$. Then, there does not exist a trading strategy with a strictly increasing value process (Albanese, 2005). Namely, for all vectors $r_i, i = 1, \ldots, n$ there are two elements $y_+, y_- \in \ell$ such that

$$\sum_{i=0}^{n} r_i \left( \bar{u}_i(y_+, W(x; y_+, v)) - \bar{u}_i(x; v) \right) > 0$$

(A.14)

while

$$\sum_{i=0}^{n} r_i \left( \bar{u}_i(y_-, W(x; y_-, v)) - \bar{u}_i(x; v) \right) < 0$$

(A.15)

Consider the differences in (2.9) as vectors whose components are labelled by $y \in \ell$. These vectors span the vector space $\ell$. Due to the hypothesis of absence of arbitrage, these vectors do not belong to the positive octant $K$ singled out by all vectors with positive component. If $i = 1$, the hyperplane $\Pi_1$ orthogonal to the vector $\zeta_1(y)$ will intersect $K$. Let $P_1$ be the orthogonal projection operator onto the hyperplane $\Pi_1$ and let us consider the vector

$$(\Pi_1 \zeta_2(y)) = \zeta_2(y) - \frac{\sum_{z \in \ell} \zeta_2(z) \zeta_1(z)}{\sum_{z \in \ell} \zeta_1(z) \zeta_1(z)} \zeta_1(y)$$

(A.16)

for $i = 1, \ldots, n$. Due to the absence of arbitrage, the hyperplane $\Pi_2 \subset \Pi_1 \cup \ell$ orthogonal to both vectors $\zeta_1$ and $\zeta_2$, also intersects the octant $K$. 

46
The above argument can be iterated \( n \) times, leading to the conclusion that for each \( x \in \Omega \), there exists a function \( g_x(y, t), y \in \ell \) such that

\[
\sum_{y \in \ell} \left( \bar{u}_i(y, W(x, y, v)) - \bar{u}_i(x, v) \right) g_x(y, t) = 0
\]  

(A.17)

for all \( i = 0, \ldots, n \). In particular, this implies that there exists a function \( G_x(y, t) \) such that

\[
\sum_{y \in \Omega} \left( \bar{u}_i(y, W(x, y, v)) - \bar{u}_i(x, v) \right) \ell(x, y) G_x(y, t) = 0
\]  

(A.18)

### A.3 Proof of the measure change theorem (2.1.2)

The proof is as follows (Albanese, 2005b)

\[
\frac{\partial F_G}{\partial t} = - \frac{1}{G^2} \frac{\partial G}{\partial t} FG - \frac{1}{G} \ell FG = \\
= - \frac{1}{G} \frac{\partial G}{\partial t} FG - \frac{1}{G} \ell GF_G = \ell G F_G
\]  

(A.19)

I then set,

\[
F_G(t_0, t_N) = \frac{1}{G(t_0) - 0} \prod_{i=0}^{N-2} \left[ F(t_i, t_{i+1}) \frac{G(t_i - 0)}{G(t_i + 0)} \right] F(t_{N-1}, t_N) G(t_N + 0)
\]  

(A.20)

Taking the limit of (A.20) as \( N \to 0 \)

\[
\frac{G(t_i - 0)}{G(t_i + 0)} = 1 - \frac{G(t_i + 0) - G(t_i - 0)}{G(t_i + 0)}
\]  

(A.21)

which leads to obtaining the Markov generator (2.15).

### A.4 Proof of the pricing kernel’s properties (2.2.1)-(2.2.3)

**Proof of Property (2.2.1).** Consider the following when assuming that \( x, y, z \in \Omega \) where \( x = y \) and \( \Delta t = (t' - t) \) is an infinitesimal change in time

\[
\exp\left( (t + \Delta t) \ell(x, y) \right) = \exp\left( t \ell(x, y) \right) \exp\left( \Delta t \ell(x, y) \right) = 
\]
\[
\begin{align*}
&= \left( \exp(t \ell) \exp(\Delta t \ell)(x, y) = \sum_z \exp(t \ell(x, z)) \exp(\Delta t \ell(z, y)) = \\
&= \sum_z \exp(t \ell(x, y))(\Delta z + \ell(z, y) \Delta t) = \\
&= \exp(t \ell(x, y) + \sum_z \exp(t \ell(x, y) \ell(z, y) \Delta t)) = \\
&\sum_{z \neq y} \exp(t \ell(x, z) \ell(z, y) \Delta t) + \exp(t \ell(x, y)) \left[ 1 + \ell(y, y) \Delta t - \sum_{z \neq y} \ell(y, z) \right] \quad (A.22)
\end{align*}
\]

By the properties (2.1.2)-(2.1.4) of the Markov operator, the following terms become

\[
\begin{align*}
\exp(t \ell(x, z) \ell(z, y) \Delta t) &\geq 0 \quad (A.23) \\
\exp(t \ell(x, y)) &\geq 0 \quad (A.24)
\end{align*}
\]

Moreover, since \( \Delta t \) is infinitesimal and \( \ell(y, y) \) is finite, hence,

\[
0 \geq \sum_{z \neq y} \ell(y, z) \leq \ell(y, y) \Delta t \quad (A.25)
\]

\[
\ell(y, y) \Delta t - \sum_{z \neq y} \ell(y, z) \geq 0 \quad (A.26)
\]

**Proof of Property (2.2.2).** I start by considering the following

\[
\frac{d}{dt} \sum_y \exp(t \ell)(x, y) = \sum_y \frac{d}{dt} \exp(t \ell)(x, y) = \quad (A.27)
\]

Then, by using the Taylor’s expansion (A.27) becomes

\[
\sum_y \frac{d}{dt} \left( 1 + t \ell + \frac{t^2}{2!} \ell + \ldots \right)(x, y) \quad (A.28)
\]

Then, after carrying on with the calculations and applying the property 2.1.3 of the Markov operator it is possible to get

\[
\sum_z \ell(x, z) \cdot 1 = \sum_z \ell(x, z) = 0 \quad (A.29)
\]

**Proof of Property (2.2.3).** This is rather straightforward. Adapting the exponential functions gives

\[
\exp(t' \ell(x, y)) = \exp \left( t \ell(x, z) + t' \ell(x, y) - t \ell(x, y) \right) = \\
= \sum_z \exp \left( t \ell(x, z) \right) \cdot \exp \left( (t' - t) \ell(x, y) \right) \quad (A.30)
\]
Appendix B

Tables and Figures

Figure [1]. A credit default swap transaction

Protection Buyer → Premium → Protection Seller
- Contingent credit
- Event payment

Figure [2]. The global credit derivatives market trend

USD bn

Source: BBA (2002) and ISDA (2005)
Figure [3]. Market shares of credit derivative products, 2003

Source: BBA Credit Derivatives Report 2003-04

Figure [4]. A general CDO securitization process

Source: Jobst (2005)
Figure [5]. European SME CDO issuance trends


Figure [6]. The European SME CDO issuance by pool location, 2003

Source: KfW Bankengruppe (2005)
Figure [7]. Credit default assumptions for European SMEs and LargeCaps

Note: Figure is adapted to the Standard & Poor’s rating scale.
Source: Standard & Poor’s (2003a; 2004c) and Moody’s (2003)

Figure [8]. The dynamic correlation structure
Figure [9]. The local drift function under the statistical measure

Figure [10]: The local volatility function under the statistical measure
Figure [11]. Calibration of the CBM on SME 1, 3 and 5-year PDs

Figure [12]. Calibration of the CBM on the transition matrix
Figure [13]. The structure of the PROMISE Mobility 2005-1 SME CLO


Figure [14]. The risk-neutral drift function

ratings
Figure [15]. The term to maturity of the CLN spreads by rating class

Figure [16]. The term structure of implied versus actual recovery rates

Figure [17]. The distance to default by rating class

Figure [18]. Specification of the local beta function, March 1st 2005
Figure [19]. Specifications of the weights and percentile levels

Figure [20]. The contagion skew
Figure [21]. The PandL distribution of the CLO, unwind barrier 0%
Figure [23]. The PandL distribution of the CLO, unwind barrier 0% and time horizons 1-10 years

Loss distribution over 1 year

Loss distribution over 2 years

Loss distribution over 3 years
<table>
<thead>
<tr>
<th>$a_x$</th>
<th>$-(a_x + b_x)$</th>
<th>$b_x$</th>
<th>eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15000</td>
<td>-0.20000</td>
<td>5.00E-02</td>
<td>-2.0999</td>
</tr>
<tr>
<td>0.15</td>
<td>-0.325</td>
<td>0.1750</td>
<td>-1.3909</td>
</tr>
<tr>
<td>7.50E-02</td>
<td>1.75E-01</td>
<td>0.1</td>
<td>-1.2342</td>
</tr>
<tr>
<td>2.00E-01</td>
<td>-4.25E-01</td>
<td>0.225</td>
<td>-1.0401</td>
</tr>
<tr>
<td>1.25E-01</td>
<td>-3.00E-01</td>
<td>0.175</td>
<td>-0.9545</td>
</tr>
<tr>
<td>0.275</td>
<td>-0.575</td>
<td>0.3</td>
<td>-0.8142</td>
</tr>
<tr>
<td>0.2</td>
<td>0.525</td>
<td>0.325</td>
<td>-0.7027</td>
</tr>
<tr>
<td>0.225</td>
<td>-0.575</td>
<td>0.35</td>
<td>-0.6012</td>
</tr>
<tr>
<td>0.25</td>
<td>-0.525</td>
<td>0.275</td>
<td>-0.5367</td>
</tr>
<tr>
<td>0.375</td>
<td>-0.775</td>
<td>0.4</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.725</td>
<td>0.425</td>
<td>-0.0030</td>
</tr>
<tr>
<td>0.325</td>
<td>-0.675</td>
<td>0.35</td>
<td>-0.0205</td>
</tr>
<tr>
<td>0.45</td>
<td>-0.925</td>
<td>0.475</td>
<td>-0.0415</td>
</tr>
<tr>
<td>0.375</td>
<td>-0.775</td>
<td>0.4</td>
<td>-0.4121</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.925</td>
<td>0.425</td>
<td>-0.3747</td>
</tr>
<tr>
<td>0.525</td>
<td>-1.075</td>
<td>0.55</td>
<td>-0.0792</td>
</tr>
<tr>
<td>0.45</td>
<td>-0.925</td>
<td>0.475</td>
<td>-0.1302</td>
</tr>
<tr>
<td>0.575</td>
<td>-1.075</td>
<td>0.5</td>
<td>-0.2963</td>
</tr>
<tr>
<td>0.6</td>
<td>-1.125</td>
<td>0.525</td>
<td>-0.1876</td>
</tr>
<tr>
<td>0.625</td>
<td>-1.275</td>
<td>0.65</td>
<td>-1.5551</td>
</tr>
<tr>
<td>0.55</td>
<td>-0.55</td>
<td></td>
<td>-0.2177</td>
</tr>
<tr>
<td>Row</td>
<td>Elements</td>
<td>Estimated Elements</td>
<td></td>
</tr>
<tr>
<td>-----</td>
<td>------------------</td>
<td>--------------------</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Λ(1,1)-Λ(11,1)</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Λ(12,1)-Λ(21,1)</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Λ(1,2)-Λ(11,2)</td>
<td>3.6E-06</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Λ(12,2)-Λ(21,2)</td>
<td>2.6E-08</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Λ(1,3)-Λ(11,3)</td>
<td>2.8E-06</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Λ(12,3)-Λ(21,3)</td>
<td>2.0E-08</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Λ(1,4)-Λ(11,4)</td>
<td>2.5E-06</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Λ(12,4)-Λ(21,4)</td>
<td>1.8E-08</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Λ(1,5)-Λ(11,5)</td>
<td>2.1E-06</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Λ(12,5)-Λ(21,5)</td>
<td>1.5E-08</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Λ(1,6)-Λ(11,6)</td>
<td>1.9E-06</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Λ(12,6)-Λ(21,6)</td>
<td>1.4E-08</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Λ(1,7)-Λ(11,7)</td>
<td>1.6E-06</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Λ(12,7)-Λ(21,7)</td>
<td>1.2E-08</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>Λ(1,8)-Λ(11,8)</td>
<td>1.4E-06</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Λ(12,8)-Λ(21,8)</td>
<td>1.0E-08</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Λ(1,9)-Λ(11,9)</td>
<td>1.2E-06</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Λ(12,9)-Λ(21,9)</td>
<td>8.8E-09</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Λ(1,10)-Λ(11,10)</td>
<td>1.1E-06</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Λ(12,10)-Λ(21,11)</td>
<td>7.9E-09</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>Λ(1,11)-Λ(11,1)</td>
<td>3.6E-15</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Λ(12,11)-Λ(21,1)</td>
<td>2.6E-17</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>Λ(1,11)-Λ(11,1)</td>
<td>6.0E-09</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Λ(12,11)-Λ(21,1)</td>
<td>-2.6E-08</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>Λ(1,11)-Λ(11,1)</td>
<td>4.1E-08</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Λ(12,11)-Λ(21,1)</td>
<td>7.5E-10</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>Λ(1,11)-Λ(11,1)</td>
<td>8.3E-08</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Λ(12,11)-Λ(21,1)</td>
<td>1.5E-09</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>Λ(1,11)-Λ(11,1)</td>
<td>8.3E-07</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Λ(11,1)-Λ(21,1)</td>
<td>1.5E-08</td>
<td></td>
</tr>
</tbody>
</table>

Table [2]. The elements of the subordinated Markov operator.
### Table [3]. European SMEs PDs by rating class and term to maturity

<table>
<thead>
<tr>
<th>Rating</th>
<th>1 YR</th>
<th>3 YR</th>
<th>5 YR</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>100.00%</td>
<td>100.00%</td>
<td>100.00%</td>
</tr>
<tr>
<td>SD</td>
<td>100.00%</td>
<td>100.00%</td>
<td>100.00%</td>
</tr>
<tr>
<td>CC</td>
<td>100.00%</td>
<td>100.00%</td>
<td>100.00%</td>
</tr>
<tr>
<td>CCC-</td>
<td>31.59%</td>
<td>54.42%</td>
<td>63.18%</td>
</tr>
<tr>
<td>CCC</td>
<td>31.59%</td>
<td>54.42%</td>
<td>63.18%</td>
</tr>
<tr>
<td>CCC+</td>
<td>31.59%</td>
<td>54.42%</td>
<td>63.18%</td>
</tr>
<tr>
<td>B-</td>
<td>12.13%</td>
<td>29.75%</td>
<td>39.16%</td>
</tr>
<tr>
<td>B</td>
<td>8.49%</td>
<td>20.31%</td>
<td>27.93%</td>
</tr>
<tr>
<td>B+</td>
<td>2.98%</td>
<td>9.05%</td>
<td>14.32%</td>
</tr>
<tr>
<td>BB-</td>
<td>1.71%</td>
<td>5.28%</td>
<td>8.76%</td>
</tr>
<tr>
<td>BB</td>
<td>1.02%</td>
<td>3.36%</td>
<td>5.84%</td>
</tr>
<tr>
<td>BB+</td>
<td>0.44%</td>
<td>1.76%</td>
<td>3.68%</td>
</tr>
<tr>
<td>BBB-</td>
<td>0.44%</td>
<td>1.56%</td>
<td>2.98%</td>
</tr>
<tr>
<td>BBB</td>
<td>0.24%</td>
<td>1.04%</td>
<td>2.12%</td>
</tr>
<tr>
<td>BBB+</td>
<td>0.18%</td>
<td>0.79%</td>
<td>1.60%</td>
</tr>
<tr>
<td>B-</td>
<td>0.11%</td>
<td>0.49%</td>
<td>1.09%</td>
</tr>
<tr>
<td>B</td>
<td>0.07%</td>
<td>0.34%</td>
<td>0.78%</td>
</tr>
<tr>
<td>BB</td>
<td>0.04%</td>
<td>0.24%</td>
<td>0.58%</td>
</tr>
<tr>
<td>BBB</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>BBB+</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>B-</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>B</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
</tbody>
</table>

Source: Standard and Poor’s (2004b)

### Table [4]. One-year Transition Matrix for European CDOs (1987-2004)

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>D</th>
<th>C</th>
<th>CC</th>
<th>CCC</th>
<th>B</th>
<th>BB</th>
<th>BBB</th>
<th>A</th>
<th>AA</th>
<th>AAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>C</td>
<td>50.0</td>
<td>-</td>
<td>50.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>CC</td>
<td>18.6</td>
<td>1.6</td>
<td>16.1</td>
<td>71.0</td>
<td>4.8</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>CCC</td>
<td>16.6</td>
<td>5.1</td>
<td>72.9</td>
<td>3.4</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>B</td>
<td>0.9</td>
<td>0.9</td>
<td>4.5</td>
<td>4.2</td>
<td>84.8</td>
<td>4.4</td>
<td>0.2</td>
<td>-</td>
<td>0.2</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>BB</td>
<td>0.1</td>
<td>0.1</td>
<td>0.7</td>
<td>0.6</td>
<td>3.0</td>
<td>92.0</td>
<td>2.3</td>
<td>0.7</td>
<td>0.5</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>A</td>
<td>-</td>
<td>0.0</td>
<td>0.2</td>
<td>0.2</td>
<td>0.5</td>
<td>2.9</td>
<td>91.3</td>
<td>3.3</td>
<td>1.5</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>AA</td>
<td>-</td>
<td>-</td>
<td>0.1</td>
<td>0.1</td>
<td>0.3</td>
<td>0.5</td>
<td>4.0</td>
<td>92.0</td>
<td>3.2</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>AAA</td>
<td>-</td>
<td>-</td>
<td>0.0</td>
<td>0.0</td>
<td>0.4</td>
<td>0.3</td>
<td>2.0</td>
<td>97.3</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Source: Standard and Poor’s (2004b)

### Table [5]. CLN spreads and recovery rates by rating class

<table>
<thead>
<tr>
<th>Rating Class</th>
<th>CLN spreads (bps)</th>
<th>Recovery Rate (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>530</td>
<td>45</td>
</tr>
<tr>
<td>BB</td>
<td>225</td>
<td>47</td>
</tr>
<tr>
<td>BBB</td>
<td>100</td>
<td>49</td>
</tr>
<tr>
<td>A</td>
<td>55</td>
<td>51</td>
</tr>
<tr>
<td>AA</td>
<td>35</td>
<td>53</td>
</tr>
<tr>
<td>AAA</td>
<td>25</td>
<td>55</td>
</tr>
</tbody>
</table>

Source: KfW (2005c) and FitchRatings (2005b)
### Table [6]. Composition of names and capital structure by rating class

<table>
<thead>
<tr>
<th>Rating Class</th>
<th>Names’ Composition (%)</th>
<th>Names’ Composition (number)</th>
<th>Notional Amount (EUR m)</th>
<th>Average Loan (EUR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>3%</td>
<td>11</td>
<td>21</td>
<td>1,917,808</td>
</tr>
<tr>
<td>BB</td>
<td>3%</td>
<td>11</td>
<td>5.3</td>
<td>484,018</td>
</tr>
<tr>
<td>BBB</td>
<td>17%</td>
<td>62</td>
<td>5.3</td>
<td>85,415</td>
</tr>
<tr>
<td>A</td>
<td>34%</td>
<td>124</td>
<td>7.5</td>
<td>60,435</td>
</tr>
<tr>
<td>AA</td>
<td>14%</td>
<td>51</td>
<td>8.3</td>
<td>162,427</td>
</tr>
<tr>
<td>AAA</td>
<td>29%</td>
<td>106</td>
<td>9.35</td>
<td>88,333</td>
</tr>
<tr>
<td>Super Senior CDS</td>
<td>-</td>
<td>-</td>
<td>650</td>
<td>-</td>
</tr>
<tr>
<td>Total</td>
<td>100%</td>
<td>365</td>
<td>707</td>
<td>-</td>
</tr>
</tbody>
</table>


### Table [7]. The CDO tranche prices, no unwind barrier, March 1<sup>st</sup> 2005

<table>
<thead>
<tr>
<th>Tranches</th>
<th>Loss Attachment</th>
<th>Detachment</th>
<th>Maximum Loss</th>
<th>Spread</th>
<th>Index Spread</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3%</td>
<td>1095</td>
<td>449.50</td>
<td>500.00</td>
<td></td>
</tr>
<tr>
<td>3%</td>
<td>6%</td>
<td>1460</td>
<td>187.80</td>
<td>189.00</td>
<td></td>
</tr>
<tr>
<td>6%</td>
<td>9%</td>
<td>1095</td>
<td>68.70</td>
<td>64.00</td>
<td></td>
</tr>
<tr>
<td>9%</td>
<td>12%</td>
<td>1825</td>
<td>42.90</td>
<td>22.0</td>
<td></td>
</tr>
<tr>
<td>12%</td>
<td>22%</td>
<td>5475</td>
<td>18.50</td>
<td>8.00</td>
<td></td>
</tr>
</tbody>
</table>

### Table [8]. The CDO tranche prices, unwind barrier of 10.5%, March 1<sup>st</sup> 2005

<table>
<thead>
<tr>
<th>Tranches</th>
<th>Loss Attachment</th>
<th>Detachment</th>
<th>Maximum Loss</th>
<th>Spread</th>
<th>Index Spread</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3%</td>
<td>872</td>
<td>506.4</td>
<td>500.00</td>
<td></td>
</tr>
<tr>
<td>3%</td>
<td>6%</td>
<td>896</td>
<td>187.5</td>
<td>189.00</td>
<td></td>
</tr>
<tr>
<td>6%</td>
<td>9%</td>
<td>772</td>
<td>65.0</td>
<td>64.00</td>
<td></td>
</tr>
<tr>
<td>9%</td>
<td>12%</td>
<td>920</td>
<td>23.9</td>
<td>22.0</td>
<td></td>
</tr>
<tr>
<td>12%</td>
<td>22%</td>
<td>1860</td>
<td>8.9</td>
<td>8.00</td>
<td></td>
</tr>
</tbody>
</table>

### Table [9]. The CDO tranche hedge ratios, unwind barrier of 10.5%, March 1<sup>st</sup> 2005

<table>
<thead>
<tr>
<th>Tranches</th>
<th>Hedge Ratio Spread (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>24.69%</td>
</tr>
<tr>
<td>3%</td>
<td>3% 6%</td>
</tr>
<tr>
<td>6%</td>
<td>6% 9%</td>
</tr>
<tr>
<td>9%</td>
<td>9% 12%</td>
</tr>
<tr>
<td>12%</td>
<td>12% 22%</td>
</tr>
</tbody>
</table>

65
Appendix C

Bibliography


FitchRatings (2005b) PPROMISE I MOBILITY 2005-1, Credit Products/Germany, Structured Finance, Pre-Sale Report.


KfW Bankengruppe (2005b) Information Memorandum: PROMISE-I Mobility 2005-1 PLC.

KfW Bankengruppe (2005c) PROMISE- Transaction and Structural Data.


McDonough, W.J. (2002) Bank Supervision and Credit Standards under Basel II: Perspectives for SMEs, Remarks of the Chairman of the Basel Committee on Banking Supervision at the Gurzenich Koln, Germany.


