Implied Correlation of synthetic CDOs with liquid markets

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Authenticity Statement

I hereby declare that this dissertation is my own work and confirm its authenticity.

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Chapter 1

Introduction

The predominant financial risk in credit portfolio management is default risk. Credit default is commonly understood as the unwillingness or inability of an obligor to pay back his debt in a timely manner. The likelihood that an obligor defaults within a given time horizon is referred to as the probability of default. The credit quality of an obligor is usually described by a credit rating, which is determined by rating agencies or by internal rating models of financial institutions.

For traded debt market prices reflect an opinion of the market participants on the default risk of the respective debt issuer. The excess yield of defaultable debt compared to non-defaultable debt, (e.g., of sovereigns) is known as credit spread. The credit spread represents the market price of risk, in particular of instantaneous default and spread volatility risk. It is the general notion of the “market-price of risk” that links pricing of securities to the financial risks of the corresponding transactions. Price discovery in the market may be a way to assess risks. But, vice versa, an assessment of risks is frequently the foundation for the pricing of transactions in financial markets. This is necessary, because price discovery in the market requires liquid markets. In the context of credit risk, e.g., loan pricing must use a fundamental risk assessment in the first place, in order to determine a proper credit spread for loans.

For the assessment of single-name credit risk by price discovery in the market, credit default swaps (CDS) have become more and more important during recent years. Today, the notional amount of traded single-name CDS for a particular reference entity often exceeds the notional amount of the corresponding outstanding debt (Felsenheimer et al., 2006a). For large debt issuers credit risk is liquidly traded today.

A key risk factor in credit risk management, equally important as default risk, is default correlation. A financial institution needs to assess financial risks not only for single transactions but for whole portfolios. It is then the risk contribution of a single transaction to the whole portfolio that determines the proper price, e.g., the credit spread for a loan. To assess portfolio credit risk and the risk contributions of individual transactions, financial institutions have developed portfolio models for credit risk. These models assess the portfolio credit risk from a fundamental point of view and may be viewed as the portfolio analogue of the rating models and methodologies for single-name credit rating. These models are mostly based on statistical modelling of historical data. The essential ingredient of portfolio credit risk models is the sta-
tistical dependency of default events. The financial strength or the credit-worthiness of an issuer depends both on idiosyncratic risk factors and systemic risk factors. The latter are the driving force behind (positive) default correlation.

In principle, a market view on portfolio credit risk is made possible by portfolio transactions. The most prominent portfolio transactions are securitisations and non-funded portfolio credit derivatives. However, price discovery for such transactions has been difficult for a long time. For securitisations the transaction costs are high and markets are not liquid. Furthermore, the bespoke character of most transactions, including pure portfolio credit derivatives, have hampered the development of a proper market view on credit correlation risk.

This has changed since the year 2003, when the first portfolio credit derivatives have emerged that are based on standardised portfolios. Since then the liquidity and the traded notional amount of portfolio credit derivatives have grown dramatically. A market view on credit portfolio risk has emerged, in particular due to the rich set of CDS indices and the related portfolio tranches. Synthetic single-tranche CDOs, based on standardised portfolios, are traded on a daily basis worldwide. The most prominent of these “index portfolios” are the iTraxx and Dow Jones CDX indices. Daily price quotes are available for credit derivative contracts based on these indices from various market makers worldwide. Even (highly speculative) structured products for the retail market that are based on the CDS index or the equity tranche of the iTraxx Europe portfolio have appeared already (ABN AMRO Bank N.V., 2004; Commerzbank AG, 2006).

In this thesis some of the pricing models for synthetic single-tranche CDOs are applied to market data covering almost two years of daily quotes that have been obtained from Bloomberg L.P. (2006).

The thesis is organised as follows. In Chapter 2 credit risk of single issuers is analysed. Default term structures are derived from financial markets data. These correspond either to the real-world measure or the risk-neutral or pricing measure. The former are obtained from historical default rates of rating agencies, whereas the latter are deduced from market quotes for single-name credit default swaps. Chapter 2 closes with an example to illustrate and discuss the relationship between risk-neutral and real-world default-probabilities.

Chapter 3 covers portfolio credit risk. After presenting some important aspects of default correlation modelling, the pricing of synthetic single-tranche collateralised debt obligations (STCDO) is briefly described, in particular several versions of the single-factor Gaussian copula model. Some descriptive statistics for the markets on iTraxx-based portfolio credit derivatives are presented. The single-factor Gaussian copula models are applied to market data for credit derivatives. Numerical results for the models are discussed, parametric dependencies of base correlation curves on model parameters are presented and market-implied base correlations are compared to dealer quotes for these parameters.
Chapter 2

Single-name credit risk

In this chapter credit curves are discussed because they are essential ingredients of pricing multi-name credit derivatives. For pricing purposes risk-neutral default term-structures are required. These are usually obtained from spread quotes for single-name credit default swaps (cf. Section 2.5). However, the general features of default term-structures are present in the real-world measure as well. Sections 2.1 to 2.4 discuss default probabilities that are based on historical default-data of rating agencies. The chapter closes with Section 2.6, a brief comparison of real-world and risk-neutral probabilities of default.

2.1 Issuer ratings

To estimate single-name default probabilities averaged historical default-rates may be examined. Taking recourse to historical data provides information about default probabilities in the real-world probability measure. In the following, default probabilities are calculated which only depend on the issuer rating of an obligor. The three major rating agencies (Fitch, Moody’s and Standard and Poor’s) report their assessment of the credit-worthiness of corporate debt issuers using the rating scales given in Table 2.1.

S&P rating grades analysed in this work refer to the “long-term local currency issuer credit ratings” of S&P. The corresponding rating category of Moody’s is the “senior unsecured debt” rating. The meaning of issuer credit ratings is described best possible (at least for the S&P ratings) by the following quote from Brady and Vazza (2004):

An issuer credit rating reflects Standard & Poor’s opinion of a company’s overall capacity to pay its obligations (that is, its fundamental credit-worthiness). This opinion focuses on the obligor’s ability and willingness to meet its financial commitments on a timely basis, and it generally indicates the likelihood of default regarding all financial obligations of the firm. It is not necessary for a company to have rated debt in order to be assigned an issuer credit rating. Although the rating on a company’s very senior forms of secured debt, particularly ones with strong covenants, may
Table 2.1: Standard system of rating grades used by three major rating agencies to publish their long term corporate debt and issuer ratings. The labels used by Standard and Poor’s and Fitch are identical. Their usual mapping to the rating scale of Moody’s Investor’s Service is given below. The rating grades are listed in decreasing order of credit-worthiness. The rating grade “D” is given to companies that have defaulted according to the default definition of the respective rating agency. Bonds rated at least “BBB−” are referred to as “investment grade” debt and otherwise as “speculative grade” debt.

<table>
<thead>
<tr>
<th>no.</th>
<th>whole letter rating</th>
<th>Fitch</th>
<th>Moody’s</th>
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<tbody>
<tr>
<td>1</td>
<td>AAA</td>
<td>AAA</td>
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<tr>
<td></td>
<td>AA+</td>
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occasionally be rated higher than the issuer credit rating on the company, specific issues are typically rated as high or lower than these ratings, depending on their relative priority within the company’s debt structure.

A default is recorded upon the first occurrence of a payment default on any financial obligation, rated or unrated, other than a financial obligation subject to a bona fide commercial dispute; an exception occurs when an interest payment missed on the due date is made within the grace period.

A default is assumed to take place on the earliest of the date Standard & Poor’s changed the ratings to 'D', 'SD', or 'R'; the date a debt payment was missed; the date a distressed exchange offer was announced; or the date the debtor filed or was forced into bankruptcy.

This means that the rating grade should correspond to a probability of default,

\[ \Pr(\tau \leq t | \rho(t_0) = i) , \]  

for a particular risk horizon \( s = t - t_0 \), the rating agency’s definition of obligor default and initial rating \( \rho(t_0) \) at time \( t_0 \). Here \( \tau \) is the random variable of the time of default, the obligor has rating \( i \) and \( \Pr(\cdot) \) denotes the probability measure.\(^1\) Note, that the risk horizon \( s \) is not explicitly mentioned in Brady and Vazza (2004). In the following it will be demonstrated that \( T = 1 \) or 2 years are values which are empirically supported by the historical default and rating transition rates, which are, in fact, provided by rating agencies themselves. Not surprisingly, the choice \( T = 1 \) year apparently corresponds to industry best practice (Bluhm, 2003). However, this finding is not officially supported by S&P as they also emphasise the following (Brady and Vazza, 2004):

It is important to note that Standard & Poor’s ratings do not imply a specific probability of default; however, Standard & Poor’s historical default rates are frequently used to estimate these characteristics.

2.2 Markov rating migrations

Standard and Poor’s and Moody’s publish averaged default and rating migration rates. An estimate of the real-world default term-structure can be deduced from this data, a method of credit risk modelling that goes back at least to the work of Jarrow et al. (1997).

Markov process The essential idea behind this approach is that rating migrations can occur at any time with a certain transition probability, which can be estimated from historical data for migration rates. Furthermore, the probability for a transition from rating grade \( i \) to \( j \) within a very short (i.e. infinitesimal) time interval \( dt \) is

\(^1\)In this thesis we will not distinguish real-world and risk-neutral measures by means of notation. It will always be clear which measure \( \Pr \) refers to.
assumed to depend only on the initial rating \( i \). This is the Markov property for the stochastic process of rating transitions. The transition probability for a time interval \( dt \) can be written in terms of a transition intensity \( \Lambda_{ij} \) as,

\[
\Pr (\rho(t+dt) = j \mid \rho(t) = i) = \delta_{ij} + \Lambda_{ij} dt \equiv [1 + \Lambda dt]_{ij}. \tag{2.2}
\]

Here \( 1 \) denotes the unit matrix, \( \delta_{ij} \) is the Kronecker delta and \( \rho(t) \) the stochastic process for the rating of the respective obligor. A reasonable transition-intensity matrix must have the following properties:

\[
\Lambda_{ij} \geq 0, \quad \text{for all } i \neq j \tag{2.3}
\]

\[
\sum_j \Lambda_{ij} = 0, \quad \text{for all initial ratings } i \tag{2.4}
\]

The positivity condition (2.3) stems from the interpretation of \( \Lambda_{ij} \) as a probability, which must be non-negative. The second condition, equation (2.4), is the conservation of probability.

Using the Markov assumption, the transition matrix for a finite time interval \( s = t - t_0 \) is formally given by iteration of (2.2), yielding the matrix-exponential form of the Markov rating-transition matrix \( \Pi(t_0, t) \),

\[
\Pi(t_0, t) = \exp s\Lambda, \quad \text{for any } s = t - t_0 > 0. \tag{2.5}
\]

The matrix-exponential is defined as the usual absolutely converging series, which can be used as well for numerical calculations.

In practice estimates for finite-time transition-probability matrices \( \Pi(t_0, t) \) are provided, e.g., by rating agencies as rating-transition frequencies \( \hat{\Pi}_{ij}(s) \). The time-interval \( s \) for the observation of the rating migrations is usually one or two calendar years. In order to obtain transition matrices for other time horizons the Markov hypothesis may be used. This approach requires that a Markov generator \( \Lambda \) can be determined, which satisfies the relations (2.3) and (2.4) as well as the following equation,

\[
\hat{\Pi}(s) = \exp s\Lambda. \tag{2.6}
\]

This is the problem of finding a Markov embedding for a given transition matrix \( \hat{\Pi}_{ij}(s) \). Several authors have discussed this problem in the context of credit risk (see e.g. Israel et al. (2001), Duffie and Singleton (2003), Bluhm (2003) and references therein). Given the statistical uncertainties of the matrix \( \hat{\Pi}(s) \) it is appropriate to search for an approximate solution of equation (2.6), i.e., to find a generator matrix \( \Lambda \) such that the exponential \( \exp s\Lambda \) is “close” to \( \hat{\Pi}(s) \) with respect to some suitable measure. A frequent choice for such a measure is the Frobenius norm \( \|A\|_F \) of a matrix \( A \), defined as (Golub and Van Loan, 1996),

\[
\|A\|_F = \sqrt{\sum_{i,j} |A_{ij}|^2}. \tag{2.7}
\]
An approximate solution of equation (2.5) is then a matrix $\Lambda$ which yields a small approximation error $\epsilon$:

$$\epsilon = \| \hat{\Pi}(s) - \exp \Lambda s \|_F \ll 1. \quad (2.8)$$

Finally, we note that in this rating-migration approach the default probability is just a migration to the rating grade “D” (cf. Table 2.1):

$$\Pr(\tau \leq s = t - t_0 | \rho(t_0) = i) = (\exp \Lambda s)_{i,D} \quad (2.9)$$

This fact elucidates the reason for the discussion of rating migrations in this work. In the context of the Markov model, rating migrations are the key to the understanding of the default term structure.

**Construction of a Markov generator** The natural attempt to find a generator $\Lambda$ is to compute the logarithm of the empirical rating-migration matrix $\hat{\Pi}(s)$ using the series expansion,

$$\log \left( \hat{\Pi} \right) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} (\hat{\Pi} - 1)^k. \quad (2.10)$$

According to standard theory of linear operators the series (2.10) is absolutely converging if the so-called spectral radius of the square matrix $\hat{\Pi} - 1$ is smaller than 1 (see e.g. Kato (1980)). For finite-dimensional matrices the spectral radius is equal to the maximum absolute value of all (possibly complex-valued) eigenvalues of the matrix $\hat{\Pi} - 1$. Israel et al. (2001) quote the following theorem: if,

$$\hat{\Pi}_{ii} > \frac{1}{2} \quad (2.11)$$

for all $i$ and $\hat{\Pi}$ is a transition matrix, i.e.,

$$\hat{\Pi}_{ij} \geq 0, \quad \text{for all } i, j, \text{ and } \quad (2.12)$$

$$\sum_j \hat{\Pi}_{ij} = 1, \quad \text{for all initial states } i, \quad (2.13)$$

then the spectral radius of $\hat{\Pi}$ is smaller than 1. This sufficient condition for the convergence of the series (2.10) is fulfilled by empirical rating-migration matrices $\hat{\Pi}$. The conditions (2.12) and (2.13) are directly related to the probabilistic interpretation of the matrix elements.

Having obtained a logarithm $\hat{\Lambda} = \log \hat{\Pi}$ of the matrix $\hat{\Pi}$, the matrices of the one-parameter semi-group $\exp s\hat{\Lambda}$, with $s \geq 0$, are not necessarily transition matrices (in the sense defined above). It can be proved that relations (2.12) and (2.13) are satisfied for all elements of the semi-group $\Pi = \exp (s\Lambda)$, with $s \geq 0$, if and only if the generator matrix $\Lambda$ satisfies (2.3) and (2.4). The idea of the formal proof is indicated by equation (2.2).

Therefore, the construction of an approximate embedding (2.8) may proceed along the following steps:
Table 2.2: Rating-transition matrices that have been analysed. The rating matrices
are not reprinted here, the references listed in the last column of the table are publicly
available.

<table>
<thead>
<tr>
<th>No.</th>
<th>Rating agency</th>
<th>averaging period</th>
<th>region/market</th>
<th>migration period</th>
<th>data source</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>S&amp;P</td>
<td>1981–2003</td>
<td>Europe</td>
<td>2 years</td>
<td>Vazza and Aurora (2004, Table 8)</td>
</tr>
</tbody>
</table>

1. Make sure that the empirical transition matrix $\hat{\Pi}$ satisfies (2.12) and (2.13) and compute the logarithm $\hat{\Lambda}$.

2. Make appropriate adjustments to $\hat{\Lambda}$ in order to obtain a generator $\Lambda$ that satisfies (2.3) and (2.4)

3. Compute the approximation error $\|\hat{\Pi} - \exp \Lambda\|_F$ to assess the quality of the adjustments in step two.

Looking for an approximate solution $\Lambda$ of the embedding problem according to equa-
tion (2.8), it must be pointed out that there may well be several different possibilities for the generator $\Lambda$. These may be of the same “quality” with respect to the approximation error. Different procedures for the second step are proposed in the literature (Israel et al., 2001) but there seems to be no “optimum” choice for such a procedure.

**Worked out examples** Approximate Markov embeddings have been computed for four different empirical transition matrices, which are listed in Table 2.2. Similar calculations are presented, e.g., in publications of Israel et al. (2001), Duffie and Singleton (2003), Bluhm (2003). The following examples extend the discussion of these authors by comparing the credit curves,

- obtained from 1- and 2-year transition matrices,

- estimated either from global or European markets and

- referring to the ratings of different rating agencies (S&P vs. Moody’s).

The credit curves will be discussed in the next section. The remainder of this section
describes the construction of the Markov generators.

For each of the empirical rating-migration matrices listed in Table 2.2 the following steps have been carried out to construct an approximate Markov embedding.

1. The published migration matrices comprise transitions to a state in which a corporate looses its rating and is classified “not rated”. These transitions must be excluded and the rows must be normalised. This is a standard procedure,
Table 2.3: Errors of the Markov generator embedding for the transition matrices listed in Table 2.2. The large embedding error for matrix M1 reflects the comparatively low statistics for the European rating transition matrix of Standard and Poor’s and the substitution of default frequencies as discussed in the text and shown in Figure 2.1.

<table>
<thead>
<tr>
<th>No.</th>
<th>error</th>
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</thead>
<tbody>
<tr>
<td>M1</td>
<td>0.0173</td>
</tr>
<tr>
<td>M2</td>
<td>0.00061</td>
</tr>
<tr>
<td>M3</td>
<td>0.00026</td>
</tr>
<tr>
<td>M4</td>
<td>0.00032</td>
</tr>
</tbody>
</table>

the associated estimation bias is believed to be small (see Israel et al. (2001), Duffie and Singleton (2003), Bluhm (2003)).

2. The last column $\hat{\Pi}_{D,i}$ of the (adjusted) rating-migration matrix $\hat{\Pi}$ is the vector of (one- or two-year) default rates. These should increase with decreasing rating quality. Due to low default-statistics especially for the rating grades of high credit-worthiness the observed default frequencies may not increase monotonically. In such cases a linear regression of the logarithms of the frequencies is used to substitute values that do not fit into the hypothesis. To assure that row sums of the calibrated matrix are again equal to one the diagonal element is adjusted. Figure 2.1 shows this calibration for the matrix M1 of Table 2.2 as a blue line. The default frequencies have been substituted for the “AAA” and “A” rating grades (indicated by filled squares). A similar method is proposed by Bluhm (2003), who replaces the whole column of default frequencies by calibrated values.

3. Next the logarithm $\hat{\Lambda}$ of the adjusted and “PD-calibrated” rating-transition matrix is computed. The default intensity column $\hat{\Lambda}_{D,i}$ is calibrated in the same way as the default rate column $\hat{\Pi}_{D,i}$ in the previous step. The red line in Figure 2.1 shows the linear fit of the logarithms of the estimated default intensities.

4. To enforce the conditions (2.3) and (2.4) for the Markov generator, all off-diagonal matrix elements of $\hat{\Lambda}$ that are negative are set to zero. After that the diagonal matrix elements of $\hat{\Lambda}$ are adjusted such that the row sums are zero (Bluhm, 2003).

5. The approximation errors that arise from this construction method for an approximate embedding are listed in Table 2.3.

The calibration plots for the migration matrices M2, M3 and M4 of Table 2.2 can be found in Section A.2 of Appendix A.
2.3 Ratings-based credit curves

In this section a series of plots is presented to compare real-world credit curves. The credit curves and hazard rates are computed within the Markov chain framework of the previous section for the matrix examples listed in Table 2.2.

Credit curves and hazard rates The credit curve is the following function that maps a risk horizon \( s = t - t_0 \) and rating grade \( i \) at time \( t_0 \) to a (cumulative) default probability:

\[
(t, i) \mapsto \Pr(\tau \leq t \mid \rho(t_0) = i).
\]

In the context of the Markov model for rating transitions this probability can be computed as follows:

\[
\Pr(\tau \leq t \mid \rho(t_0) = i) = \exp(t - t_0) \Lambda_{i,D}.
\]

A time-dependent quantity related to the default-probability term-structure is the (instantaneous) forward default rate \( h(t, i; t_0) \), defined implicitly (Duffie and Singleton, 2003) by the relation,

\[
\Pr(\tau \leq t \mid \rho(t_0) = i) = 1 - \exp - \int_{t_0}^{t} h(s, i; t_0) \, ds,
\]

and equivalently by,

\[
h(s, i; t_0) = \frac{\partial_t \Pr(\tau \leq t \mid \rho(t_0) = i)}{1 - \Pr(\tau \leq t \mid \rho(t_0) = i)}
\]
In the literature $h(s, i; t_0)$ is also called a continuous hazard rate of default (Schönbucher, 2003a, p. 57) or a default intensity (Andersen and Sidenius, 2005). The analogy to the instantaneous forward rate in interest rate modelling is apparent. In the context of the Markov model for rating migrations and default the hazard rate can be computed explicitly. Due to the absolute convergence property of the exponential series the time derivative of the default probability is given by,

$$\partial_t \left( \exp(t - t_0)\Lambda \right)_{i,D} = \Lambda \exp(t - t_0)\Lambda_{i,D}.$$  \hfill (2.18)

Therefore, the hazard rate depends only on the initial rating $i$ and, due to the time-homogeneity of the Markov model, on the time difference $t - t_0$. It is given by:

$$h(t - t_0, i) = \frac{\Lambda \exp(t - t_0)\Lambda_{i,D}}{1 - \exp(t - t_0)\Lambda_{i,D}}.$$  \hfill (2.19)

**Worked out examples** In this paragraph the credit curves (2.15) and hazard rates (2.19) are discussed that are directly obtained from the Markov generators constructed in Section 2.2 for the transition matrices listed in Table 2.2.

In Figure 2.2 cumulative default probabilities and hazard rates are compared which have been deduced from the S&P migration matrices. The Figures 2.3 and 2.4 are similar plots. Only the curves for the investment grade credit ratings (“AAA” to “BBB”) and the best speculative grade whole-letter rating (“BB”) are shown. Debt issuers in iTraxx Europe portfolios, which are investigated in Chapter 3, are required to have such high-quality ratings. Therefore, only the credit curves for high credit qualities are of prime interest in this work. The time intervals presented in Figure 2.2 range from zero to approximately ten years. The standard maturities of credit default swaps in OTC markets are approximately 1, 3, 5, 7 and 10 years. These maturities are indicated by vertical lines. In general all credit curves and hazard rates are nearly exponentially increasing. The monotonous increase of the hazard rate reflects that the risk of default for good credit qualities on time intervals well above one year is mainly due to the expected migration to inferior rating grades with increased default likelihood.

In Figure 2.2 credit curves are compared that correspond to different geographical regions. European debt issuers seem to have a higher default probability compared to the global market. In particular the hazard rates at the short end of the credit curve are much larger in the European market. However, this observation may be due to the PD calibration which was applied due to the low default statistics (cf. Figure 2.1). Both curves in Figure 2.2 derive from two-year migration matrices.

Figure 2.3 shows the difference between credit curves and hazard rates for Markov generators estimated from one- and two-year migration matrices of S&P. The curves are quite close to each other. The general tendency for most time intervals and initial ratings is, however, that default probabilities from two-year migration matrices are greater than those from the one-year matrix. A similar statement is true for the corresponding hazard rates. As the underlying matrices refer to the same observation period (1981–2003) and are both derived from Standard & Poor’s default data base,
Figure 2.2: Cumulative default probabilities (a) and hazard rates (b) from Standard and Poor’s migration matrices. Graphs obtained from a global migration matrix (dot and dash line) and a European debt migration matrix (dashed line) are compared. In both cases 2-year migration matrices have been selected to construct the corresponding Markov generators and related credit curves (transition matrices M1 and M2 in Table 2.2).
Figure 2.3: Similar to Figure 2.2, but comparing credit curves estimated by means of 1 and 2-year global rating-transition matrices of Standard and Poor’s respectively (transition matrices M2 and M3 in Table 2.2).
Figure 2.4: Similar to Figure 2.2, but comparing credit curves estimated by means of 1-year global rating-transition matrices of Standard and Poor’s and Moody’s respectively (transition matrices M3 and M4 in Table 2.2).
the differences of the credit curves should be attributed in particular to the non-Markov behaviour of rating migrations. Another factor will be statistical error and the artifacts due to the construction of the Markov generator.

In order to compare the credit curves that are implied by migration matrices of different rating agencies, Figure 2.4 has been created. The curves are based on one-year migration matrices of S&P and Moody’s respectively (matrices M3 and M4 in Table 2.2). The comparison of credit curves and hazard rates is based on the standard mapping of rating grades, as defined in Table 2.1. The difference of the curves is striking and illustrates great uncertainties of modelling credit curves based on Markov rating migrations: the values according to Moody’s are much lower in particular for the very-high quality issuers. Qualitatively this is explained by smaller empirical rating-migration and default rates in the migration matrix of Moody’s. S&P measures higher one-year default rates as illustrated in Figures A.2 and A.3 on page 55.

2.4 Rating distributions

The significant difference between the credit curves of Moody’s and S&P, discussed in the previous Section 2.3, raises the question whether the standard mapping of rating grades of Table 2.1 is appropriate.

To track down possible differences a small portfolio of European debt issuers has been considered. As particular attention is paid in this thesis to the European debt issuers that are most liquidly traded in the CDS market, the portfolio comprises the 153 European names comprised in iTraxx Europe CDS indices of series 1 to 5 (cf. Table A.1 of the Appendix A). Many of these issuers have rating grades from several different rating agencies, as all iTraxx Europe names are major European debt issuers. An issuer rated by several different rating agencies frequently obtains different ratings from the agencies. Of course, rating agencies are believed to have different methodologies to determine their rating grades.

Long term issuer ratings (of Standard & Poor’s) and senior unsecured debt ratings (of Moody’s and Fitch) have been retrieved from the Bloomberg Professional service in August 2006 for the iTraxx Europe names (Bloomberg L.P., 2006). Table 2.4 shows the two-dimensional rating distribution relating grades of Standard & Poor’s and Moody’s Investor’s Service. The distribution does not show a significant drift away from the diagonal. This could be conjectured given the pronounced difference of the empirical credit curves presented in the preceding Section 2.3. This confirms the analysis that ratings-based credit curves may suffer from large statistical uncertainties.

The Tables A.3 and A.4 are the corresponding two-dimensional rating distributions, which relate the rating grades of Fitch Ratings to ratings of S&P and Moody’s respectively. Again, a dispersal around the diagonal of the tables is observed. A systematic shift, however, cannot be asserted, without having applied special statistical tests.
Table 2.4: Two-dimensional distribution of the credit ratings assigned by Moody’s and Standard and Poor’s. The table compares issuer ratings for the 153 companies included in at least one of the first five series of the DJ iTraxx Europe CDS portfolios (cf. Table A.1 in the Appendix). Companies not rated by either one or both rating agencies are classified “nr”. Depending on availability, the “senior unsecured debt” rating of Moody’s is chosen and otherwise the “issuer rating” of Moody’s. The Standard and Poor’s rating is the “long term issuer credit rating” for local or foreign currency debt of the respective obligor. The table was compiled from Bloomberg data of 27 August 2006.

![Two-dimensional distribution of credit ratings](image)

Moody’s

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Sum: 1 1 1 3 17 16 14 23 28 26 9 3 1 1 10
2.5 Risk-neutral credit curves

Credit default swaps  Defaultable bonds trading at par have interest payments that are greater than those of equivalent non-defaultable or government bonds. The coupon difference is referred to as the credit spread. Given the importance of credit spreads for financial markets worldwide, there is a huge amount of literature on this subject. At least for liquid bonds the a large portion of the credit spread compensates for the risk of default of the bond issuer (cf. discussion below).

The idea of credit default swaps (CDS) is to strip the default-risk related cash flows from defaultable obligations. CDS have therefore two payment legs: the premium or fixed leg and the protection or floating leg. The fixed leg is a series of credit-spread coupon-payments until maturity of the CDS or until the reference obligor defaults, whichever occurs earlier. According to OTC market conventions payment dates are quarterly: on 20 March, 20 June, 20 September and 20 December. For the floating leg physical settlement is the most common variant: the protection seller pays the notional of the CDS, (a few weeks) after default occurs. The protection seller and receives from the protection buyer, by physical delivery, an obligation issued by the defaulted party.

We note that a position in a CDS cannot be closed completely by entering an off-setting transaction such that the protection legs cancel each other. In general, the premium legs of different CDS transactions on the same reference name will have different credit spreads, yielding a series of net payments. These payments are still subject to default risk even if the protection legs cancel each other exactly.

For large international corporates credit default protection is liquidly traded in the OTC markets. The outstanding notional amount of credit default swaps has become so large in recent years that it exceeds the outstanding notional amount of credit itself (Felsenheimer et al., 2006a, Ch. 1).

Credit spreads in the CDS and the cash markets differ slightly on the short time scale but move together in the long run (Zhu, 2004). The differences are mainly due to market inefficiency.

CDS pricing  According to the general theorem of asset pricing theory the price of any derivative security, and the price of credit default swaps in particular, is given by the expectation value of the sum of discounted cash flows of the contract. The CDS cash flows are random since they are depending on the occurrence of default events and on the timing of default. In the most general setting the risk-free interest-rates are stochastic as well and are not even statistically independent of the default event.

To price a CDS an expectation value has to be calculated with respect to a risk-neutral probability measure (or pricing measure). The theory of pricing credit derivatives is described in detail in the standard reference book of Schönbucher (2003a). Here, we do not start from first principles but use some approximations, required in practice, right from the beginning. The crucial notion from asset pricing theory is that of risk-neutral default probabilities.

The value of a credit default swap, with notional value $N$ and CDS spread $s$, is equal to the sum of the present values of the premium and protection legs. Let
$T_1 < T_2 \ldots < T_n$ be the (quarterly) payment dates of the CDS and $T_0$ the valuation date. From the point of view of the protection buyer, the value of the premium leg is the sum of the risky annuity,

$$- N_S \sum_{i=1}^{n} B(T_0, T_i) \Delta(T_{i-1}, T_i) \Pr(\tau > T_i \mid \mathcal{F}_{T_0}),$$

(2.20)

and the default accrual term,

$$- N_S \sum_{i=1}^{n} B(T_0, \bar{T}_i) \frac{1}{2} \Delta(T_{i-1}, T_i) \Pr(T_{i-1} < \tau \leq T_i \mid \mathcal{F}_{T_0}).$$

(2.21)

The value of the protection leg is given by the following expression:

$$N(1 - R) \sum_{i=1}^{n} B(T_0, \bar{T}_i) \Pr(T_{i-1} < \tau \leq T_i \mid \mathcal{F}_{T_0})$$

(2.22)

Here $\bar{T}_i = \frac{1}{2}(T_{i-1} + T_i)$ denotes the date between two spread payment dates, $B(T_0, T_i)$ is the discount factor and $\Delta(T_{i-1}, T_i)$ the day-count factor for the period from $T_{i-1}$ until $T_i$. The day-count convention for the CDS market is Actual/360. Furthermore, $\Pr(\tau > T_i \mid \mathcal{F}_{T_0})$ denotes the risk-neutral probability of no default until time $T_i$ and $\Pr(T_{i-1} < \tau \leq T_i \mid \mathcal{F}_{T_0})$ is the risk-neutral probability of default within the time interval $(T_{i-1}, T_i]$. All probabilities are conditioned on $\mathcal{F}_{T_0}$, the information available at time $T_0$. $\tau$ is the random variable for the default time of the obligor which is the reference name of the CDS. Finally, $R$ is the expectation value for the (random) recovery rate.

The market quotes fair value spreads for CDS contracts (par CDS spreads). If a CDS deal is made the spread is determined such that the value of the CDS is zero. According to the equations (2.20) to (2.22) the fair spread is given by (omitting the conditioning on $\mathcal{F}_{T_0}$ in the following):

$$s = \left\{ \frac{(1 - R) \sum_{i=1}^{n} B(T_0, \bar{T}_i) [\Pr(\tau \leq T_i) - \Pr(\tau \leq T_{i-1})]}{\sum_{i=1}^{n} B(T_0, T_i) \Delta(T_{i-1}, T_i) [1 - \Pr(\tau \leq T_i)]} \right\} /$$

$$+ \sum_{i=1}^{n} B(T_0, \bar{T}_i) \frac{1}{2} \Delta(T_{i-1}, T_i) [\Pr(\tau \leq T_i) - \Pr(\tau \leq T_{i-1})]$$

(2.23)

The fair credit spread $s$ will depend in particular on the maturity date $T_n$ of the CDS contract. To determine the fair spread it is tempting to model the default probability term structure $\Pr(\tau \leq T_i)$, possibly starting from real-world probabilities. As risk-neutral probabilities need to reflect the market-price of risk, apart from the actuarial default probabilities, they are expected to be larger than the real-world probabilities discussed in Section 2.3. In principle, larger default probabilities will decrease the
value of the fixed leg and increase the value of the protection leg, yielding a fair spread that is greater than a fair spread deduced from real-world probabilities. The relation between actual default probabilities and credit spreads is still subject to academic research (see e.g. Denzler et al. (2005)). An example for that is discussed in Section 2.6 below.

**Bootstrapping the credit curve** In practice, however, formula (2.23) is used to deduce from CDS spreads the market view on risk-neutral default probabilities. Given a series of $k$ expiry dates $T_{n_1} < T_{n_2} < \ldots < T_{n_k}$ and related credit spreads $s_{n_1}, \ldots, s_{n_k}$ a vector of default probabilities,

$$\Pr(\tau \leq T_{n_i} | \mathcal{F}_{T_0}), \quad \text{with } i = 1, \ldots, k,$$

(2.24)

by solving a system of equations of the form (2.23) iteratively for the probabilities, starting with $T_{n_1}$. The method is well-known as “bootstrapping” from interest rate theory (see e.g. Hull (2000)).

The forward default probabilities (Duffie and Singleton, 2003, p. 50),

$$1 - \frac{\Pr(T_{n_i} > \tau | \mathcal{F}_{T_0})}{\Pr(T_{n_i} > \tau | \mathcal{F}_{T_0})} = \frac{\Pr(T_{n_{i-1}} < \tau \leq T_{n_i} | \mathcal{F}_{T_0})}{1 - \Pr(\tau \leq T_{n_{i-1}} | \mathcal{F}_{T_0})},$$

(2.25)

with $i = 1, \ldots, k$ and $n_0 = 0$, are then easily calculated. Risk-free stepwise constant hazard rates can be derived from the forward default probabilities. Two issues arise:

1. Probabilities $\Pr(\tau \leq T_i | \mathcal{F}_{T_0})$ cannot be computed for each of the payment dates $T_i$ with $i = 1, \ldots, n$. Liquid credit spreads from the OTC markets exist at the most for the 1, 3, 5, 7 and 10-year maturities whereas the payment dates are quarterly.

2. Usually, the risk-neutral expectation value(s) $R$ for the default recovery rate must be provided as an input parameter to the bootstrapping procedure. (In general $R$ could be time-dependent as well.)

The first issue is well-known from interest rate modelling and, therefore, a wealth of literature exists how to interpolate rates, i.e., spreads in the present context (see e.g. Schönbucher (2003a)).

The second issue is usually “solved” by choosing a constant numerical value for $R$, which is partly motivated by historical observations. JP Morgan proposes $R = 0.4$ (Ahluvalia et al., 2004) which corresponds to the long term historical mean, as shown in Figure 2.5. In fact, it turns out that the sensitivity of multi-name credit-derivatives pricing on the recovery rate is small (O’Kane, 2004).

A statistical dependence of default probabilities and recovery rates is observed in historical data (Hamilton and Varma, 2006) and is in fact neglected in equation (2.23). It is investigated from a modelling point of view by Schönbucher (2003b), who implements a two-factor Hull-White model on a tree. He draws the following conclusion:
Figure 2.5: Historic recovery rates according to Moody’s Investor’s Service. The figure is copied from Hamilton and Varma (2006, p. 13) and shows the annual average of senior-unsecured-debt recovery-rates from 1982 until 2005. The long-term mean is remarkably close to 40%. However, the real-world recovery-uncertainty is pronounced, even if annual averages are considered.

The recovery rate is one of the most uncertain input parameters in the [pricing] model and it can be seen that its influence is much larger than the influence of the correlation. It will therefore be more important to improve the estimate of the recovery rate than the correlation.

Rule of thumb In this work we follow a more simple approach to estimate risk-neutral default probabilities from CDS spreads than bootstrapping equation (2.23). It was checked for realistic market data that the rule of thumb described in the following is sufficiently precise to estimate default probabilities, given the width of the bid-ask spreads in the OTC market. The reason is that bootstrapping credit curves for 125 different names of the iTraxx portfolios requires CDS spreads for different maturities for each name. Available maturities will differ for the various names and, moreover, for the different trading dates. Therefore, it becomes an intricate task to derive stable credit curves for all names and a series of consecutive valuation dates, both from the point of view of the numerical method and the data availability. This is beyond the scope of the present work.

To obtain a more simple estimate of risk-neutral default probabilities, we make several simplifying assumptions:

1. Neglect discounting:
   \[ B(T_0, T_i) \lessapprox 1, \text{ for all payment dates } T_i \]  \hfill (2.26)

2. Constant day-count fractions due to regular quarterly payments
   \[ \Delta(T_{i-1}, T_i) \approx \bar{\Delta} = \frac{4}{365.25}, \text{ for all consecutive dates } T_{i-1} \text{ and } T_i \]  \hfill (2.27)
3. Constant hazard rate assumption

\[ \Pr(\tau \leq T_i) \approx 1 - \exp(-h \Delta(T_i, T_0)), \text{ for all payment dates } T_i \quad (2.28) \]

Using these approximations and the series expansion \( \exp(h \bar{\Delta}) = 1 + h \bar{\Delta} + o(h^2 \bar{\Delta}^2) \), the risky annuity in equation (2.20) simplifies to,

\[ -\frac{Ns}{h} \Pr(\tau \leq T_n). \quad (2.29) \]

Similarly, the default accrual (2.21) becomes,

\[ -Ns \frac{\bar{\Delta}}{2} \Pr(\tau \leq T_n), \quad (2.30) \]

and the protection leg (2.22) is simply,

\[ -N(1 - R) \Pr(\tau \leq T_n). \quad (2.31) \]

Noting that \( 2\bar{\Delta}^{-1} = 182.625 \gg h \) is clearly satisfied for investment grade credit, the default accrual term can be omitted. We arrive at the following equations that are used in this work to estimate implied default probabilities from CDS spread quotes,

\[ h = \frac{s}{1 - R} \quad \text{and} \quad \Pr(\tau \leq T_i) = 1 - \exp(-h \Delta(T_i, T_0)), \quad (2.32) \]

for \( i = 1, \ldots, n \). It should be emphasised here that \( s \) is the spread for a CDS with maturity date \( T_n \). The drawback of this approximation is clearly that the probability \( \Pr(\tau \leq T_i) \) depends on the maturity \( T_n \) of the CDS from which the spread \( s \) is taken. This unsatisfactory dependence arises because the upward sloping term structure of credit spreads, observed in the market, is ignored (cf. Figure 2.6 below). The hazard rate \( h \) is also referred to as the “clean spread”, while \( s \) is the “dirty spread”.

### 2.6 Relating actual and risk-free probabilities

Figure 2.6 shows the historical evolution of CDS spreads for the British wholesale and retail company Boots Group PLC, which became Alliance Boots PLC after the take-over of Alliance UniChem on 31 July 2006. It demonstrates that credit spreads increase with increasing maturity of CDS contracts, in agreement with the upward sloping real-world credit curves of Section 2.3.

The time evolution of credit spreads shows signatures of idiosyncratic risk as well as systemic risk. The rating downgrades, which are purely idiosyncratic events, have an immediate impact on credit spreads. Systemic influence is qualitatively recognisable by comparing Figure 2.6 with Figure 3.1 on page 38. The latter figure shows the general development of credit spreads in the European market.

For the five-year risk horizon implied default probabilities for Boots Group PLC have been computed using equation (2.5). The results of the corresponding spreadsheet calculations are shown in Table 2.5. Two different periods of time are considered.
Figure 2.6: The recent history of CDS spreads for Boots Group PLC. Different maturities are distinguished by the colour of their curves. Five different maturities are plotted: 1, 3, 5, 7, and 10 years (cf. ticker symbols in the upper left corner of the plot). Two curves are shown for each maturity, the lower one corresponds to the “bid” spread and the upper curve to the “ask” spread. Vertical lines indicate the dates at which S&P adjusted the long-term issuer rating of Boots Group PLC. Clearly, rating downgrades seem to have a noticeable effect on default swap spreads. Data source for CDS spreads and rating migrations: Bloomberg L.P. (2006)
Table 2.5: Ranges of CDS spreads for Boots Group PLC and associated cumulative probabilities of default (risk-neutral measure). The CDS-spread range is the range of the Bloomberg “mid” quote during the period of time stated in the second and third column. This periods correspond exactly to the time between consecutive rating changes (cf. Figure 2.6). The implied probabilities of default are computed using the rule of thumb (2.32) and the recovery rate stated. The time fraction was computed according to the Act/360 day count convention for a time interval of 5 1/8 actual years. This is the mean maturity of standard 5y CDS in the OTC market.

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<td>40%</td>
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</tbody>
</table>

Table 2.6: Cumulative probabilities of default for a risk horizon of 5 1/8 years as obtained from the real-world credit curves of Section 2.3 (real-world measure). The probabilities are given in basis points (bp). The values printed in bold are identical to those shown in Figures 2.2 and 2.3 for the whole-letter rating grades of S&P. The values shown in the columns “A-” to “BBB+” are computed by an exponential interpolation between the values for the whole letter ratings, using the values in the second row for the abscissa (cf. Figure 2.1).

<table>
<thead>
<tr>
<th>S&amp;P rating</th>
<th>(A)</th>
<th>A-</th>
<th>(A/BBB)</th>
<th>BBB+</th>
<th>(BBB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>abscissa for interpolation</td>
<td>3</td>
<td>3.333</td>
<td>3.5</td>
<td>3.667</td>
<td>4</td>
</tr>
<tr>
<td>M1 S&amp;P, 2y, Europe</td>
<td>121 bp</td>
<td>193 bp</td>
<td>244 bp</td>
<td>309 bp</td>
<td>493 bp</td>
</tr>
<tr>
<td>M2 S&amp;P, 2y, global</td>
<td>95  bp</td>
<td>158 bp</td>
<td>204 bp</td>
<td>263 bp</td>
<td>437 bp</td>
</tr>
<tr>
<td>M3 S&amp;P, 1y, global</td>
<td>82  bp</td>
<td>136 bp</td>
<td>174 bp</td>
<td>223 bp</td>
<td>368 bp</td>
</tr>
<tr>
<td>range (rounded)</td>
<td>140-190 bp</td>
<td></td>
<td></td>
<td>220-310 bp</td>
<td></td>
</tr>
<tr>
<td>mean value (rounded)</td>
<td>100 bp</td>
<td>210 bp</td>
<td></td>
<td>430 bp</td>
<td></td>
</tr>
</tbody>
</table>

that represent periods during which the S&P long-term local-currency issuer-rating of Boots Group PLC was not changed. Using the standard recovery rate assumption of 40%, the risk-neutral default probability for the time horizon of 5 1/8 years has been approximately 390 to 740 basis points during the period of an “A-” rating. Similarly, it has been approximately 240 to 1070 basis points during the period of the “BBB+” rating. Both periods extend over approximately one year and the ranges reflect the movements of the credit spreads during the corresponding period (cf. Figure 2.6).

Similarly, real-world probabilities according to the S&P rating-migration matrices have been determined. The results of the spreadsheet calculations are shown in Table 2.6. The credit curves presented in Section 2.3 are based on whole-letter ratings, which is a too coarse scale here. Therefore, it is necessary to interpolate between the “A” and “BBB” credit curves to obtain an estimate for the rating notches “A-” and “BBB+”. The method used here is exponential interpolation: a linear interpolation of the logarithms of cumulative default probabilities, which is suggested by Figure 2.1.

For the “A-” rating a cumulative 5y default probability in the range from 100 to 210 basis points is expected. This is a rough and conservative estimate. The
corresponding implied probabilities observed for Boots Group PLC in 2004/2005 while being rated “A-”, 390 to 740 basis points as reported above, are distinctly larger. Similarly, for the rating “BBB-” a cumulative 5y default probability approximately in the range from 210 to 430 is expected. In 2005/2006 the market-implied default probabilities for Boots Group PLC have been in the range from 240 to 1070, while being rated “BBB+”. Although these implied probabilities have been above the real-world probabilities most of the time during the respective period, they seem to have been practically equal to the historical rate in the second quarter of 2006. This may be due to a different market opinion on the recovery rate, compared to the value that has been assumed here. However, it seems that a systematic decrease of CDS spreads has been observed in the markets in Q2 2006, as illustrated in Figure 3.1.

The difference between historical default rates and market-implied default probabilities has been subject to academic analysis in the past and it is still subject to vigorous discussion (see e.g. Zhu (2004), Hull et al. (2005)). Admittedly, the default definitions of the rating agencies differ slightly from the standard credit-derivatives definitions of ISDA (2003). However, there is no doubt that the distinctly larger risk-neutral probabilities reflect the market price of risk. The protection seller is compensated for taking various types of risk by receiving a risk premium: risk of instantaneous default (jump to default), default timing risk, spread risk, liquidity risk and recovery risk.

A further analysis of this topic is beyond the scope of this work, in order to turn to implied default correlation in the next Chapter 3.
Chapter 3
Portfolio credit risk

3.1 Introduction

The foundation of portfolio management is risk diversification. In the context of credit portfolio management diversification is closely related to the correlation of defaults. However, contrary to the equity markets or investments in different asset classes, there seems to be no anti-correlation of default risk. A typical feature of credit portfolio models is the positive correlation of default events and credit spread movements. Credit risk is thought to have two contributions, a systemic and an idiosyncratic component. The systemic component is the driving force for positive default correlation and is “somehow” related to macro-economic conditions and developments. This rationale is a feature of all current portfolio credit-risk models (see e.g. Crouhy et al. (2000), Bluhm et al. (2003) and Duffie and Singleton (2003)).

Several different approaches exist to model the statistical dependence of credit defaults. The two principal models for single-name credit risk modelling, namely the reduced-form or hazard rate models and the structural or Merton-type models, can be extended to the multi-name problem. In the first case simply the default intensities (or stochastic hazard rates) are correlated to obtain correlated defaults (Schönbucher and Schubert, 2001; Duffie and Singleton, 2003). For Merton-type models the default-driving factors, like asset-returns, are correlated, yielding a statistical dependence of the default events. For multiple risk horizons this was apparently discussed first by Li (2000). His approach leads to the Gauss copula statistical-dependency for the default times, which can be extended to more general copula-based models.

As default correlation is a principal risk factor in credit risk management, the emergence of standardised portfolios during the recent years is of particular interest. Due to greatly increased liquidity in markets for portfolio credit derivatives, for the first time a market view on default correlation is established. Credit correlation can be traded and price discovery is no longer hampered by the bespoke character of most portfolio credit derivatives before standardisation. Amato and Gyntelberg (2005) describe it in the following way:

One of the most significant developments in financial markets in recent years has been the creation of liquid instruments that allow for the trading
of credit risk correlations. Prime among these instruments are CDS index tranches. Broadly put, index tranches give investors, i.e., sellers of credit protection, the opportunity to take on exposures to specific segments of the CDS index default loss distribution. Each tranche has a different sensitivity to credit risk correlations among entities in the index. One of the main benefits of index tranches is higher liquidity. […]

The standardisation of index tranches may prove to be a significant further step towards more complete markets. Credit risk correlations have always been key risk components in portfolios of credit-risky securities. However, up until now, standardised products for the trading of credit risk correlations have not been available. The emergence of index tranches therefore fills a gap in the ability of the markets to transfer certain types of credit risks across individuals and institutions.

Today credit correlation desks seem to use market implied correlation in a similar way as implied volatility is used, e.g., in the equity derivatives markets. The standard model for equity derivatives is the Black-Scholes model. It is even used for purposes like vega-hedging, a risk-type not known in plain Black-Scholes theory. Also, pricing of exotic options is based on implied volatility surfaces, obtained from market prices for plain vanilla options (see e.g., Samuel (2002)). This approach has frequently been described as “put the wrong numbers into the wrong formula to get the right price”. As the analogous standard model for correlation trading single-factor Gauss copula models seem to become the choice for many market participants, in combination with the base correlation framework (Andersen and Sidenius, 2005).

In this chapter different forms of this standard model are discussed and applied to market data. Section 3.2 briefly outlines the foundations of default-dependence modelling. The subsequent Section 3.3 introduces synthetic CDOs and the pricing of these contract. A short description of the standardised CDS indices and related products is given in Section 3.4. Ths Sections 3.5 and 3.6 present results from numerical calculations with a C++ code briefly described in Appendix B. Market-implied base correlations are computed for a large set of market data and three different variations of the single-factor Gauss copula model. Finally, Section 3.7 closes this thesis with some proposals for possible future research.

### 3.2 Modelling default dependence

From the modelling point of view, correlation of default events can be achieved by modelling a multivariate distribution of default times $\tau_1, \ldots, \tau_n$ of $n$ different debt issuers, i.e., by specifying the joint probability distribution function,

$$F(t) = F(t_1, \ldots, t_n) = \Pr(\tau \leq t | \mathcal{F}_{t_0}). \tag{3.1}$$

Here $\tau = (\tau_1, \ldots, \tau_n)$, $t = (t_1, \ldots, t_n)$ and $\tau \leq t$ means that $\tau_k \leq t_k$ for all $k = 1, \ldots, n$. The $n$ different marginal distributions of $F(t)$,

$$F_k(t_k) = F(\infty, \ldots, \infty, t_k, \infty, \ldots, \infty) = \Pr(\tau_k \leq t_k | \mathcal{F}_{t_0}), \tag{3.2}$$
are the single-name credit curves.

**Copula concept** A copula function is simply the link between the multivariate
distribution function $F(t)$ and the marginal distributions $F_k(t_k)$. More precisely,
according to Nelson (1999, Definition 2.10.6) an $n$-dimensional copula is a function,

$$C : [0, 1]^n \rightarrow [0, 1],$$

(3.3)

that has the following properties:

1. For every vector $n$-dimensional vector $u \in [0, 1]^n$ with $u_k = 0$ for at least one
   $k \in \{1, \ldots, n\}$ it follows $C(u) = 0$.

2. For every vector $n$-dimensional vector $u \in [0, 1]^n$ with $u_k = 1$ for all $k$ except a
   single $k'$ it follows $C(u) = u_k'$.

3. For any $n$-box $[a, b] \subseteq [0, 1]^n$ (i.e., vectors $a, b \in [0, 1]^n$ with $a \leq b$) the $n$-th
   order difference $\Delta^b_a C(u)$ of $C$ on $[a, b]$ is greater than or equal to zero,

$$\Delta^b_a C(u) \equiv \Delta^b_{a_n} \cdots \Delta^b_{a_1} C(u) \geq 0.$$  

(3.4)

Here the first-order difference operator $\Delta^b_{a_k} C(u)$ is defined as,

$$\Delta^b_{a_k} C(u) = C(u_1, \ldots, u_{k-1}, b_k, u_{k+1}, \ldots, u_n) - C(u_1, \ldots, u_{k-1}, a_k, u_{k+1}, \ldots, u_n).$$

(3.5)

The $n$-th order difference $\Delta^b_a C(u)$ of $C$ on $[a, b]$ is also called the $C$-volume of $[a, b]$.

According to the well-known theorem of Sklar (see e.g. Nelson (1999, Theorem
2.10.9)), any distribution function (3.1) can be written in terms of the marginal
 distributions (the credit curves) and an $n$-dimensional copula,

$$F(t_1, \ldots, t_n) = C(F_1(t_1), \ldots, F_n(t_n)).$$

(3.6)

The copula is unique if the marginal distributions are continuous, which can be assumed in the present context of default time modelling. Conversely, it can be proved
that the function,

$$C(u_1, \ldots, u_n) = F(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)),$$

(3.7)

is a copula for any joint probability distribution function $F(.)$ with marginal distributions $F_k(.)$ and their inverse functions $F_k^{-1}(.).$

For the modelling of statistically dependent default times this means that the “right” copula must be determined, assuming that the marginal distributions are given. Sklar’s theorem means that any model for the statistical dependency of default
times can be stated in terms of a copula function. In practice it is necessary to have
tractable models that allow for an efficient numerical evaluation.
Gaussian copula and factor copulas  The modelling of the statistical dependence for the pricing and hedging of portfolio credit derivatives is still debated (see e.g. Hull and White (2006)). It seems, however, that many financial institutions use models based on the Gauss, the student-t copula or related factor-copula models for pricing and risk management (see e.g. Andersen et al. (2003), Kakodkar et al. (2004), Felsenheimer et al. (2004), Turc and Very (2004), Andersen and Sidenius (2005)).

The Gauss copula model has been implemented for numerical calculations of this thesis. More specifically the Gaussian factor copula with a single factor has been used. It has apparently emerged as the basic standard model in the financial industry and a simplification of it has entered the Basel II capital framework. We have to skip the description here and refer to the introductions of Li (2000), Finger (2000), Gregory and Laurent (2003) or Andersen and Sidenius (2005).

The principal idea behind a factor copula is conditional independence. In the present context this means that the cumulative joint default probability for \( n \) different names factorises, conditional on a vector of "systematic factor" random variables \( S \):

\[
Pr(\tau \leq t | \mathcal{F}_{T_0} \wedge S) = Pr(\tau_1 \leq t_1 | \mathcal{F}_{T_0} \wedge S) \cdots Pr(\tau_n \leq t_n | \mathcal{F}_{T_0} \wedge S).
\]

(3.8)

The random variables \( S = (S_1, \ldots, S_n) \) are normally distributed with zero mean and standard deviation equal to one, \( S_j \sim N(0,1) \) for \( j = 1, \ldots, n \). In this work only the single-factor case is used in numerical calculations, i.e., the vector \( S \) is one-dimensional. In the single-factor Gauss copula the dependence is parametrised by \( n \) correlation parameters \( \beta_i \) for \( i = 1, \ldots, n \). A further simplification used in the following is the constant correlation for all names, \( \beta_i = \sqrt{\rho} \) for all \( i \) and some correlation constant \( \rho \in [0,1) \).

### 3.3 Synthetic single-tranche CDOs

During recent years the market for credit derivatives has grown enormously (e.g. Wolcott (2004)). This applies to the plain single-name CDS market as well as to the portfolio credit derivatives market. The latter has benefitted in particular from standardisation (cf. Section 3.4 below). A particular portfolio credit derivative is the so-called synthetic single-tranche CDO (STCDO).\(^1\)

**STCDOs**  A synthetic CDO is a portfolio credit derivative. Similar to single-name credit default swaps it is composed of a fixed and a floating payment leg.

Its contract specification is related to a portfolio or pool of \( n \) single-name CDS, each with notional amount \( N_k \) and a loss given default \( L_k \), for \( k = 1, \ldots, n \). We assume that all CDS have the same maturity and \( L_k = N_k(1 - R_k) \) with an issuerspecific recovery rate \( R_k \). The loss until some risk horizon \( t \) in the CDS pool, relative

---

\(^1\)The well-known acronym CDO means **collateralised** debt obligation. In fact, synthetic CDOs have similar payment legs like some of the classical funded CDOs. However, there is no collateral in a synthetic CDO transaction, as it is a pure credit derivative.
to the total notional $N_{\text{CDO}}$, is given by the following random variable $l(t)$,

$$l(t) = \frac{1}{N_{\text{CDO}}} \sum_{k=1}^{n} L_k 1_{\tau_k \leq t} = \frac{\sum_{k=1}^{n} L_k 1_{\tau_k \leq t}}{\sum_{k=1}^{n} N_k}. \quad (3.9)$$

Here $1_{\tau_k \leq t}$ is the default indicator jump process, which depends on $t$:

$$1_{\tau_k \leq t} = \begin{cases} 1 & \text{if } \tau_k \leq t, \\ 0 & \text{otherwise}. \end{cases} \quad (3.10)$$

The stochastic loss process $l(t)$ depends on default correlation, i.e., the statistical dependence of the default times $\tau_k$.

A synthetic single-tranche CDO (STCDO) is defined in terms of (percentage) lower and upper tranche attachment points, $a$ and $b$ respectively. The initial tranche notional, before any losses occur, is equal to $(b - a)N_{\text{CDO}}$. Losses that take place inside the tranche interval $[aN_{\text{CDO}}, bN_{\text{CDO}}]$ are covered by the floating or protection leg of the STCDO. More precisely, the relative loss of the tranche at time $t$ is given by,

$$l_{[a,b]}(t) = \frac{[l(t) - a]^+ - [l(t) - b]^+}{b - a}. \quad (3.11)$$

Here the common notation $[.]^+ = \max(., 0)$ is used. The tranche loss, as a function of the total loss in the CDS pool, is similar to the combined payoff of a long call option with strike $a$ and a short position in a call option with strike $b$ (call spread on the portfolio loss distribution). The fractional (or normalised) tranche loss $l_{[a,b]}(t)$ is bounded by 0 and 1. The protection leg pays whenever a loss occurs, i.e., when the increasing stochastic process $l_{[a,b]}(t)$ jumps. In practice some weeks may pass between the occurrence of a default event and the actual payment. Until maturity of the trade, the fixed or premium leg pays a fixed coupon on the (time-dependent) remaining tranche notional $(b - a)N_{\text{CDO}}(1 - l_{[a,b]}(t))$ after accounting for payments on the protection leg. Here, the relative tranche notional-value at time $t$ is $1 - l_{[a,b]}(t)$.

At least for the standardised trades, payment dates are fixed by the market convention for the quarterly payments of single-name CDS (i.e. the $20^\text{th}$ of the last month of each quarter of a year).

**Pricing of portfolio credit derivatives** A straight forward and the most general approach to value portfolio credit derivatives is a Monte Carlo simulation. This is a viable method if the payoff of a portfolio credit derivative is a function of the default times and other pricing parameters, like risk-free interest rates, recovery rates, etc.. According to standard arbitrage pricing theory, the present value at time $T_0$ is then the following expectation value (cf. Andersen et al. (2003)),

$$V(T_0) = \mathbb{E} \left( f(\tau_1, \ldots, \tau_n; r, R_1, \ldots, R_n, \ldots) \mid \mathcal{F}_{T_0} \right). \quad (3.12)$$
Assuming that the default times $\tau_1, \ldots, \tau_n$ are the only random variables, the present value $V(T_0)$ is the $n$-dimensional integral,

$$V(T_0; r, R_1, \ldots) = \int_{T_0}^{+\infty} \cdots \int_{T_0}^{+\infty} f(t_1, \ldots, t_n; r, R_1, \ldots) \phi_{T_0}(t_1, \ldots, t_n) \, dt_1 \cdots dt_n, \quad (3.13)$$

Here $\phi_{T_0}(t_1, \ldots, t_n)$ is the probability density corresponding to the joint probability distribution of the default times $\tau_1, \ldots, \tau_n$, given in terms of the marginal distributions and a copula. Several copulas exist for which a numerical Monte Carlo evaluation of the integral (3.13) is feasible. In particular random samples of $\tau_1, \ldots, \tau_n$ can be drawn very easily for the Gaussian copula. However, for fast computations the computational effort is usually too great for a large number of names (say 20 names, depending on computing resources, of course).

In recent years several approaches have been proposed to value portfolio credit derivatives without Monte Carlo simulations. In this work the method of Andersen et al. (2003) has been implemented and is therefore briefly described. Other approaches have been proposed, e.g., by Debuysscher and Szegö (2003) or Hull and White (2004). A description of these competing methods is beyond the scope of this work.

**Portfolio loss framework for STCDOs**  STCDOs have a payoff that depends only on the (random) path $l_{[a,b]}(t)$ of the tranche loss process, which simplifies the computation of the present value (3.12) if the joint distribution function of the default times $\tau_1, \ldots, \tau_n$ is a factor copula. Conditional independence can then be used to evaluate the distribution of $l_{[a,b]}(t)$ efficiently for some particular time $t$, without recourse to Monte Carlo integration.

The present value $V(T_0)$ is the expectation value of a function $g$, which takes the stochastic process $l_{[a,b]}(.)$ as one of its arguments,

$$V(T_0) = \mathbb{E} \left( g(l_{[a,b]}(.); r, R_1, \ldots) \bigg| \mathcal{F}_{T_0} \right). \quad (3.14)$$

For practical calculations the probability distributions of the random variables $l_{[a,b]}(t)$ can only be computed for a finite set of times. Following Andersen et al. (2003) and McGinty and Ahluwalia (2004) we use the payment dates $T_i$ of the fixed leg of the STCDO, with $i = 1, \ldots, m$. The last payment date $T_m$ coincides with the maturity of the contract. Using this approximation, we calculate an expectation value like,

$$\mathbb{E} \left( \tilde{g}(l_{[a,b]}(T_1), \ldots, l_{[a,b]}(T_m); r, R_1, \ldots) \bigg| \mathcal{F}_{T_0} \right), \quad (3.15)$$

to determine the present value of the STCDO. The function $\tilde{g}(.)$ must be a sum of two terms, $\tilde{g}_{\text{fixed}}$ and $\tilde{g}_{\text{float}}$, representing the fixed and floating legs of the STCDO respectively.

The protection leg pays whenever a default occurs between two spread payment dates $T_i-1$ and $T_i$. In the mean default(s) will occur at time $\bar{T}_i = \frac{1}{2}(T_i + T_{i-1})$. The payment amount is proportional to the difference of the fractional tranche notional.
values at the times \( T_{i-1} \) and \( T_i \), i.e., the loss during that time interval. Hence we obtain,

\[
\tilde{g}_{\text{float}} = N_{\text{CDO}} (b - a) \sum_{i=1}^{m} B(T_0, T_i) \left\{ l_{[a,b]}(T_i) - l_{[a,b]}(T_{i-1}) \right\} . \tag{3.16}
\]

Alternatively, it could be assumed that the payment of the floating-leg cash-flow(s) is delayed with respect to the default event. Therefore, McGinty and Ahluwalia (2004) use the discount factor \( B(T_0, T_i) \) instead of \( B(T_0, \bar{T}_i) \), which further simplifies their implementation.

The fixed spread that is paid for the accrual period \([T_{i-1}, T_i]\) is proportional to the tranche notional value during that period, taking into account defaults occurring in that period. In the discrete time approximation we use the average of the notional values at the times \( T_{i-1} \) and \( T_i \) respectively. As in the previous chapter, the day-count factor is denoted by \( \Delta(T_{i-1}, T_i) \) and the coupon rate by \( s \). From the point of view of the protection buyer the sum of the discounted cash flows of the fixed leg is then given by:

\[
\tilde{g}_{\text{fixed}} = -N_{\text{CDO}} (b - a) s \sum_{i=1}^{m} B(T_0, T_i) \Delta(T_{i-1}, T_i) \left\{ 1 - l_{[a,b]}(T_{i-1}) + 1 - l_{[a,b]}(T_i) \right\} . \tag{3.17}
\]

The present value \( V(T_0) \) of a STCDO from the perspective of the protection buyer can be calculated by:

\[
\frac{V(T_0)}{N_{\text{CDO}} (b - a)} = E \left( \sum_{i=1}^{m} B(T_0, \bar{T}_i) \left\{ l_{[a,b]}(T_i) - l_{[a,b]}(T_{i-1}) \right\} \mid \mathcal{F}_{T_0} \right) \\
- E \left( s \sum_{i=1}^{m} B(T_0, T_i) \Delta(T_{i-1}, T_i) \left\{ 1 - l_{[a,b]}(T_{i-1}) + 1 - l_{[a,b]}(T_i) \right\} \mid \mathcal{F}_{T_0} \right) \tag{3.18}
\]

\[
= \sum_{i=1}^{m} B(T_0, \bar{T}_i) \left\{ e_{i-1}^{[a,b]} - e_i^{[a,b]} \right\} - \frac{s}{2} \sum_{i=1}^{m} B(T_0, T_i) \Delta(T_{i-1}, T_i) \left\{ e_{i-1}^{[a,b]} + e_i^{[a,b]} \right\} .
\]

Here, \( e_i^{[a,b]} \) is the expectation value,

\[
e_i^{[a,b]} = E \left( 1 - l_{[a,b]}(T_i) \mid \mathcal{F}_{T_0} \right) \in [0, 1]. \tag{3.19}
\]

Its computation is the main task in order to determine the present value or the par spread of a STCDO. The STCDO par spread is clearly given by:

\[
s = \frac{\sum_{i=1}^{m} B(T_0, \bar{T}_i) \left\{ e_{i-1}^{[a,b]} - e_i^{[a,b]} \right\}}{\frac{1}{2} \sum_{i=1}^{m} B(T_0, T_i) \Delta(T_{i-1}, T_i) \left\{ e_{i-1}^{[a,b]} + e_i^{[a,b]} \right\}}. \tag{3.20}
\]

In STCDO trading par spreads \( s \) are quoted in most cases. For the equity tranches with \( a = 0 \) the coupon rate is fixed and the protection seller receives an upfront
payment. The upfront payment is quoted as a fraction of the tranche notional value (in percent) and can be determined by means of equation (3.18).

The expectation value (3.19) has an important property, which is used in the base correlation framework (see below). The following relation is easily verified:

\[(b - a) e_i^{[a,b]} = b e_i^{[0,b]} - a e_i^{[0,a]} .\] (3.21)

It means that the building blocks of STCDO pricing are the expectation values \(e_i^{[0,a]}\) of (hypothetical) equity tranches with upper attachment point \(a\). This property is independent of the correlation model which is actually used for pricing.

**Expected tranche notional values for factor-copulas** In the following the conditioning on the algebra \(\mathcal{F}_{T_0}\), i.e., “all information available at time \(T_0\)” will be dropped from the notation. To compute the expectation value \(e_i^{[a,b]}\) a factor-copula is assumed.

In the numerical calculations presented in the remainder of this chapter, the following three different cases of decreasing complexity are considered:

1. The single-factor Gaussian copula model with name-independent correlation parameter \(\rho\). The model uses individual default probabilities for the different names of a STCDO. It will be referred to as the SFGC model.

2. A simplification of this model is to assume portfolio homogeneity: all names of the portfolio are assumed to have the same default term structure the same weight in the portfolio and the same recovery rate \(R\). This model will be referred to as the HPGC model (homogeneous pool Gaussian copula model).

3. Since the HPGC model still depends on the number of names in the portfolio, another simplification is possible. Dropping the granularity of the portfolio, i.e., taking the limit \(n \to \infty\) leads to LHPGC model (large homogeneous pool Gaussian copula model). It is described, e.g., by (Vasicek, 1987), McGinty and Ahluwalia (2004) or Felsenheimer et al. (2006b). Its principal parameters are a single default probability (or default term structure), a correlation parameter and a name-independent recovery rate.

In the cases 2. and 3., the HPGC and LHPGC models respectively, the computation of the expectation values \(e_i^{[a,b]}\) is facilitated enormously. Due to the factorisation of the conditional joint default probability (3.8) and only a single default probability for all names, the conditional (fractional) loss-distribution for the complete portfolio is a binomial distribution:

\[
\Pr \left( l(T) = j \frac{1-R}{n} \mid S = s \right) = \binom{n}{j} p^j (1-p)^{n-j} .
\] (3.22)

Here, \(n\) is the number of names in the portfolio, \(j = 0, \ldots, n\) is the number of defaults and the probability \(p\) is the default probability (for some time horizon \(T\) conditional
on the systematic factor $S$ of the single-factor Gauss copula (Andersen et al., 2003):

$$p \equiv p(s, T) \equiv \Pr(\tau \leq T \mid S = s) = \Phi\left(\frac{\Phi^{-1}(\Pr(\tau \leq T)) - \sqrt{\rho} s}{\sqrt{1 - \rho}}\right)$$

(3.23)

$\Phi(.)$ and $\Phi^{-1}(.)$ denote the cumulative normal distribution and its inverse function respectively. The unconditional (fractional) loss distribution for the complete portfolio is given by the following $n + 1$ one-dimensional integrals:

$$\Pr(l(T) = j \frac{1-R}{n}) = \int_{-\infty}^{\infty} \frac{\exp(-\frac{s^2}{2})}{\sqrt{2\pi}} \left(\begin{array}{c} n \\ j \end{array}\right) p(s, T)^j (1 - p(s, T))^{n-j} \, ds.$$  

(3.24)

For the numerical evaluation of integrals of this type (for other conditional loss distributions) three different quadrature formulas have been tried: a simple trapezoidal rule, Gauss-Legendre quadrature and Gauss-Hermite quadrature (Press et al., 1992). Numerical experiments indicate that the trapezoidal rule is converging slightly faster than the Gauss-Legendre quadrature and distinctly faster than the Gauss-Hermite quadrature (since the loss probabilities as a function of $s$ are bounded on $(-\infty, \infty)$).

The unconditional loss distribution for the tranche loss $l_{[a,b]}(T_i)$ at time $T_i$ is easily obtained using equation (3.11):

$$\Pr(l_{[a,b]}(T_i) = 0) = \Pr(l(T) \leq a) = \sum_{0 \leq j \leq \frac{na}{1-R}} \Pr(l(T) = j \frac{1-R}{n}),$$

$$\Pr(l_{[a,b]}(T_i) = \frac{j(1-R)-na}{n(b-a)}) = \Pr(l(T) = j \frac{1-R}{n}), \text{ for } \frac{na}{1-R} < j < \frac{nb}{1-R},$$

$$\Pr(l_{[a,b]}(T_i) = 1) = \Pr(l(T) \geq b) = \sum_{\frac{nb}{1-R} \leq j \leq n} \Pr(l(T) = j \frac{1-R}{n}).$$

(3.25)

The expectation value $e_{[a,b]}^{[a,b]}$ can be calculated by summation over the distinct loss values.

For the LHPGC model the limit $n \to \infty$ must be taken and the conditional loss distribution (3.22) becomes a Dirac delta distribution,

$$\Pr(l(T) = l \mid S = s) = \delta(l - (1 - R) p(s, T)).$$

(3.26)

Again, integration over the systematic factor $s$ yields the unconditional distribution. Here, we compute the probability distribution for the (fractional) portfolio loss $l(T)$
at time $T$ (conditioned on all information at time $T_0 < T$ and no losses at time $T_0$):

$$
\Pr(l(T) \leq l) = \int_0^l \int_{-\infty}^\infty \delta(l' - (1 - R)p(s, T)) \frac{\exp - \frac{s^2}{2}}{\sqrt{2\pi}} \, ds \, dl' 
$$

$$
= \int_{-\infty}^\infty \frac{\exp - \frac{s^2}{2}}{\sqrt{2\pi}} \chi(l' \geq (1 - R)p(s, T)) \, ds 
$$

$$
= \int_{-\infty}^\infty \frac{\exp - \frac{s^2}{2}}{\sqrt{2\pi}} \chi(s \geq \frac{\Phi^{-1}(\Pr(\tau \leq T)) - \sqrt{1 - \rho} \Phi^{-1}\left(\frac{l}{1 - R}\right)}{\sqrt{\rho}}} \, ds 
$$

$$
= \Phi\left(\frac{\sqrt{1 - \rho} \Phi^{-1}\left(\frac{l}{1 - R}\right) - \Phi^{-1}(\Pr(\tau \leq T))}{\sqrt{\rho}}\right). 
$$

(3.27)

This nice result is due to Vasicek (1987). Unfortunately, for the computation of the expectation value $e_{[a,b]}^m$ in the LHPGC approximation it cannot be used. Similar to the SFGC model, a one-dimensional integral must be determined numerically:

$$
e_{[a,b]}^i = 1 - \int_0^1 \left[\frac{l' - a}{b - a}\right]^+ \int_{-\infty}^\infty \frac{\exp - \frac{s^2}{2}}{\sqrt{2\pi}} \delta(l' - (1 - R)p(s, T)) \, ds \, dl' 
$$

$$
= 1 - \frac{1}{b - a} \int_{-\infty}^\infty \frac{\exp - \frac{s^2}{2}}{\sqrt{2\pi}} \left\{ [(1 - R)p(s, T) - a]^+ - [(1 - R)p(s, T) - b]^+ \right\} \, ds 
$$

$$
= \frac{1}{b - a} \int_{-\infty}^\infty \frac{\exp - \frac{s^2}{2}}{\sqrt{2\pi}} \left\{ \max((1 - R)p(s, T), b) - \max((1 - R)p(s, T), a) \right\} \, ds. 
$$

(3.28)

The implementation of the LHPGC model of McGinty and Ahluwalia (2004) uses another approximation. Only the expectation value $e_{[a,b]}^m$ for the final date $T_m$, the maturity of the STCDO, are computed by evaluating an integral like (3.28). The respective expectation values for intermediate times $T_i$, with $i = 1, \ldots, m - 1$, are computed using an exponential interpolation between $T_0$ and $T_m$.

For the SFGC model we have to skip further details and must refer to the literature, in particular to the article of Andersen et al. (2003), and the C++ implementation given in Section B.2 of the Appendix. The conditional loss distribution is not binomial: the default probabilities have individual values for each name in the portfolio. In principle, notionals and recovery rates could be heterogeneous in the portfolio as well. Conditional independence of the default probabilities is used again, to determine a discretised conditional loss distribution for each $T_i$ using a recursion method. The loss distribution is discretised by considering only a discrete set of loss values, which are multiples of some suitably chosen loss unit $u$. Similar to the HPGC model the expectation values $e_{[a,b]}^i$ can be derived from the unconditional loss distributions, obtained by numerical integration.

**Remarks** The portfolio loss distribution (3.27) for normalised losses $l \in [0, 1]$ has only few parameters that determine its shape completely, the recovery rate $R$ and
the correlation parameter \( \rho \) and the default probability. It would be too optimistic to believe that this provides sufficient flexibility to describe the risk-neutral loss distributions which are observed in the market. Adding further parameters in the HPGC and SFGC models, namely the portfolio granularity parameter \( n \) and individual probabilities and recovery rates, should not be sufficient either.

In the literature other copulae and more complicated correlation parameters have been investigated with success. However, JP Morgan has proposed the base correlation framework for STCDOs. STCDO pricing models like LHPGC, HPGC or SFGC are used to compute market-implied correlation curves \( \rho(a) \). Correlation becomes a risk factor like volatility in the Black-Scholes theory for equity options. Apparently this method has become the market-standard (Andersen and Sidenius, 2005). These pricing models have to be used for hedging and risk management of portfolio credit derivatives as well. Depending on the hedging method, a very large number of greeks has to be calculated (Andersen et al., 2003; Kakodkar et al., 2003; Turc and Very, 2004; Andersen and Sidenius, 2005). This is a strong argument for comparatively simple models as numerical efficiency becomes crucial. Another problem of more complicated models of default correlation may be the requirement to obtain reasonable estimates for an increased number of model parameters, without making arbitrary choices.

**Base correlation** Within the concept of market-implied correlations, a base correlation is defined as the correlation parameter \( \rho(a) \) that yields the correct price (or market quote) for the equity tranche \([0,a]\). The base correlation \( \rho(a) \) depends on the upper attachment point \( a \) of the equity tranche and, of course, on the model used to determine tranche prices. The concept of base correlation can be applied for any pricing model that is based on a factor copula with a single correlation parameter which parametrises the copula function (including e.g. single-factor student-t copulas).

Unfortunately there is only a single equity tranche traded in the market, typically with upper attachment point of 3% or 4%. All other tranches that are liquidly traded have lower attachment points greater than zero. However, assuming that a base-correlation curve \( \rho(a) \) is available, the corresponding prices for common STCDO tranches can be determined by virtue of equation (3.21). A single-tranche spread then depends on two correlation values, \( \rho(a) \) and \( \rho(b) \), corresponding to the lower and upper tranche attachment points \( a \) and \( b \) respectively.

Conversely, base correlations can be “bootstrapped” from the market quotes for a series of adjacent portfolio tranches by iteration. Assume that market quotes are given for \( N \) different tranches \([a_{j-1}, a_j]\) with attachment points \( a_j, j = 0, 1, \ldots, N \), and \( a_0 = 0 \). The base correlation \( \rho(a_1) \) for the equity tranche can be calculated such that the market quote is matched (which is usually an upfront payment in the premium leg). A numerical root finding procedure is required for that. Having determined the base correlation value \( \rho(a_{j-1}) \) the next point in the correlation curve \( \rho(a_j) \) can be deduced from the market quote for the \( j \)-th tranche \([a_{j-1}, a_j]\). Note, however, that a solution of this problem is not guaranteed to exist.

The market favours base correlations over compound correlations. A compound
correlation is the correlation parameter $\rho_{\text{compound}}(a, b)$ that prices a STCDO for the portfolio-loss interval $[a, b]$ correctly. As it depends on two parameters it represents a correlation surface. This surface must be calibrated with same number of market quotes as the more simple base-correlation curve. Another problem is the existence of compound correlations. In many market situations in the past, market quotes could not be matched by any value for the compound correlation, in particular for the mezzanine tranches. This problem exists for base correlations as well, however, it seems to be less severe in practice. Compound correlations have been calculated using the C++ code of Section B.2. They will not be discussed further in this work.

3.4 CDS indices and standardised index tranches

In order to increase liquidity in the portfolio credit-derivatives market, market participants worldwide have created products, which are based on standardised portfolios, the CDS indices. Presently, the most prominent index portfolios are published by the International Index Company, a firm founded in 2001 by major international investment banks.

iTraxx Europe CDS indices In this work we consider the “Dow Jones iTraxx Europe” CDS indices, published by the International Index Company (International Index Company Ltd., 2006). The principal iTraxx Europe index portfolio comprises 125 European debt issuers. According to the index rules, these issuers have to be distributed up among the industry sectors given in Table 3.1 and must have investment-grade credit quality. Within each industry sector the International Index Company determines, by means of dealer polls, those names in the CDS market that are most liquidly traded. The issuers in the iTraxx Europe index portfolios are equally weighted, i.e., the weight is 0.8% in the principal iTraxx Europe indices. The portfolio composition is adjusted (approximately) semi-annually, as detailed in Table 3.2. Since June 2004 until September 2006 six different iTraxx Europe series have been created. The composition of the index portfolios is listed in Table A.1 of Appendix A for series 1 to 5 of the iTraxx Europe.

The principal virtue of the CDS indices is standardisation and the possibility to trade credit derivatives on standard portfolios. Before the creation of these index portfolios, STCDO trades have been bespoken. Prices have not been transparent, since there was not enough liquidity for price discovery in the market due to the bespoke character of portfolio credit derivative trades.

Credit derivatives based on iTraxx indices The most liquidly traded credit derivative base on the iTraxx Europe CDS index is the portfolio CDS. This is essentially a 0–100% STCDO: a single coupon rate, the “index spread”, is paid to buy

2The industry sectors define corresponding iTraxx sub-indices. However, these have not been considered in this work. A lot of other indices exist, covering other geographical regions and lower credit qualities. CDS indices for North America, the CDX indices, are published by the Dow Jones company.
Table 3.1: Industry sectors of the DJ iTraxx Europe CDS indices.

<table>
<thead>
<tr>
<th>number of names</th>
<th>industry sector</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>Auto (automotive industries)</td>
</tr>
<tr>
<td>30</td>
<td>Consumers (consumer products and retail)</td>
</tr>
<tr>
<td>20</td>
<td>Energy (energy and natural resources)</td>
</tr>
<tr>
<td>20</td>
<td>Industrials (chemicals, engineering, health-care, defence, etc.)</td>
</tr>
<tr>
<td>20</td>
<td>TMT (telecommunication, media and technology)</td>
</tr>
<tr>
<td>25</td>
<td>Financials (financial institutions)</td>
</tr>
</tbody>
</table>

Table 3.2: Roll-over and expiry dates of single-name CDS and of multi-name credit derivatives based on the DJ iTraxx Europe indices. The first column gives the quarterly dates at which the expiry dates of credit default swaps have been rolled over since 2004. The second column lists the corresponding expiry dates for the 5-year CDS contracts. The time-to-expiry of single name CDS is 5\(\frac{1}{4}\) years initially and 5 years when the contract is rolled over again. Clearly, the roll-over date is every 20th of the last month of a quarter (trading-day adjusted). Except for the first two series of the DJ iTraxx Europe, also iTraxx-based derivatives have exhibited regular roll-over and expiry dates in the past. But a new series was introduced semi-annually only. Hence 5-year iTraxx-based derivatives have been rolled when their time-to-expiry was already 4\(\frac{3}{4}\) years (for series 3 to 6). However, there is another distinction between the single-name CDS and multi-name iTraxx-based derivatives: the latter are still traded after rolling over to a new series, thereby successively increasing the number of traded series’ in the market. Everything which has been explained here applies correspondingly to the 3y, 7y and 10y contracts. (Gisdakis, 2006; Bloomberg L.P., 2006; International Index Company Ltd., 2006; Markit Group Ltd., 2006)

<table>
<thead>
<tr>
<th>roll-over date CDS</th>
<th>expiry date 5y CDS</th>
<th>iTraxx Europe series</th>
<th>expiry date 5y iTraxx</th>
<th>initial time-to-expiry 5y iTraxx</th>
</tr>
</thead>
<tbody>
<tr>
<td>19 Mar 04</td>
<td>19 Jun 09</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21 Jun 04</td>
<td>21 Sep 09</td>
<td>1</td>
<td>20 Sep 09</td>
<td>5(\frac{1}{4}) y</td>
</tr>
<tr>
<td>20 Sep 04</td>
<td>21 Dec 09</td>
<td>2</td>
<td>20 Mar 10</td>
<td>5(\frac{3}{4}) y</td>
</tr>
<tr>
<td>20 Dec 04</td>
<td>19 Mar 10</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21 Mar 05</td>
<td>21 Jun 10</td>
<td>3</td>
<td>20 Jun 10</td>
<td>5(\frac{1}{4}) y</td>
</tr>
<tr>
<td>20 Jun 05</td>
<td>20 Sep 10</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20 Sep 05</td>
<td>20 Dec 10</td>
<td>4</td>
<td>20 Dec 10</td>
<td>5(\frac{1}{4}) y</td>
</tr>
<tr>
<td>20 Dec 05</td>
<td>21 Mar 11</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20 Mar 06</td>
<td>20 Jun 11</td>
<td>5</td>
<td>20 Jun 11</td>
<td>5(\frac{1}{2}) y</td>
</tr>
<tr>
<td>20 Jun 06</td>
<td>20 Sep 11</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20 Sep 06</td>
<td>20 Dec 11</td>
<td>6</td>
<td>20 Dec 11</td>
<td>5(\frac{1}{2}) y</td>
</tr>
</tbody>
</table>
Figure 3.1: The complete history of iTraxx Europe index CDS spreads for contracts with initial maturity of approximately 5 years. All series are shown which have been traded since the first portfolio series in 20 June 2004 (cf. Table 3.2). The spreads are much tighter compared to spreads of single-name default-swaps (cf. Figure 2.6). Data source for iTraxx index spreads: Bloomberg L.P. (2006)
Figure 3.2: Spread dispersion for CDS in the iTraxx Europe indices. Four different dates and iTraxx Europe series respectively are shown. Note that the mean spread of all single-name CDS of names in the index portfolio are approximately equal to the index spread itself.
protection on all names. The coupon accounts for defaults in the portfolio, i.e., the notional amount of the contract decreases as defaults occur. Figure 3.1 shows the index spread history for the iTraxx Europe series 1 to 5. The rather small bid/ask spreads, in comparison to Figure 2.6, demonstrate the superior liquidity of the index swaps.

Further descriptive statistics of market data related to the iTraxx Europe indices provides Figure 3.2. It illustrates the spread dispersion for the single-name CDS spreads in the iTraxx Europe portfolios for series 2 to 5.

Also available in the OTC market are the common STCDO tranches. The standard series of adjacent portfolio tranches for the iTraxx index portfolios have upper attachment points of 3%, 6%, 9%, 12% and 22%. The principal maturities of the contracts are 5, 7 and 10 years.

A detailed description of the iTraxx contracts and quotation conventions is Felsenheimer et al. (2004). We skip further details here. Table A.2 in Appendix A lists the most important Bloomberg ticker symbols for the iTraxx Europe index swaps and the standardised STCDOs.

### 3.5 Historical implied correlations

The LHPGC model has been used to compute market-implied base correlations for all iTraxx Europe STCDOs, with maturities of 5, 7 and 10 years and for the series 3, 4 and 5. Six of the nine possible plots are shown in Figures 3.3 and 3.4.

The results are based on the following parameters:

- Zero interest rates for discounting the cash flows, \( r = 0 \). However the C++ code of Appendix B can be used for calculations with \( r > 0 \). It turns out that the interest rate sensitivity of STCDOs is rather small.
- Recovery rate \( R = 0.4 \).
- The risk-neutral default term structure for the LHPGC model is deduced from the index CDS spread. (The index CDS spread can be put into the model to compute the spread of the \([0,1]\)-tranche. The result is very close to the input spread and does not depend on the correlation parameter.)
- The “mid” quotes of Bloomberg are used, which seem to be the mean values of the “bid” and “ask” quotes. These quotes are taken for the STCDO spreads as well as for the index CDS spreads.

Figure 3.3 presents the base-correlation curves as a function of time for the 5y-contracts (initial maturity 5\(\frac{1}{4}\) years). Given the large time interval of more than one year, the implied correlations are fairly stable, in particular for the equity tranche, with notable exceptions. But they clearly show volatility, demonstrating that base correlation must be seen as a risk factor in the Gauss copula and base correlation framework.
Figure 3.3: Implied base correlations according to the LHPGC model for three different iTraxx Europe index series. The correlations are shown as a function of time and (equity) tranche upper attachment points. The labels of the abscissa are identical to the corresponding series start dates. During the first half year the indices tranches are most actively traded. The initial maturity is $5\frac{1}{4}$ years in all three cases.
Figure 3.4: Similar to Figure 3.3, but showing three different contract-maturities for the iTraxx Europe series 4 only. The time interval shown covers approximately 10 months.
The time before and after the 20 March 2006 is most remarkable. In particular for the series 3, implied base correlations that would fit the quotes provided by Bloomberg do not exist.

For the series 4 a regime of distinctly increased volatility is observed after 20 March 2006. Furthermore, it is not clear why the data for series does not begin on 20 March, the respective series start date. These evidence may be a hint to insufficient data integrity. It is difficult to assess the data integrity of the historic quotes obtained from Bloomberg (cf. Appendix A).

For the iTraxx Europe series 4, however, an alternative data source has been available for the 5y STCDO spreads and the corresponding index (Gisdakis, 2006). The comparison of internal data of the HVB iTraxx Trading desk with the corresponding Bloomberg data revealed significant differences after 22 March 2003 on every trading day and only very rarely significant differences during the trading days before that date. This supports the conjecture that the increased volatility of the implied correlations may be partly due to data integrity problems and/or limited liquidity (associated with large bid/ask spreads).

Figure 3.4 compares the implies volatility history for the different initial maturities of 5, 7 and 10 years of the iTraxx Europe series 4. The implied correlations are remarkably constant during the first half year after series start. The different maturities show similar patterns, although differences are clearly visible. Most notably, the 6% base correlation is quite low for the 10y maturity compared to the other two maturities. The converse is true for the 3% correlation. One has to keep in mind that the model used here makes very strong assumptions, e.g., neglecting the proper discounting of cash flows by using zero interest rates will have a greater impact for the 10y STCDOs the for the 5y ones.

3.6 Parametric dependencies

In their daily note “Daily Credit Briefing” HVB publishes, among other credit related information, spread quotes and implied correlations for STCDO products, which are based on the most recent iTraxx Europe series (Bayerische Hypo- und Vereinsbank AG, 2006). Table 3.3 is copied from that customer information and Table 3.4 lists the corresponding data retrieved from Bloomberg.

In order to understand the pricing model, HVB iTraxx Trading uses to quote their base correlations, Figure 3.5 shows the difference between the base correlations of HVB and the corresponding ones determined by means of the LHPGC model. The difference is striking. The base correlation curves are approximately parallel to each other for all maturities. In this Section we make the same assumptions on the computational parameters as in the previous Section 3.5.

A reason for the difference could be the approximation inherent in the LHPGC model, namely the large pool assumption and the homogeneity assumption. In order

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3Significant means here a relative difference of the tranche spreads of more than 5% for almost all tranches. Deviations of more that 10% even for the equity tranche occurred frequently after 22 March.
Table 3.3: Typical dealer quotes for standardised tranches of the iTraxx Europe S5 index. The table is taken from the “Daily Credit Briefing”, a daily publication of the market maker HVB and corresponds to the trading day 7 July 2006 (Bayerische Hypo-und Vereinsbank AG, 2006). The 0%-3% tranche is quoted as an upfront payment in percent of the deal notional. According to market conventions the spread payment of this tranche is fixed to 500 bp. For all other tranches and the index default swap the running spread is quoted in basis points (bp). The base correlations given her are related to the proprietary and undisclosed pricing model of HVB iTraxx Trading.

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Bid</th>
<th>Offer</th>
<th>Delta</th>
<th>Base Corr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-3%</td>
<td>19.5%</td>
<td>20%</td>
<td>24.5%</td>
<td>12%</td>
</tr>
<tr>
<td>3-6%</td>
<td>66</td>
<td>69</td>
<td>5.5</td>
<td>21%</td>
</tr>
<tr>
<td>6-9%</td>
<td>18</td>
<td>21</td>
<td>1.5</td>
<td>28.5%</td>
</tr>
<tr>
<td>9-12%</td>
<td>6.5</td>
<td>8.5</td>
<td>0.75</td>
<td>33%</td>
</tr>
<tr>
<td>12-22%</td>
<td>2.5</td>
<td>3.5</td>
<td>0.3</td>
<td>52%</td>
</tr>
</tbody>
</table>

*0-3 Tranche quoted as upfront +500bp running. All other tranches are running spreads in bp.

Table 3.4: Data source: Bloomberg L.P. (2006)

<table>
<thead>
<tr>
<th>iTraxx Europe series (maturity)</th>
<th>tranche</th>
<th>bid</th>
<th>ask</th>
<th>Bloomberg ticker</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 (5y) index spread</td>
<td>30.0</td>
<td>30.6</td>
<td>ITRXEB55</td>
<td></td>
</tr>
<tr>
<td>5 (5y) 0%-3%</td>
<td>20.0</td>
<td>20.5</td>
<td>CT354363</td>
<td></td>
</tr>
<tr>
<td>5 (5y) 3%-6%</td>
<td>69.2</td>
<td>70.4</td>
<td>CT354367</td>
<td></td>
</tr>
<tr>
<td>5 (5y) 6%-9%</td>
<td>18.8</td>
<td>20.6</td>
<td>CT354371</td>
<td></td>
</tr>
<tr>
<td>5 (5y) 9%-12%</td>
<td>5.4</td>
<td>6.9</td>
<td>CT354375</td>
<td></td>
</tr>
<tr>
<td>5 (5y) 12%-22%</td>
<td>2.6</td>
<td>3.7</td>
<td>CT354379</td>
<td></td>
</tr>
</tbody>
</table>

| 5 (7y) index spread             | 40.1    | 41.2 | ITRXEB75 |
| 5 (7y) 0%-3%                    | 38.0    | 38.4 | CT354411 |
| 5 (7y) 3%-6%                    | 172.3   | 175.7| CT354415 |
| 5 (7y) 6%-9%                    | 51.6    | 52.5 | CT354419 |
| 5 (7y) 9%-12%                   | 25.2    | 28.0 | CT354423 |
| 5 (7y) 12%-22%                  | 5.7     | 6.7  | CT354427 |

| 5 (10y) index spread            | 51.1    | 52.3 | ITRXEB05 |
| 5 (10y) 0%-3%                   | 48.5    | 49.0 | CT354387 |
| 5 (10y) 3%-6%                   | 493.0   | 498.0| CT354391 |
| 5 (10y) 6%-9%                   | 112.7   | 116.0| CT354395 |
| 5 (10y) 9%-12% (missing)        | (missing) | (missing) | CT354845 |
| 5 (10y) 12%-22%                 | 17.0    | 19.7 | CT354403 |
Figure 3.5: Base correlations published by (Bayerische Hypo- und Vereinsbank AG, 2006) for the iTraxx series 5 STCDOs for the trading day 7 July 2006 (blue line) in comparison with results from the LHPGC model (green line).
to investigate whether portfolio granularity or spread dispersion in the index portfolio are the reasons for the observed difference, the impact of these factors is studied in the following two paragraphs.

Impact of portfolio granularity  
Within the assumption of a homogeneous portfolio, the assumption of a large pool is dropped in Figure 3.6. The spread data of Table 3.3 is used for the calculations. The risk-neutral credit curve is computed by formula (2.32) from the “Ref” spread. In Figure implied correlations are shown that correspond to various levels of portfolio granularity, i.e., the number of names in the hypothetical portfolio ranges from 50 to infinity. Results for the finite values are calculated with the C++ implementation of the SFGC model (using the same single-name spread for all names in the portfolio). The results for the infinite value are calculated with the C++ implementation of the LHPGC model.4

It is seen that implied correlation decreases as the granularity of the portfolio increases (i.e. the number of names decreases). This effect is less pronounced for the larger attachment points than for the lower ones. In particular, the base correlation of the equity tranche exhibits the greatest sensitivity to portfolio granularity.

Impact of credit spread dispersion  
The calculations of the previous paragraph have been repeated for \( n = 125 \), but with the actual spread dispersion of the respective trading day, 7 July 2006. The results are shown in Figure 3.7 as a red base-correlation curve. Individual single-name credit curves have been determined from CDS “mid” quotes for the names of the iTraxx Europe series 5 portfolio (cf. Table A.1 of Appendix A). Note that these calculations with a realistic spread dispersion (as observed in the market) already require Bloomberg data for 375 different contracts or Bloomberg ticker symbols. For some names CDS quotes are not available in the Bloomberg Professional System for the trading day 7 July 2006. As a substitute the median of the CDS quotes that can be retrieved is used.

To make the base correlation curve comparable to the calculations that assume a homogeneous portfolio with the single spread according to the “Ref” spread in Table 3.3 (green curve in Figure 3.7), the market spreads for single-name CDS have been multiplied by a constant factor \( \approx 1 \). This factor was determined such that the 0–100% tranche has the same spread for a calculation with a single “Ref” spread as for a calculation using the adjusted individual single-name CDS spreads. As has been pointed out earlier, the factor does not depend on the base correlation curve used to compute the 0-100% tranche spreads by means of the SFGC model.5

Figure 3.7 shows that the assumption of portfolio homogeneity regarding the credit curves reduces the implied correlations. Therefore, it partially compensates the impact of the large-pool assumption, which increases implied correlation (cf. Figure 3.6).

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4This is emphasised here also for the reason that the agreement of the calculation results in the limit of infinite many obligors, \( n \to \infty \), in Figure 3.6 is a successful test for the implementation of the SFGC model.

5Without this adjustment the deviation shown in Figure between the red and green curves would have been more pronounced. However this could be the other way around as well: it depends on the magnitude of the “Ref” spread compared to the single-name spreads taken from the market.
Figure 3.6: Effect of various levels of portfolio granularity on implied correlation. The case of “infinitely many” obligors in the portfolio is the limit which is calculated by means of the LHPGC model. The underlying data is that of Table 3.3. The dotted vertical line indicates the actual portfolio granularity of 125 names in the iTraxx Europe index portfolio.
Figure 3.7: Impact of credit spread dispersion on implied correlations according to the single-factor Gaussian copula model. The implied correlations of HVB iTraxx Trading are shown as well. The assumption of portfolio homogeneity regarding the credit curves reduces the implied correlations.
Figure 3.7 demonstrates as well that base-correlation quotes of HVB cannot be explained by means of the SFGC model. Several further assumptions of the SFGC model could be relaxed in order to “reverse-engineer” the pricing model HVB. It may be conjectured that the HVB model uses a single-factor copula that shows greater tail dependence of the default times. This is achieved, e.g., by a student-t factor-copula. Investigations like this must be deferred to research subsequent to this work.

3.7 Outlook

A principal topic of this Chapter is the computation of implied correlation for liquidly traded (standardised) STCDOs, using single-factor Gaussian copula models for the modelling of the statistical dependence of default events. For trading desks an important issue is the risk management of STCDOs. Hedging of STCDOs may require the computation of many greeks. A simple hedging strategy tries to hedge the systematic spread risk in the portfolio tranches using the related iTraxx index swaps. This hedging ignores idiosyncratic spread risk and spread dispersion. Refinements of the hedging by index CDS can be thought of. Different hedging schemes seem to exist in the market and their historical performance could be investigated by hedging simulations based on historical data.

Also, the influence of the pricing model chosen could be investigated. More complicated pricing models and different factor-copulae could be used as well and the reverse-engineering of the proprietary pricing models, e.g., of HVB could be carried further.

For the computation of greeks, a method has been published by Andersen et al. (2003); Andersen and Sidenius (2005), which uses the recursive computation method, which is already implemented in C++ for the SFGC model (cf. Appendix B). The intruding extension of the existing C++ code and the availability of historical spread data suggest to continue the work presented in this thesis with hedging simulations. Although this is a matter of interest to trading desks and the corresponding risk management units in investment banks, it seems that such simulations have not been made publicly available yet. However, such calculations depend on a good data quality. Therefore, data integrity will be of concern in these investigations, as it was conjectured above that the Bloomberg data may not always be reliable.
Appendix A

Market data

For multi-name credit-derivatives analytics the necessary amount of data is large compared to other branches of mathematical finance. Furthermore, data for market prices is not publicly available in general as credit derivative markets are pure OTC markets. Commercial data vendors provide market spreads for single- and multi-name credit derivative contracts. A principle commercial data provider for credit derivatives data is Markit Group Ltd. (2006). Unfortunately, historic market spreads from Markit were not available for this thesis. Instead most of the market data was taken from the commercial data vendor Bloomberg L.P. (2006).

A.1 CDS, iTraxx indices and tranches

Market data and company information have been retrieved from the Bloomberg Terminal system. Data has been downloaded to Microsoft Excel workbooks using the Bloomberg Excel-API, which is available on Bloomberg terminals. In the Bloomberg data base, a unique identification string for a bond, a share, an interest rate, a credit default swap, an index or any other object is referred to as a Bloomberg ticker symbol. Often Bloomberg ticker symbols do not follow simple conventions of construction and it is difficult to look up these ticker symbols: one has to know what to look for.

Hundreds of different ticker symbols were required to retrieve the data for this work. Tables of ticker symbols had to be compiled in a time-consuming effort. Table A.1 lists the 153 obligors that have been the constituents of the DJ iTraxx Europe portfolios from June 2003 to September 2006, comprising the series 1 to 5 (excluding the so-called “crossover” indices). The table contains essential information for this work: the obligors, which are characterised by their Bloomberg corporate debt ticker and their entity name, the index composition provided in columns 3 to 7, the Bloomberg ticker of the 5-year single-name CDS (which is linked to the corporate debt ticker), and the Bloomberg equity ticker (which is necessary to retrieve the issuer ratings of the respective obligor).

Ticker symbols change over time, e.g., due to corporate actions and major company restructuring. Therefore Table A.1 is a snapshot, dated from July 2006. Unfortunately such a table is not publicly available from the data sources (Bloomberg
It was compiled exclusively for this work.

Table A.1: Index composition of the series 1 to 5 of the iTraxx Europe portfolios.

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<td>−</td>
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<td>−</td>
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<td>−</td>
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<td>+</td>
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<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
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<td>STEV FR</td>
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<td>Suez SA</td>
<td>+</td>
<td>+</td>
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<td>+</td>
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<td>SCAB SS</td>
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<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
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<td>RUKN VX</td>
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Ticker symbols for iTraxx Europe STCDOs and the index CDS are listed in Table A.2. The data availability for the tranche spreads of the series 1 and 2 is poor, market quotes are provided only for a small fraction of dates. However, spreads for the index CDS are available for all series for most of the respective trading days. Unfortunately, most time series suffer from (obviously) erroneous data and must be cleansed in a pragmatic way.

Table A.2: Bloomberg ticker symbols for the synthetic CDO tranches of the iTraxx Europe portfolios. Surprisingly, such a table is neither published by Bloomberg L.P. (2006) nor made publicly available by International Index Company Ltd. (2006) or Markit Group Ltd. (2006). Therefore, it was compiled exclusively for this thesis.

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<td>2 (5y)</td>
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<td>5 (5y)</td>
<td>ITRXEB05</td>
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<td>20.09.2013</td>
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A.2 Supplementary figures and tables

Figures A.1 to A.3 supplement the discussion in Section 2.2 on the calibration of the PD-curve of migration matrices.

Tables A.3 and A.4 provide descriptive statistics for the iTraxx Europe index portfolios. They supplement the corresponding rating distribution of Table 2.4 of Section 2.4.

![Graph showing calibration curves for default probabilities](image)

Figure A.1: Calibration curves for default probabilities, similar to Figure 2.1 on page 10, but for matrix M2 of Table 2.2 on page 8: Standard and Poor’s, global market, 2y.
Figure A.2: Calibration curves for default probabilities, similar to Figure 2.1 on page 10, but for matrix M3 of Table 2.2 on page 8: Standard and Poor’s, global market, 1y.

Figure A.3: Calibration curves for default probabilities, similar to Figure 2.1 on page 10, but for matrix M4 of Table 2.2 on page 8: Moody’s, global market, 2y.
Table A.3: Two-dimensional distribution of the credit ratings assigned by Standard and Poor’s and Fitch, similar to Table 2.4 on page 16. The “senior unsecured debt” rating of Fitch is shown.

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<th>Fitch</th>
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<td>AA+</td>
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<tr>
<td>AA-</td>
<td>1</td>
</tr>
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<td>A+</td>
<td>1</td>
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<td>A</td>
<td>6</td>
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<td>5</td>
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Table A.4: Two-dimensional distribution of the credit ratings assigned Fitch and Moody’s, similar to Table 2.4 on page 16. The “senior unsecured debt” rating of Fitch and Moody’s is analysed.

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Appendix B

Implementation of numerical calculations

B.1 General remarks

The challenge of implementing the computations presented in this thesis comprised the combination of efficient numerical computations with a large amount of market data, which itself required a comparatively complex data structure.

Core numerical computations have therefore been coded in C++. The C++ sources have been used to compile a shared library, i.e. a dynamically linked library (.dll) in a Microsoft Windows and a shared object library (.so) in a Linux operating-system environment respectively. The “analytics kernel” created in this way could then be loaded by either the R or the S-PLUS software packages. These packages provide in particular (Venables et al. (2006); Venables and Ripley (2000) and Venables and Ripley (2002)),

- a high-level (interpreted) scripting language (known as the R and S programming languages respectively),
- efficient and flexible means to create graphics and plots and
- various methods to access external data sources, i.e., tables in text files, spreadsheets or data bases).

A listing of the C++ source code of the mathematical core of the numerical calculations is provided in the following Section B.2. This source code represents only a small fraction of the software, written exclusively for this work, to run all the computations presented. Approximately two thousand lines of code in the R and S scripting languages have been used for the following tasks:

- Data preprocessing and cleansing, simple analysis and data integrity checks for the market data obtained from Bloomberg L.P. (2006) and HVB (Gisdakis, 2006) as well as several scripts to organise the data download itself.
• Numerical calculations, i.e., accessing combining the market data and calling the C++ shared library with the appropriate parameters.

• Finally, creating the plots presented in this work.

Furthermore, several Excel workbooks have been prepared for the following purposes:

• Compile iTraxx portfolio membership tables using publicly available data from International Index Company Ltd. (2006) and Markit Group Ltd. (2006).

• Compile tables with hundreds of different Bloomberg ticker symbols (partly using the Bloomberg API).

• Download historic market data and other attributes (e.g. credit ratings) for these items (using the Bloomberg API).

• Perform various data consistency checks (in combination with R/S scripts), including the comparison of data obtained from Bloomberg and HVB iTraxx Trading respectively.

B.2 C++ pricing analytics

Scientific results should be reproducible and conclusions must be understandable. For this reason the most important part of the numerical code used for this work is provided here.

The inverse cumulative normal distribution is computed according to a rational function approximation published by Acklam (2005). Gaussian quadrature points and weights are computed according to the procedures published in Press et al. (1992) (which have been implemented in C++ here). The functions to compute implied base and compound correlations from market quotes of tranche spreads require a numerical solution of a system of equations. The algorithm which is used here for that purpose, “Ridder’s root”, is also described by Press et al. (1992).

Note, that only the LHPGC and SFGC models have been implemented. The granular HPGC model is just a special case of the SFGC model, although the former allows for a much more efficient implementation (cf. Section 3.3). The LHPGC model was tested successfully with data from in (Felsenheimer et al., 2004, p. 27) and McGinty and Ahluwalia (2004).

```c++
// //////////////////////////////////////////////////////////////////////////////
//
// C++ code: singlence_factor_model.cpp, v
// purpose: Single-factor Gaussian-copula STCDO pricing analytics
// author: Tim Brunne, Frankfurt am Main, tim.brunne@gmx.de
// $Date: 2006/09/30 16:09:11 $
// $Author: tim $
// $Revision: 1.15 $
//
```
```cpp
#include <algorithm>
#include <cmath>
#include <numeric>

#include <boost/numeric/ublas/vector.hpp>
#include <boost/numeric/ublas/vector_proxy.hpp>
#include <boost/numeric/ublas/matrix.hpp>
#include <boost/numeric/ublas/matrix_proxy.hpp>

using namespace boost::numeric::ublas;

// Gaussian probability distribution: \mathcal{N}(\mu = 0, \sigma = 1)

inline double p_norm(const double& q) { return erfc(-q / MSQRT2) / 2; }

inline double q_norm(const double& p) {
    // The inverse of the standard Gaussian distribution:
    // An approximation published by: Peter J. Acklam, pjacklam@online.no,
    // http://home.online.no/~pjacklam. The following implementation is based
    // on the corresponding MATLAB code, written by P J Acklam himself.

    const double a[6] = {
        -3.969683028665376e+01, 2.209460984245205e+02,
        -2.759285104469687e+02, 1.838577518672690e+02,
        -3.066479806614716e+01, 2.506628277459239e+00
    };
    const double b[5] = {
        -5.447609879822406e+01, 1.615858368580409e+02,
        -1.556989785988666e+02, 6.680131188771972e+01,
        -1.328068155288572e+01
    };
    const double c[6] = {
        -7.784894002430293e-03, -3.223964580411365e-01,
        -2.400758277161838e+00, -2.549732539343734e+00,
        4.37466414464968e+00, 2.938163982698783e+00
    };
    const double d[4] = {
        7.784695709041462e-03, 3.224671290700398e-01,
        2.445134137142996e+00, 3.754408661907416e+00
    };

    register double q, t, u;

    if (_isnan(p)) || p > 1.0 || p < 0.0) return NAN;
    if (p == 0.0) return -INFINITY;
    if (p == 1.0) return INFINITY;

    q = (p > 0.5 ? 1 - p : p);

    if (q > 0.02425) {
        // Rational approximation for central region.
        u = q - 0.5;
        t = u * u;
        u = * (((a[0] * t + a[1]) * t + a[2]) * t + a[3]) * t + a[4]) * t + a[5]) / (((b[0] * t + b[1]) * t + b[2]) * t + b[3]) * t + b[4]) * t + 1;
    } else {
        // Rational approximation for tail region.
    }

    return u;
}
```
\[
\begin{align*}
t &= \sqrt{-2 \cdot \log(q)}; \\
u &= (((c[0] \cdot t + c[1]) \cdot t + c[2]) \cdot t + c[3]) \cdot t + c[4] \cdot t + c[5]) \\
&\quad / (((d[0] \cdot t + d[1]) \cdot t + d[2]) \cdot t + d[3]) \cdot t + 1;
\end{align*}
\]

// The relative error of the approximation has absolute value less than 1.15e-9. One iteration of Halley's rational method (third order) gives full machine precision...

\[
\begin{align*}
t &= \text{erfc}(u / \sqrt{2}) / 2 - q; \\
u &= u - t / (1 + u * t / 2); \\
p &= (p > 0.5 ? -u : u);
\end{align*}
\]

// Gaussian quadrature (Gauss–Hermite and Gauss–Legendre)

```cpp
void gauss_hermite (vector<double>& x, vector<double>& w)
{
// The function determines vectors x and w, the abscissas and weights of the n-point Gauss–Hermite quadrature formula. Abscissa values are returned in descending order. The number of points n is equal to the initial size of z.
{
    const size_t maxit = 30;
    const double dsqrt_pi = sqrt(M_SQRTPi / 2.);
    size_t n = x.size();
    w.resize(n);

    // The roots are symmetric about the origin, so we have to find only half of them. Loop over the desired roots:
    for (size_t i = 0; i < (n + 1) / 2; ++i) {
        // (A) Initial guesses for the root values z
        double z;
        if (i == 0)
            z = sqrt(2 * n + 1) - 1.85575 * pow(2 * n + 1, -0.16667);
        else if (i == 1)
            z = 1.14 * pow((double)n, 0.426) / z;
        else if (i == 2)
            z = 1.86 * z - 0.86 * x[0];
        else if (i == 3)
            z = 1.91 * z - 0.91 * x[1];
        else
            z = 2.0 * z - x[i-2];

        // (B) Refinement by Newton's method.
        size_t its;
        double pp;
        for (its = 0; its < maxit; ++its) {
            // Loop up the recurrence relation to get the Hermite polynomial evaluated at z.
            double p1 = dsqrt_pi;
            double p2 = .0;
            double p3;
            for (size_t j = 1; j <= n; j++) {
                p3 = p2;
                p2 = p1;
                p1 = z * sqrt(2. / j) * p2 - sqrt((j - 1.) / j) * p3;
            }
        }
    }
}
```
// p1 is now the desired Hermite polynomial.
// We next compute its derivative pp using p2,
// the polynomial of one lower order.
pp = sqrt(2.*n) * p2;
double x1 = z;

// Newton's formula for the root and convergence condition
z = z1 - p1 / pp;
if (fabs(z - z1) <= 3e-14) break;

// (C) If the iteration was successful:
// - Store the root and its symmetric counterpart
// - Compute the weight and its symmetric counterpart
if (its >= maxit) {
    x = w = zero_vector<double>(n);
    return;
}

x[i] = z;
x[n - 1 - i] = -z;
w[i] = 2. / (pp * pp);
w[n - 1 - i] = w[i];
}

void gauss_legendre(
    const double& lb, const double& ub, vector<double>& x, vector<double>& w)
// The function determines vectors x and w, the abscissas and weights of the
// n-point Gauss-Legendre quadrature formula. Abscissa values are returned in
// descending order. The number of points n is equal to the initial size of x.
// The interval of integration is determined by the lower and upper bounds
// lb and ub respectively.
// {
    size_t n = x.size();
w.resize(n);

    size_t m = (n + 1) / 2;
double xm = 0.5 * (ub + lb);
double xl = 0.5 * (ub - lb);

    // The roots are symmetric about the origin, so we have to find
    // only half of them. Loop over the desired roots: for each root start
    // with an initial guess and enter a refinement loop using Newton's
    // method.
    for (size_t i = 0; i < m; ++i) {
        double z = cos(M_PI * (i + 0.75) / (n + 0.5));
double xl, pp;
do {
            double p1 = 1, p2 = 0, p3;
            for (size_t j = 1; j <= n; ++j) {
                // recurrence relation: obtain Legendre polynomial at z.
p3 = p2;
p2 = p1;
p1 = ((2 * j - 1) * z * p2 - (j - 1) * p3) / j;
            }
            // compute pp, the derivative of p1 at z, by a standard relation
            // involving also p2, the polynomial of one lower order.
            pp = n * (z * p1 - p2) / (z * z - 1);
xl = z;
z = xl - p1 / pp;
} while (fabs(z - xl) > 3e-14);
// scale root, put in symmetric counterpart, compute the weights.
x[i] = xm - xl * z;
\[ x[n - 1 - i] = x_m + x_l \ast z; \]
\[ w[i] = 2 \ast x_l / ((1 - z \ast z) \ast pp \ast pp); \]
\[ w[n - 1 - i] = w[i]; \]

// Root finding

double ridders_root (  
    double (*func)(double) ,  
    const double& lb ,  
    const double& ub ,  
    const double& epsilon = 1e-10 ,  
    size_t maxit = 300 )  
{
    double fl = (*func)(lb);  
    double fh = (*func)(ub);  
    if (fl * fh < 0) {  
        double xl = lb;  
        double xh = ub;  
        double xroot = NAN;  
        for (size_t j = 0; j < maxit; ++j) {  
            // First of two function evaluations: interval mid point  
            double xm = 0.5 \ast (xl + xh);  
            double fm = (*func)(xm);  

            // Update unless calculation has converged  
            double s = sqrt(fm \ast fm - fl \ast fh);  
            if (s == 0.0)  
                return xroot;  
            double x = xm + (xm - xl) \ast (((fl >= fh ? 1.0 : -1.0) \ast fm / s));  
            if (std::abs(x - xroot) <= epsilon)  
                return xroot;  

            // Second of two function evaluations: possible root  
            // Keep the root bracketed on next iteration  
            double f = (*func)(x);  
            if (fm \ast f <= 0) {  
                xl = xm; xh = x;  
                fl = fm; fh = f;  
            } else if (fl \ast f <= 0) {  
                xh = x;  
                fh = f;  
            } else if (fh \ast f <= 0) {  
                xl = x;  
                fl = f;  
            } else  
                return NAN;  

            xroot = x;  
            if (std::abs(xh - xl) <= epsilon) return xroot;  
        }  
    // exceeding maximum iterations  
    return NAN;  
    }
    else if (fl == 0.0) return lb;  
}
else if (fh == 0.0) return ub;
else return NAN;
}

// //////////////////////////////////////////////////////////////////////////////
// Loss probabilities for a pool of obligors with dependent defaults
// (Single-factor Gaussian-copula model)
inline vector<double>
sfg_prob (const vector<double>& p, const double& rho, const double& x)
{
    double denominator = sqrt(1 - rho);
    double sqrt_rhox = sqrt(rho) * x;

    vector<double> _p (p.size());
    for (int i = 0; i < _p.size(); i++)
        _p[i] = p_norm((q_norm(p[i]) - sqrt_rhox) / denominator);
    return _p;
}

inline vector<double>
loss_prob (const vector<size_t>& w, const vector<double>& p)
// Probability of loss in a pool of statistically independent obligors,
// using a recursion relation for discretized loss. Reference:
// [1] L Andersen, J Sidenius and S Basu, All your hedges in one basket,
// Risk Magazine, November 2003
{
    size_t l_max = std::accumulate(w.begin(), w.end(), 0);
    vector<double> pout = zero_vector<double>(l_max + 1);
    pout[0] = 1;
    l_max = 0;

    // Loop up the recurrence/recursion relation
    for (size_t k = 0; k < p.size(); ++k) {
        l_max += w[k];
        for (int l = l_max; l >= 0; --l)
            pout[l] = pout[l] * (1 - p[k])
                + (l < w[k] ? 0 : pout[l - w[k]] * p[k]);
    }
    return pout;
}

vector<double> sfg_loss_prob (const vector<size_t>& w, const vector<double>& p, const double& rho, const size_t& nq)
// Single-factor Gaussian-copula loss distribution
{
    // absicssas and weights for Gauss quadrature
    vector<double> xq(nq), wq(nq);
    gauss_hermite(xq, wq);
    xq *= M_SQRT2;
    wq *= M_SQRTPI / 2;
    gauss_legendre(-8, 8, xq, wq);
    for (size_t i = 0; i < nq; ++i)
        wq[i] *= M_SQRTPI / 2 / M_SQRT2 * exp(-xq[i] * xq[i] / 2);
// compute conditional default probabilities and subsequently
// the loss probabilities for all values x[i]

size_t l_max = std::accumulate(w.begin(), w.end(), 0);

matrix<double> p_lx (l_max + 1, nq);

for (size_t i = 0; i < nq; ++i)
    column(p_lx, i) = loss_prob(w, sfg_prob(p, rho, xq[i]));

// compute unconditional loss probability by Gauss–Hermite quadrature
return prod(p_lx, wq);

} //////////////////////////////////////////////////////////////////////////////

// Loss expectation–values for a set of adjacent portfolio tranches

vector<double> et_losses (const vector<double>& p,
    const double& u,
    const vector<double>& a)
{
    if (a.size() < 1) return vector<double>(0);
    vector<double> el = zero_vector<double>(a.size() - 1);
    if (p.size() > 1)
    // loop tranches
    for (size_t i = 0; i < el.size(); ++i) {
        size_t l_min = std::min(p.size() - 1, (size_t)(a[i] / u) + 1);
        size_t l_max = std::min(p.size() - 1, (size_t)(a[i + 1] / u));
        // full tranche loss
        el[i] = (a[i + 1] - a[i]) * std::accumulate(p.begin() + l_max + 1, p.end(), .0);
        // partial tranche loss
        for (size_t l = l_min; l <= l_max; el[i] += (u * l - a[i]) * p[l++]);
    }
    return el;
}

/////////////////////////////////////////////////////////////////////////////

// JPMorgan standardized LHPGC model
// [1] McGinty et al., A model for base correlation calculation,
// J. P. Morgan Chase & Co., March 2004

// (fractional) tranche notional–values for a series of tenor dates

matrix<double> lhpgc_ftnv(
    const vector<double>& a,
    const vector<double>& rho,
    const vector<double>& tau,
    const double& s,
    const double& rec,
    const double& ppy,
    const bool& daf)
{
    const size_t ntau = tau.size();
    // quadrature: abscessas and weights
    // trapezoidal rule a la JPMorgan (seems to be most efficient)
    const size_t nq = 80;
    vector<double> xq(nq), wq(nq);

    // compute unconditional loss probability by Gauss–Hermite quadrature
    return prod(p_lx, wq);
}
for (size_t i = 0; i < nq; ++i) {
    xq[i] = 8.0 - 16.0 / (nq - 1) * i;
    wq[i] = 8.0 / (nq - 1) * M_2SQRTPI / M_2SQRT2 * exp(-xq[i] * xq[i] / 2);
}

// gauss_hermite(xq, wq);
// xq = M_2SQRT2;
// wq = M_2SQRTPI / 2;

// gauss_legendre(-8, 8, xq, wq);
// for (size_t i = 0; i < nq; ++i)
// wq[i] = M_2SQRTPI / 2 / M_2SQRT2 * exp(-xq[i] * xq[i] / 2);

// expected loss of all equity tranches (corresponding to the
// regular-tranche upper attachment-points and using the
// appropriate base correlation values)
// - loop tranches (inside loop of payment dates)
// - deduce (expected) fractional tranche notional-values for
// all payment dates (with or w/o approximation)

matrix<double> ftvv(a.size(), ntau);
const double time_to_maturity = std::accumulate(tau.begin(), tau.end(), .0);

for (size_t j = ntau; j > 0;) {
    double time_to_pay_date = std::accumulate(tau.begin(), tau.begin() + j, .0);
    j--;

    if (!daf || j == ntau - 1) {
        // compute (expected) fractional tranche notional-values at
        // particular pay date (or maturity) by numerical integration.
        // LARGE POOL APPROXIMATION

        vector<double> etel = zero_vector<double>(a.size());
        double pd_quantile
            = q_norm(1 - exp(-s / 1e4 / (1 - rec) * time_to_pay_date));
        //
        //
        for (size_t i = 0; i < a.size(); ++i) {
            double denominator = sqrt(1 - rho[i]);
            double sqrt_rho = sqrt(rho[i]);
            for (size_t k = 0; k < xq.size(); ++k) {
                double etel_x;
                etel_x = (pd_quantile - sqrt_rho * xq[k]) / denominator;
                etel_x = std::min(p_norm(etel_x) * (1 - rec), a[i]);
                etel[i] += etel_x * wq[k];
            }
            ftv(i, j) = (i == 0 ? 1 - etel[0] / a[0]
                          : 1 - (etel[i] - etel[i - 1]) / (a[i] - a[i - 1]));
        }
    } else {
        // compute (expected) fractional tranche notional-values at
        // particular pay date by exponential decay approximation

        double exponent = time_to_pay_date / time_to_maturity;
        for (size_t i = 0; i < a.size(); ++i)
            ftv(i, j) = pow(ftv(i, ntau - 1), exponent);
    }
}

return(ftv);

vector<double> lhpqc_quotes {
    const vector<double>& a // [0,1] tranche attachment-points (increasing)
const vector<double>& fs // [0,1) fixed tranche spread (yields upfront payment)
const vector<double>& rho // [0,1) base corr. s of the corresp. Ist-loss tr.
const vector<double>& tau // consecutive time intervals between valuation / payment / maturity dates (time fraction Act/360)
const double&s // (BP) CDS index spread (Actual/360 quarterly)
// default values correspond to standardized LHPGC model:
const double& rec = 0.4 // recovery rate
const double& rfr = 0 // risk-free rate (Actual/360 quarterly)
const double& ppy = 4 // payments per year, ie compounding frequency
const bool& daf = true // to use or not to use decay approximation
const bool& jpm = true // default time approximation as in JPM model

const size_t ntau = tau.size();

matrix<double> ftv = lhpge_ftv(a, rho, tau, s, rec, ppy, daf);

// calculation of discount factors (for fixed and floating leg)
vector<double> df_fixed (ntau), df_float (ntau);

df_fixed[0] = pow(1 - rfr / ppy, ppy * tau[0]);
for (size_t i = 1; i < ntau; ++i)
  df_fixed[i] = df_fixed[i - 1] * pow(1 - rfr / ppy, ppy * tau[i]);

df_float[0] = pow(1 - rfr / ppy, ppy * tau[0] / 2);
for (size_t i = 1; i < ntau; ++i)
  df_float[i] = df_float[i - 1] * pow(1 - rfr / ppy, ppy * (tau[i - 1] + tau[i]) / 2);

if (jpm) df_float = df_fixed;

// loop tranches to compute market quotes (upfront fee or tranche spread)
vector<double> quotes (a.size());
for (size_t i = 0; i < a.size(); ++i) {
  double risky_duration = 0;
  double contingent_leg = 0;
  for (size_t j = 0; j < ntau; ++j) {
    double incremental_loss = (j == 0 ? 1 - ftv(i, 0) : ftv(i, j - 1) - ftv(i, j));
    risky_duration += tau[j] * ftv(i, j) * df_fixed[j];
    contingent_leg += incremental_loss * df_float[j];
  }
  // for dfxxx = 1 and daf == true we must have:
  // contingent_leg == 1 - ftv(i, ntau - 1)

  // return spreads in basis points and upfront fee(s) as percent number(s)
  quotes[i] = (fs[i] < 0
              ? 1e4 * contingent_leg / risky_duration
              : 1e2 * (contingent_leg - fs[i] * risky_duration) );
}

return quotes;

// //////////////////////////////////////////////////////////////////////////////
// Single-factor Gaussian-copula model (w/o LARGE pool approximation)
// (i.e. with pool granularity / granular model / inhomogeneous model)
// Reference:
// [1] L Andersen, J Sidenius and S Basu, All your hedges in one basket,
// Risk Magazine, November 2003
// Here: assuming a portfolio composed of equally weighted names!
// //
// (fractional) tranche notional=values for a series of tenor dates
//
// matrix<double> sfgc_ftnv(
//   const vector<double>& a, // [0,1] tranche attachment-points (increasing)
//   const vector<double>& rho, // (BP) single-name CDS spreads
//   const vector<double>& tau, // consecutive time intervals between valuation / payment / maturity dates (time fraction Act/360)
//   const vector<double>& s, // (and a particular correlation parameter rho[i])
//   const vector<size_t>& w, // loss weights of single names
//   const double& u, // loss unit
//   const double& ppy // payments per year, ie compounding frequency
// )
//
//    matrix<
//      const vector<double>& a, const vector<double>& rho, const vector<double>& tau, const vector<double>& s, const vector<
//      size_t>& w, const double& u, const double& ppy
//    )
//
//  // quadrature: absccissas and weights
//  // trapezoidal rule a la JPMorgan (seems to be most efficient)
//  const vector<double> xq(nq), wq(nq);
//  for (size_t i = 0; i < nq; ++i) {
//    xq[i] = 8.0 - 16.0 / (nq - 1) * i;
//    wq[i] = 8.0 / (nq - 1) * M_SQRTPI / M_SQRT2 * exp(-xq[i] * xq[i] / 2);
//  }
//  
//  // compute (expected) fractional tranche notional=values at
//  // particular pay date (or maturity) by numerical integration
//  // (exposures). Assertion: unit_loss_rate * max[w[i]] <= 1
//
//  const double unit_loss_rate = u * s.size();
//  const size_t l_max = std::accumulate(w.begin(), w.end(), 0);
//  for (size_t j = ntau; j > 0; j--)
//    double time_to_pay_date = std::accumulate(tau.begin(), tau.end() + j, .0);
//    
//    vector<double> pd(s.size());
//    for (size_t i = 0; i < s.size(); ++i)
//      pd[i] = 1 - exp(-s[i] / 1e4 / (w[i] * unit_loss_rate) * time_to_pay_date);
//      
//    vector<double> etel = scalar_vector<double>(a.size(), s.size());
//    for (size_t i = 0; i < a.size(); ++i)
//      // --- THE MOST TIME-CONSUMING PART OF THE CALCULATION ---
//      // conditional default-probabilities & conditional loss-probabilities for all values xq[k]
//      // (and a particular correlation parameter rho[i])
//      matrix<double> prob_lx(l_max + 1, nq);
//      for (size_t k = 0; k < nq; ++k)
//        column(prob_lx, k) = loss_prob(w, sfg_prob(pd, rho[i], xq[k]));
//    }
//    // unconditional loss probabilities and equity-tranche ELs
//    vector<double> prob_l = prod(prob_lx, wq);
//    size_t l_max_tranche = std::min(prob_l.size() - 1, (size_t)(a[i] / u));
//    etel[i]
= a[i] * std::accumulate(prob_l.begin() + l_max_tranche + 1 , prob_l.end(), .0);
for (size_t l = 1; l <= l_max_tranche; ++l)
etel[i] += u * l * prob_l[l];

ftnv(i, j) =
( i == 0 ? 1 - etel[0] / a[0] : 1 - (etel[i] - etel[i - 1]) / (a[i] - a[i - 1]) );
}
return(ftnv);
}

vector<
double>
sfgc_quotes(
    const vector<double>& a, // [0,1] tranche attachment–points (increasing)
    const vector<double>& fs, // [0,1] fixed tranche spread: quote is upfront payment
    const vector<double>& rho, // [0,1] base corr. s of the corresp. Ist–loss tr.
    const vector<double>& tau, // consecutive time intervals between valuation / payment / maturity dates  (time fraction Act/360)
    const vector<double>& s, // (BP) single–name CDS spreads
    const vector<size_t>& w, // loss weights of single names
    const double& u, // loss unit
    // default values correspond to standardized LHPGC model:
    const double& rfr = 0 // risk–free rate (Actual/360 quarterly)
    const double& ppy = 4 // payments per year, ie compounding frequency
)
{
    const size_t ntau = tau.size();
    if (a.size() < 1) return vector<double>();

    matrix<double> ftnv = sfgc_ftnv(a, rho, tau, s, w, u, ppy);

    // calculation of discount factors (for fixed and floating leg)
    vector<double> df_fixed (ntau), df_float (ntau);
    df_fixed[0] = pow(1 - rfr / ppy, ppy * tau[0]);
    for (size_t i = 1; i < ntau; ++i)
        df_fixed[i] = df_fixed[i - 1] * pow(1 - rfr / ppy, ppy * tau[i]);
    df_float[0] = pow(1 - rfr / ppy, ppy * tau[0] / 2);
    for (size_t i = 1; i < ntau; ++i)
        df_float[i] = df_float[i - 1] * pow(1 - rfr / ppy, ppy * (tau[i - 1] + tau[i]) / 2);

    // loop tranches to compute market quotes (upfront fee or tranche spread)
    vector<double> quotes(a.size());
    for (size_t i = 0; i < a.size(); ++i) {
        double risky_duration = 0;
        double contingent_leg = 0;
        for (size_t j = 0; j < ntau; ++j) {
            double incremental_loss = (j == 0 ? 1 - ftnv(i, 0) : ftnv(i, j - 1) - ftnv(i, j));
            risky_duration += tau[j] * ftnv(i, j) * df_fixed[j];
            contingent_leg += incremental_loss * df_float[j];
        }
    }

    // return spreads in basis points and upfront fee(s) as percent number(s)
    quotes[i] = (fs[i] < 0 ? 1e4 * contingent_leg / risky_duration : 1e2 * (contingent_leg - fs[i] * risky_duration) );
}
return {quotes};
}

vector<double> sfgc_pvs(
    const vector<double>& a,    // [0,1] tranche attachment points (increasing)
    const vector<double>& fs,   // [0,1] fixed tranche spread: quote is upfront payment
    const vector<double>& rho,  // [0,1] base corr. of the correspond. 1st-loss tr.
    const vector<double>& q,    // market quote of the corresponding (proper) tranche
    const vector<double>& tau,  // consecutive time intervals between valuation / payment / maturity dates (time fraction Act/360)
    const vector<double>& s,    // (BP) single-name CDS spreads
    const vector<double>& w,    // loss weights of single names
    const double& u,            // loss unit
    // default values correspond to standardized LHPGC model:
    const double& rfr = 0,       // risk-free rate (Actual/360 quarterly)
    const double& ppy = 4        // payments per year, ie. compounding frequency
) {
    const size_t ntau = tau.size();
    if (a.size() < 1) return vector<double>(0);

    matrix<double> ftv = sfgc_ftv(a, rho, tau, s, w, u, ppy);
    // calculation of discount factors (for fixed and floating leg)
    vector<double> df_fixed(ntau), df_float(ntau);
    df_fixed[0] = pow(1 - rfr / ppy, ppy * tau[0]);
    for (size_t i = 1; i < ntau; ++i)
        df_fixed[i] = df_fixed[i - 1] * pow(1 - rfr / ppy, ppy * tau[i]);
    df_float[0] = pow(1 - rfr / ppy, ppy * tau[0] / 2);
    for (size_t i = 1; i < ntau; ++i)
        df_float[i] = df_float[i - 1] * pow(1 - rfr / ppy, ppy * (tau[i - 1] + tau[i]) / 2);
    // loop tranches to compute market quotes (upfront fee or tranche spread)
    vector<double> pvs(a.size());
    for (size_t i = 0; i < a.size(); ++i) {
        double risky_duration = 0;
        double contingent_leg = 0;
        for (size_t j = 0; j < ntau; ++j) {
            double incremental_loss = (j == 0 ? 1 - ftv(i, 0) : ftv(i, j - 1) - ftv(i, j));
            risky_duration += tau[j] * ftv(i, j) * df_fixed[j];
            contingent_leg += incremental_loss * df_float[j];
        }
    }
    // ONLY THE FOLLOWING LINES ARE DIFFERENT COMPARED TO 'sfgc_quotes()' 
    // return: present values of the STCDO as (seen from the fixed leg receiver)
    pvs[i] = (fs[i] < 0 ? q[i] / 1e4 * risky_duration - contingent_leg : q[i] / 100 + fs[i] * risky_duration - contingent_leg);
}
return(pvs);

// JPMorgan (standardized) LHPGC model (see above) & single-factor Gaussian
struct lhpgcPars {
    vector<double> a;
    vector<double> fs;
    vector<double> q;
    vector<double> rho;
    vector<double> tau;
    double s;
    double rec;
    vector<double> svec;
    vector<
        size_t
    > w;
    double u;
    double rfr;
    double ppy;
    bool daf;
    bool jpm;
} global_gcq_pars;

LHPGC base correlations solver

double lhpgc_quote_delta_base(double rho_m) {
    size_t m = global_gcq_pars.q.size() - 1;
    global_gcq_pars.rho[m] = rho_m;
    return lhpgc_quotes(
        global_gcq_pars.a,
        global_gcq_pars.fs,
        global_gcq_pars.q,
        global_gcq_pars.tau,
        global_gcq_pars.s,
        global_gcq_pars.rec,
        global_gcq_pars.rfr,
        global_gcq_pars.ppy,
        global_gcq_pars.daf,
        global_gcq_pars.jpm)[m] - global_gcq_pars.q[m];
}

vector<double> lhpgc_base_correls {
    const vector<double>& a
    , const vector<double>& fs
    , const vector<double>& q
    , const vector<double>& tau
    , const double& s
    , const double& rec = 0.4
    , const double& rfr = 0
    , const double& ppy = 4
    , const bool& daf = true
    , const bool& jpm = true
} {
    vector<double> rho = scalar_vector<double>(a.size(), -1);
    global_gcq_pars.tau = tau;
    global_gcq_pars.s = s;
    global_gcq_pars.rec = rec;
    global_gcq_pars.rfr = rfr;
    global_gcq_pars.ppy = ppy;
    global_gcq_pars.daf = daf;
    global_gcq_pars.jpm = jpm;
    for (size_t m = 0; m < a.size();)
    {

    
}
double rho_m = ridders_root(&lhpgc_quote_delta_base, 0, 0.999999, 1e-7);
if (std::isnan(rho_m)) rho[m - 1] = rho_m;
else break;
}
return rho;

}  

// Single-factor GC base correlation solver  
// double sfgc_quote_delta_base(double rho_m)
{
    size_t m = global_gcq_pars.q.size() - 1;
global_gcq_pars.rho[m] = rho_m;
    return sfgc_quotes(
        global_gcq_pars.a,
        global_gcq_pars.fs,
        global_gcq_pars.q,
        global_gcq_pars.tau,
        global_gcq_pars.svec,
        global_gcq_pars.w,
        global_gcq_pars.u,
        global_gcq_pars.rf,
        global_gcq_pars.ppy)
[m] - global_gcq_pars.q[m];
}
vector<vector<double>> sfgc_base_correls {
    const vector<vector<double>>& a // [0,1] tranche attachment points (increasing)
    , const vector<vector<double>>& fs // [0,1] fixed tranche spread; quote is upfront payment
    , const vector<vector<double>>& q // market quote of the corresponding (proper) tranche
    , const vector<vector<double>>& tau // consecutive time intervals between valuation /
    // payment / maturity dates (time fraction Act/360)
    , const vector<double>& s // (BP) single-name CDS spreads
    , const vector<double>& w // loss weights of single names
    , const double& u // loss unit
    // default values correspond to standardized LHPGC model:
    , const double& rf = 0 // risk-free rate (Actual/360 quarterly)
    , const double& ppy = 4 // payments per year, ie compounding frequency
}
vector<double> rho = scalar_vector<vector<double>>(a.size(), -1);
global_gcq_pars.tau = tau;
global_gcq_pars.svec = s;
global_gcq_pars.w = w;
global_gcq_pars.u = u;
global_gcq_pars.rf = rf;
global_gcq_pars.ppy = ppy;
for (size_t m = 0; m < a.size();)
    +m;
global_gcq_pars.a = subrange(a, 0, m);
global_gcq_pars.fs = subrange(fs, 0, m);
global_gcq_pars.q = subrange(q, 0, m);
global_gcq_pars.rho = subrange(rho, 0, m);

double rho_m = ridders_root(&lhpgc_quote_delta_base, 0, 0.999999, 1e-4);
if (std::isnan(rho_m)) rho[m - 1] = rho_m;
else break;
return rho;

}  
  
//  
// LHPGC compound correlation solver  
//
double lhpgc_quote_delta_comp(double rho)  
{
  size_t m = global_gcq_pars.q.size();
  return lhpgc_quotes(  
    global_gcq_pars.a,
    global_gcq_pars.fs,
    scalar_vector<double>(m, rho),
    global_gcq_pars.tau,
    global_gcq_pars.s,
    global_gcq_pars.rec,
    global_gcq_pars.rf,
    global_gcq_pars.ppy,
    global_gcq_pars.daf,
    global_gcq_pars.jpm)[m − 1] − global_gcq_pars.q[m − 1];
}

vector<double> lhpgc_comp_corrsls(  
  const vector<double>& a,  
  const vector<double>& fs,  
  const vector<double>& q,  
  const vector<double>& tau,  
  const double& s,  
  const double& rec = 0.4,  
  const double& rfr = 0,  
  const double& ppy = 4,  
  const bool& daf = true,  
  const bool& jpm = true)  
{  
  vector<double> rho = scalar_vector<double>(a.size(), −1);
  global_gcq_pars.tau = tau;
  global_gcq_pars.s = s;
  global_gcq_pars.rec = rec;
  global_gcq_pars.rf = rfr;
  global_gcq_pars.ppy = ppy;
  global_gcq_pars.daf = daf;
  global_gcq_pars.jpm = jpm;

  for (size_t m = 0; m < a.size(); ++m) {  
    size_t k = (m > 0 ? m − 1 : 0);
    size_t l = (m > 0 ? m + 1 : 1);
    global_gcq_pars.a = subrange(a, k, l);
    global_gcq_pars.fs = subrange(fs, k, l);
    global_gcq_pars.q = subrange(q, k, l);

    for (double rho_max = 0.249999; rho_max < 1; rho_max += 0.15) {  
      double rho_m = ridders_root(&lhpgc_quote_delta_comp, 0, rho_max, 1e−5);
      if (!isnan(rho_m)) {  
        rho[m] = rho_m;
        break;
      }
    }

  }

  return rho;
}
References


Acknowledgements

iucundi acti labores — Cicero

no pain, no gain

A lot has happened between my first warm-up to write this thesis and its completion. After finishing all modules and essays for the Mathematical Finance course during the summer 2003, it took me three years until completion of this thesis. Our son Ferdinand Brunne was born in 2003 and grew up. My wife Minette finished and defended her doctoral thesis in medicine and started to work as a radiologist again. The family moved from Berlin to Frankfurt and Minette and I both moved on to new workplaces. During the three years, I was trying to catch up on thesis work several times but other duties called for my attention shortly after.

Therefore, I am much indebted to the Department of Continuing Education, the teaching staff and course directors for their appreciation and acceptance of numerous applications for extensions and, likewise, to my former employers Arthur Andersen and d-fine for their financial support. I am grateful to Dr Hans-Peter Deutsch who initiated the advanced training programme at Arthur Andersen in collaboration with Oxford University. Despite the strain and difficulties alluded here it was fun and a great pleasure having had the opportunity to participate in the Mathematical Finance programme.

The course on credit risk taught by Dr Dominic O’Kane of Lehman Brothers, London, during one of the course modules in 2003 sparked my interest in the subject of this thesis. I regret that this work has shown little progress during the years Dominic patiently acted as my official thesis supervisor.

When I decided to make a final attempt to write an MSc thesis during Trinity 2006, I was very happy that Dr Philip Gisdakis of HVB in Munich immediately agreed to supervise the final term. Thanks Philip!

In the last paragraph the reader will rightly expect words of thanks to the family. As my part-time studies in Oxford including thesis work had to be done mostly while working full-time simultaneously, my wife and son often regretfully accepted that I was busy during a time we could have shared otherwise. Thank you so much for your love and patience!
The scope of this thesis is the analysis of market-implied credit default correlation. Implied correlations are computed for synthetic single-tranche CDOs on standardised portfolios. Such STCDOs are the first portfolio credit derivatives that are liquidly traded worldwide since about 2003.

The first Chapter of the thesis compares actual and risk-neutral credit curves for single-name credit risk. Actual default term-structures are deduced from historical rating-migration data. The pricing of single-name CDS is discussed briefly and risk-neutral default probabilities, implied by CDS spreads, are compared to the real-world probabilities of default. It is found that the estimation of real-world credit term-structures has large uncertainties. The market-implied default probabilities are very volatile but usually well above the real-world probabilities.

The second Chapter briefly introduces default correlation modelling, STCDO pricing with single-factor Gauss-copula models and the base correlation framework. Using this pricing framework market implied base-correlations are computed for historic price data covering more than year of daily quotes. The dependency of implied base-correlations on the model parameters portfolio granularity and spread dispersion is investigated. Dealer quotes for implied correlation are compared to correlations computed with the models of this thesis. It is found that increasing portfolio granularity decreases implied base-correlations in the homogeneous-pool single-factor Gauss-copula model. The effect is most pronounced for the equity tranches. Taking into account credit-spread dispersion seems to increase implied base-correlations. Dealer quotes of correlation could not be reproduced using the pricing framework of this thesis, which indicates that they are based on proprietary and undisclosed pricing models.