Option Pricing for Discrete Hedging and Non-Gaussian Processes

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The Black-Scholes option pricing method is correct under certain assumptions, among others continuous hedging and a log-normal underlying process. If any of these two assumptions is not fulfilled, a riskless replication of an option is in general not possible. To handle this case, a pricing method was proposed by Bouchaud and Sornette. Similar to Black-Scholes, a hedging portfolio is considered. The hedging strategy is such that the risk of the hedging portfolio is minimized. An option price is then deduced from this hedging strategy. Since a risk remains, the price includes a risk premium.

In this thesis, a new alternative method is presented, which instead of minimizing the portfolio risk minimizes the option price. This makes the option most competitive on the market. For the option writer, the ratio of return to risk is, by definition of the method, the same as for the Bouchaud-Sornette approach.

Both methods were compared with each other. For typical options, differences of up to 10% of the price were found. The risk premium as well as fat tails in the underlying process give rise to volatility smiles for both methods. Furthermore, it was found that both methods are consistent with Black-Scholes pricing. The results converge towards the Black-Scholes result in the continuous time limit for a log-normal process.
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Chapter 1

Introduction

A milestone in the development of derivative pricing was the discovery by Black and Scholes in 1973 that under certain assumptions options can be replicated by an investment in the underlying and a cash balance[1]. In an arbitrage free economy, the price of the derivative is then directly given by the value of the replicating portfolio. Since then, a whole pricing theory was build on the replication method. The unique price from this valuation is called the fair price.

To get a unique price, amongst others, the following conditions must hold [2, 3, 4]:

- Either trading can be done continuously in time or the underlying process follows a binominal model. In a binominal model, only two states for the underlying price change exist for each time step. In this case, the risk can be eliminated completely even for discrete hedging. In a trinominal model with three possible price changes per time step, a risk remains.

- The process of the underlying has to be known for infinitesimal time steps. It has to be a semi-martingale. In the standard Black-Scholes model, a log-normal process is assumed.

Obviously, these conditions are not fulfilled in the real world. Hedging can only take place in discrete time steps. And the binomial model is not sufficient to describe the underlying price movements because of fat tails in the price distributions. Fat tails cannot be reproduced with the binominal model since the binominal model converges towards a Gaussian probability distribution for infinitesimal small time steps.

Another problem comes from the fact, that continuous hedging requires the underlying process to be known for infinitesimal time steps. This information cannot be extracted directly from market data where one has access only to price changes for finite time intervals. So, one has to find a process which is a semi-martingale and
which fits to market data. It is not guaranteed that such a process always exist. If it does exist, the process might not have been studied in the literature yet, so that the valuation is not trivial.

This thesis studies the case where hedging is not continuous. This is a more realistic assumption than continuous hedging and also makes it easy to model the underlying process. Since only discrete time intervals are considered, the underlying price change has to be known only for these time steps and therefore can be directly taken from market data. This removes restrictions on the underlying process and makes it easy to include special features of the underlying process, like fat tails.

For such a case, Bouchaud and Sornette proposed an approach to option pricing in 1994 [5, 6]. It is based on the Black-Scholes idea to build up a replication portfolio. The replication should be as perfect as possible. In the sense of Bouchaud and Sornette, this means that the hedging strategy minimizes the risk of the hedging portfolio. The fair option price is then given by the replication strategy. Since a risk remains, a risk premium has to be added to the fair price.

The shortcoming of the Bouchaud-Sornette method is that there is no argument why their option price should be accepted by the market. In the Black Scholes world, there is the no-arbitrage argument which guarantees that the price of the replicated derivative is the market price. Such an argument is missing for the Bouchaud-Sornette approach.

In this thesis another, new method is presented. As the other methods, it is based on a hedging portfolio for which an optimal hedging strategy has to be found. By definition of the method, it will give the same ratio of return to risk as the Bouchaud-Sornette approach. But instead of minimizing the risk, the option writer minimizes the option price. This makes the option most competitive on the market and provides the argument why this price should be the market price. This argument, of course, only holds as long as all investors have the same risk preference. In general, a model-independent, unique price doesn’t exist whenever a risk remains for the option writer.
Chapter 2

Basic Concepts

2.1 Asset Price Model

The most basic input to a derivative pricing method is the description of the underlying. Describing the underlying means to have a model for the statistical movement of the underlying price $S(t)$ with time. Such models will be discussed in this section. Since this thesis has its focus on comparing pricing models, only assets will be considered as underlying, which is the simplest case.

2.1.1 Statistical Variable for the Price Process

The very first thing in working out a statistical asset price model is to decide which statistical variable to use. There are two obvious choices, namely absolute price differences

$$\Delta S_i = S(t_i) - S(t_{i-1})$$

(2.1)

with $t_i = t_{i-1} + \Delta t$ or price returns

$$R_i = \frac{S(t_i)}{S(t_{i-1})}$$

(2.2)

which are relative changes. The reason why to think about the choice of variables is to simplify the model. A simple model would be based on independent identically distributed (i.i.d.) random variables. A time series of independent random variables $x_i$ has the property that the correlation between the variables at different times vanish

$$E[x_i, x_j] = \delta_{ij} E[x_i^2]$$

(2.3)

Here, $E[...]$ stands for the expectation value. The variables are identically distributed if the moments $E[x_i^n]$ are independent on time, so the moments are equal for any $i$. 
Studies on data show that for short time steps absolute changes are closer to an i.i.d. variable, while for longer times (about one month on liquid markets) this is the case for returns [7]. In this thesis, returns will be chosen although the time scale given by the rehedging frequency of options is smaller than a month. But this choice allows to compare with the standard Black-Scholes model for a log-normal process. To be precise, the logarithm, log $R_t$, of the return will be used instead of the return itself, so that the variable is additive:

$$\log R_{t_i,t_k} = \log \frac{S(t_k)}{S(t_j)} = \sum_{i=j+1}^{k} \log \frac{S(t_i)}{S(t_{i-1})} = \sum_{i=j+1}^{k} \log R_i$$

(2.4)

It is more convenient to work with an additive random variable and the logarithm restricts the asset price to positive values.

Another question is which time scale to use. In equations 2.1 and 2.2, the argument $t_i$ should be understood as the trading time in days with equidistant time intervals. Instead, one could also think about using the real time, including weekends, or to use the number of transactions as a time measure [8]. Using the trading time is the standard choice for asset models.

Apart from choosing the random variable and the time scale, a probability density to describe the distribution of the values of the random variable will complete the model. The probability density will be addressed in the next section.

2.1.2 Probability Density of the Process

2.1.2.1 Stable Distributions

Suppose there is an additive i.i.d. random variable $x_i$, for which the probability density $p(x)$ is given. If $x_i$ describes a price change then it is of interest how the overall price change

$$X_n = x_1 + x_2 + \cdots + x_n$$

(2.5)

for several time steps is distributed. The probability distribution $p_n(X)$ for the sum is obtained by the $n^{th}$ autoconvolution of $p(x)$. The autoconvolution can be easily done by applying a Fourier transformation to the probability density

$$\phi(q) = \int_{-\infty}^{\infty} p(x) e^{iqx} \, dx$$

(2.6)

A convolution corresponds to a multiplication in Fourier space. Therefore, the Fourier transformation of $p_n(X)$ is

$$\phi_n(q) = [\phi(q)]^n$$

(2.7)
and the inverse transformation yields

\[ p_n(X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\phi(x)]^n e^{-iqx} \, dq \quad (2.8) \]

The question of what happens if \( n \) goes to infinity is subject to a central limit theorem. For any probability density \( p(x) \) with tails \( p(x \to \pm \infty) \propto c_{\pm}/|x|^{1+\alpha} \) and \( 0 < \alpha \leq 2 \), the function \( \log \phi_n(q) \) will converge towards a member of a class of functions \[ \log \phi(q) = \begin{cases} 
\mu q - \gamma |q|^{\alpha} & \alpha \neq 1 \\
\mu q - \gamma |q|^{1+\alpha} / (2 \pi) \ln |q| & \alpha = 1 
\end{cases} \quad (2.9) \]

with \( 0 < \alpha \leq 2, \gamma \geq 0, \mu \) any real number, and \(-1 \leq \beta \leq 1 \). This class of functions describes Lévy processes which have the following characteristics \[8\]

- The process is stable.

  A stable function has the same functional form after convolution. Examples are the Gaussian and Lorentzian distributions.

- For \( \alpha = 2 \) the process is a Gaussian and for \( \alpha = 1, \beta = 0 \) a Lorentzian process.

  Apart from these processes, only for \( \alpha = 0.5, \beta = 1 \) (Lévy-Smirnov) the analytic form of the probability distribution is known.

- The probability distribution has power law tails for \( \alpha \neq 2 \):

  \[ p(x) \propto \frac{1}{|x|^{1+\alpha}} \quad \text{for large } |x| \quad (2.10) \]

- The variance is infinite for \( \alpha \neq 2 \). Only the Gaussian has finite variance.

- Lévy functions are the only attractors for probability functions \( p_n(X) \) in the limit \( n \to \infty \).

From the statements above, we can conclude that in continuous time any price process of i.i.d. random variables with tails \( p(x \to \pm \infty) \propto c_{\pm}/|x|^{1+\alpha} \) and \( 0 < \alpha \leq 2 \) will result in a Lévy distribution for finite time intervals. If the variance of a process is finite, it will converge towards a Gaussian process. In this case, the variance of \( X_n \) will scale like \( n \), e.g. for \( n = 2 \)

\[ \text{Var}[X_2] = E \left[ (x_1 + x_2 - E[x_1 + x_2])^2 \right] \quad (2.11) \]

\[ = 2 \cdot E \left[ x^2 \right] + 2 \cdot E \left[ x_1 \cdot x_2 \right] - 2 \cdot E \left[ (x_1 + x_2) \cdot E[x_1 + x_2] \right] + 4 \cdot E[x]^2 \]

\[ = 2 \cdot E \left[ x^2 \right] + 2 \cdot E \left[ x \right]^2 - 8 \cdot E \left[ x \right]^2 + 4 \cdot E \left[ x \right]^2 \]

\[ = 2 \cdot E \left[ x^2 \right] - 2 \cdot E \left[ x \right]^2 = 2 \cdot \text{Var}[x] \]

\[ = 2 \cdot \text{Var}[x] \]
If each $x_i$ is a price change for one equidistant time step $\Delta t$, the variance scales with $\Delta t$.

### 2.1.2.2 Real Price Dynamics

In the literature, sophisticated analysis of price changes can be found [7, 8, 10]. The main results are illustrated by Figure 2.1 where daily log-returns of the IBM stock are shown:

- A Gaussian distribution fails to describe the fat tails in data.
- The variance is finite.
- It is not a Lévy distribution since the variance is finite and the distribution doesn’t follow a Gaussian.

From these observations on data and the statements from the last section, we can conclude that in continuous time price changes cannot be described by i.i.d. random variables. However, if we give up to work in continuous time, the assumption of i.i.d. random variables is reasonable, e.g. the return for two days is quite well described by a convolution of twice the daily return distribution.

When coming to a comparison of pricing methods later in this thesis, the effect of exponential tails, as apparent in figure 2.1, will be studied and compared to a Gaussian process.

### 2.2 The Hedging Portfolio

#### 2.2.1 Self-Financing Trading Strategies

In the Black Scholes world, the risk of a derivative can be hedged away completely. This means, that the writer of a derivative can set up a self-financing portfolio which consists of a short position in the derivative, a time dependent position in the underlying and a cash balance. The amount of the underlying can be adjusted by trading in such a way that the risk of the whole portfolio vanishes. All methods described in this thesis are based on the optimisation of the trading strategy for this portfolio. The basis for this optimisation will be set in this section by discussing self-financing trading strategies and by deriving features of the hedging portfolio.

A self-financing trading strategy for an asset is defined as follows:
Figure 2.1: Daily log-return distribution for IBM stocks (1962-2000) compared to a Gaussian distribution and a Gaussian distribution with exponential tails. The Gaussian distribution with exponential tails fits the data well with a $\chi^2$ per degree of freedom of 0.93.

- At initial time $t_0$, the trading portfolio is empty. The trading portfolio is defined as the part of the hedging portfolio which consists of the asset position and the cash balance.

- At any time between $t_0$ and $T$, a finite amount of the asset can be bought or sold. Short selling is allowed. If working in discrete time, investments can be done only for finite time intervals.

- The positive or negative cash return from the trading in the underlying is invested at the spot rate.

Obviously, the value of the trading portfolio at time $T$ only depends on the investment strategy, the spot price in the past and present, and the interest rate, which is assumed to be constant here. An explicit formula for the value of the trading portfolio will be derived now. The following naming conventions are used:
\( B(t_0, t) \) The discount factor for the time period from \( t_0 \) to \( t \). In discrete time with equidistant time intervals and constant spot rate, we define \( B_l \equiv B(t_0, t_l) = B^l(t_0, t_1) = B^l_1 \equiv B^l \). This, however, is only an approximation, since the discount factor depends on the difference in calendar days and not on the difference in trading days. A proper definition of the discount factor could be used, but for the purpose of comparing the pricing methods it is not essential.

\( S(t) \) The asset price at time \( t \). In discrete time: \( S_l \equiv S(t_l) \).

\( A(t) \) The cash balance from the investment in the asset. In discrete time: \( A_l \equiv A(t_l) \).

\( \phi(S, t) \) The amount of the underlying asset hold at time \( t \). In discrete time: \( \phi_l \equiv \phi(S_l, t_l) \).

The trading strategy \( \phi(S, t) \) is defined to be independent on the cash balance \( A(t) \). The hedging strategy should not depend on the amount of cash in the portfolio, e.g. adding cash to the portfolio at any time should not change the strategy. However, if the option is path dependent, the hedging strategy will be path dependent as well. In this thesis, path dependent options are not addressed.

With the naming conventions, the value of the trading portfolio is given by:

\[
H(t) = \phi(S, t) \cdot S(t) + A(t) \quad (2.12)
\]

where the first term on the right hand side is due to the value of the assets hold and the second term is the cash balance. In discrete time at maturity \( T \), this is equivalent to:

\[
H_m = \phi_m \cdot S_m + A_m \quad (2.13)
\]

with \( T \equiv t_m \).

From this equation, the term \( A_m \) can now be eliminated. The requirement of self-financing is equivalent to

\[
A_l - A_{l-1} = \underbrace{A_{l-1} \cdot (B^{-1} - 1)}_{\text{interest}} - \underbrace{(\phi_l - \phi_{l-1}) \cdot S_l}_{\text{hedging adjustment}} \quad (2.14)
\]

\[\Rightarrow A_l = B^{-1} \cdot A_{l-1} - (\phi_l - \phi_{l-1}) \cdot S_l \]

Together with the fact that the initial investment at time \( t_0 \) is \( \phi_0 S_0 = -A_0 \), it follows that

\[
A_m = -\phi_0 S_0 B^{-m} - \sum_{l=1}^{m} (\phi_l - \phi_{l-1}) \cdot S_l \cdot B^{l-m} \quad (2.15)
\]
Substituting this in equation 2.13 removes the explicit cash balance term \( A_m \) from the expression for the portfolio value:

\[
H(T) = \phi_m S_m + A_m
\]

\[
= \phi_m S_m - \phi_0 S_0 B^{-m} - \sum_{l=1}^{m} (\phi_l - \phi_{l-1}) S_l B^{l-m}
\]

(2.16)

For the optimization of the trading strategy \( \phi \), it is convenient to rearrange this formula to

\[
H(T) = \sum_{l=0}^{m-1} \phi_l (S_{l+1} - B^{-1} S_l) B^{l-m+1}
\]

(2.17)

With the naming conventions

\[
\tilde{S}_l \equiv S_l B^{l-m}
\]

\[
\Delta \tilde{S}_l \equiv \tilde{S}_{l+1} - \tilde{S}_l \equiv S_{l+1} B^{l-m+1} - S_l B^{l-m}
\]

this simplifies to

\[
H(T) = \sum_{l=0}^{m-1} \phi_l \Delta \tilde{S}_l
\]

(2.19)

### 2.2.2 The Value of the Hedging Portfolio

The hedging portfolio \( \Pi \) consists of two part, the short position of the option and the trading portfolio \( H(t) \), discussed in the last section. For the short position in the option, one has to take into account the premium \( V_0 = V(t_0) \), which is received by the writer, and the actual value of the option \( V(t) \) at time \( t \). Hence, the portfolio value is given by:

\[
\Pi(t) = V_0 \cdot B^{-1}(t_0, t) - V(t) + H(t)
\]

(2.20)

where the interest for the premium is included. For discrete time, the portfolio value at maturity is

\[
\Pi(T) = V_0 \cdot B^{-m} + K(T) = V_0 \cdot B^{-m} - V(T) + H(T)
\]

\[
= V_0 \cdot B^{-m} - V(T) + \sum_{l=0}^{m-1} \phi_l \Delta \tilde{S}_l
\]

(2.21)

where

\[
K(T) \equiv H(T) - V(T)
\]

(2.22)

is the part which depends on the movement of the asset price.
2.2.3 The Moments of the Portfolio Value

The portfolio value $\Pi(t)$ depends on a specific path for the asset price. But, what is needed to price an option at time $t_0$ is not the portfolio value for one possible path, but expectation values. The first moment of the portfolio value follows directly from equation 2.21:

$$\langle \Pi(T) \rangle_0 = V_0 \cdot B^{-m} - \langle V(T) \rangle_0 + \sum_{l=0}^{m-1} \langle \phi_l \Delta \tilde{S}_l \rangle_0 \quad (2.23)$$

Brackets $\langle ... \rangle_0$ are a short form for the expectation value

$$\langle f(S_0, ..., S_m) \rangle_0 = \int f(S_0, ..., S_m) \prod_{i=1}^{m} p(S_i|S_{i-1}) \, dS_1...dS_m \quad (2.24)$$

and more general with an index $k$

$$\langle f(S_0, ..., S_m) \rangle_k = \frac{1}{p(S_k|S_0)} \int f(S_0, ..., S_m) \prod_{i=1}^{m} p(S_i|S_{i-1}) \, dS_1...dS_{k-1} \, dS_{k+1}...dS_m \quad (2.25)$$

$p(S_k|S_0)$ is the conditional probability density to observe $S(t_k) = S_k$ at time $t_k$ given that the asset price is $S_0$ at time $t_0$. Through this probability, the assumption on the price process, as described in section 2.1, enters in the option pricing. Further on, the shorter notation $p_{k|0} \equiv p(S_k|S_0)$ will be used.

In addition to the first moment, also the risk will be considered later. The risk squared

$$\mathcal{R}^2 = \langle \Pi(T)^2 \rangle_0 - \langle \Pi(T) \rangle_0^2 \quad (2.26)$$

contains the second moment of the portfolio value.

It should be mentioned that the risk does not depend on the option premium $V_0$ since cash return is risk-free. This can be explicitly seen by substituting for

$$\langle \Pi^2(T) \rangle_0 = V_0^2 B^{-2m} + 2V_0 B^{-m} \langle K(T) \rangle_0 + \langle K^2(T) \rangle_0 \quad (2.27)$$

in equation 2.26

$$\mathcal{R}(T) = \sqrt{\langle \Pi^2(T) \rangle_0 - \langle \Pi(T) \rangle_0^2} = \sqrt{\langle K^2(T) \rangle_0 - \langle K(T) \rangle_0^2} \quad (2.28)$$

The right hand side is independent on $V_0$. 

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Chapter 3

Option Pricing Methods

3.1 Overview

Before giving a short overview of the concept of the pricing methods, the assumptions made are repeated:

- Trading is done in discrete time. The time intervals are equidistant.
- A process for the underlying asset is given according to section 2.1. There is no bid-ask spread.
- Short selling is allowed.
- The interest rate $r$ is constant. The interest term structure is flat.
- The underlying stock pays no dividends.
- Transaction costs are neglected.
- The derivative style is European. The payoff is path independent.

Dividends and transaction costs are neglected to keep the problem simple. For a discussion on the effect of dividends and transaction costs, see references [7, 11].

The pricing procedures, described in this section, are similar to the Black-Scholes method. Both of the discussed methods consider a hedging portfolio, as described in section 2.2. The option price is then deduced from

1. an equation for the option premium as a function of the trading strategy. This equation is the same for the Black-Scholes, Bouchaud-Sornette, and the new method.

2. a global strategy, like minimizing the risk, to fix the trading strategy.
3. an economic argument why the derived price is the market price. In the Black-Scholes world this the no-arbitrage argument.

The equation for the option price will be discussed in the next section. The other two points are described afterwards, for each method separately.

### 3.2 The Option Price Equation

For the general case where the option writer cannot hedge the risk completely, the option premium will contain some risk reward. Following the ideas from portfolio theory, we assume that the risk premium is proportional to the risk. This means that the value of the hedging portfolio should grow proportional to the risk of the portfolio

\[
(\Pi(T))_0 - \overbrace{\langle \Pi(t_0) \cdot B^{-m} \rangle}_0 = \lambda \cdot R(T) \tag{3.1}
\]

The factor \(\lambda\) is the *price of risk* for this portfolio which should be larger than the market price of risk\(^1\). This is not the first place where a price of risk enters in the pricing model. Also in the asset process, a price of risk \(\lambda_S\) for the asset is implicitly included by the drift and volatility:

\[
\langle \tilde{S}(T) - \tilde{S}(t_0) \rangle_0 = \lambda_S \cdot \sqrt{ \langle (\tilde{S}(T) - \tilde{S}(t_0))^2 \rangle_0 - \langle \tilde{S}(T) - \tilde{S}(t_0) \rangle^2_0 } \tag{3.2}
\]

For the portfolio \(\Pi\), there is only one source of risk from the asset process. Therefore, \(\lambda\) should be close to \(\lambda_S\).

Apart from equation 2.26, there could be other ways how investors measure their risk. If they are more risk averse, they might use the fourth moment of the portfolio value. Also every investor will have a slightly different view on the price of risk he would charge. This will lead to a price spread, as long as the option cannot be replicated exactly as in the Black-Scholes world.

From equation 3.1 an expression for the option premium at time \(t_0\) can be derived. Since \(R\) is independent on \(V_0\), the price is

\[
V_0 \cdot B^{-m} = \lambda \cdot R(T) - \langle K(T) \rangle_0 = \lambda \cdot \sqrt{ \langle K^2(T) \rangle_0 - \langle K(T) \rangle^2_0 } - \langle K(T) \rangle_0 \tag{3.3}
\]

\(^1\)The definition of \(\lambda\) differs from the way how the market price of risk is defined in portfolio theory since it is based on differences in the portfolio value and not on returns (ratio of values). It is necessary to work with differences since the portfolio value is zero at \(t_0\).
In this equation, the hedging strategy enters through the definition of $K(T)$:

$$\langle K(T) \rangle_0 = \langle H(T) \rangle_0 - \langle V(T) \rangle_0 = \sum_{l=0}^{m-1} \langle \phi_l \Delta \tilde{S}_l \rangle_0 - \langle V(T) \rangle_0$$  \hfill (3.4)$$

The fair (game) option price $V_0^{fp}$ is, by definition, given when no risk premium is charged, $\lambda = 0$:

$$V_0^{fp} = -B^m \cdot \langle K(T) \rangle_0 = B^m \cdot \left( \langle V(T) \rangle_0 - \sum_{l=0}^{m-1} \langle \phi_l \Delta \tilde{S}_l \rangle_0 \right)$$  \hfill (3.5)$$

In the Black-Scholes world, $V_0^{fp}$ is the option market price since the Black-Scholes trading strategy eliminates the risk completely.

To get an option price from the price equation, the hedging strategy has to be known. The determination of the hedging strategy will be discussed in the following.

### 3.3 The Bouchaud-Sornette Approach

#### 3.3.1 The Optimal Hedging Strategy

In the Bouchaud-Sornette approach, the global strategy is to minimize the risk $R$ of the hedging portfolio as defined in equation 2.28. This is equivalent to minimize the risk squared. Setting the derivative of $R^2$ with respect to $\phi_k$ to zero, gives

$$0 = \left( \langle K(T) \frac{\partial K(T)}{\partial \phi_k} \rangle_k - \langle K(T) \rangle_0 \frac{\partial K(T)}{\partial \phi_k} \right)_k$$  \hfill (3.6)$$

The derivative $\partial K/\partial \phi_k$ is a functional derivatives since $\phi$ is a function of $S$. It is equal to

$$\begin{align*}
\frac{\partial K(T)}{\partial \phi_k} &= \left( \frac{\partial H(T)}{\partial \phi_k} - \frac{\partial V(T)}{\partial \phi_k} \right)_{\Delta \tilde{S}_k = 0} = \Delta \tilde{S}_k
\end{align*}$$  \hfill (3.7)$$

which, substituting in equation 3.6, yields

$$\begin{align*}
\langle V(T) - H(T) \rangle_0 \cdot \langle \Delta \tilde{S}_k \rangle_k &= \langle [V(T) - H(T)] \cdot \Delta \tilde{S}_k \rangle_k \\
\left[ \langle V(T) \rangle_0 - \sum_{l=0}^{m-1} \langle \phi_l^* \Delta \tilde{S}_l \rangle_0 \right] \cdot \langle \Delta \tilde{S}_k \rangle_k &= \langle V(T) \Delta \tilde{S}_k \rangle_k - \sum_{l=0}^{m-1} \langle \phi_l^* \Delta \tilde{S}_l \cdot \Delta \tilde{S}_k \rangle_k \\
&= \langle V(T) \Delta \tilde{S}_k \rangle_k - \phi_k^* \langle \Delta \tilde{S}_k^2 \rangle_k - \sum_{\substack{l=0 \atop l \neq k}}^{m-1} \langle \phi_l^* \Delta \tilde{S}_l \cdot \Delta \tilde{S}_k \rangle_k
\end{align*}$$  \hfill (3.8)$$
Resolving for \( \phi_k^* \) gives an involved equation for the hedging strategy

\[
\phi_k^* = \frac{1}{\langle \Delta \tilde{S}_k^2 \rangle_k} \left\{ \langle V(T) \Delta \tilde{S}_k \rangle_k \right. \\
+ \langle \Delta \tilde{S}_k \rangle_k \left[ \sum_{l=0}^{m-1} \langle \phi_l^* \Delta \tilde{S}_l \rangle_0 - \langle V(T) \rangle_0 \right] - \sum_{l \neq k} \langle \phi_l^* \Delta \tilde{S}_l \cdot \Delta \tilde{S}_k \rangle_k \} 
\]

(3.9)

Explicit formulas for the different expectation values involved are given in appendix A.

After solving equation 3.9 for the optimal trading strategy \( \phi^* \), the option price is obtained from equation 3.3.

### 3.3.2 Options in a Risk Neutral World

In general, equation 3.9 for the hedging strategy can only be solved numerically (see section 4.3.1). But in a risk neutral world, the asset price increases on average with the risk neutral interest rate

\[
\langle \Delta \tilde{S}_l \rangle_l = \langle S_{l-1} \rangle_l B^{l-n+1} - S_l B^{l-n} = 0 
\]

(3.10)

and a simple formula for the hedging strategy can be derived from equation 3.9:

\[
\phi_k^* = \frac{\langle V(T) \Delta \tilde{S}_k \rangle_k}{\langle \Delta \tilde{S}_k^2 \rangle_k} 
\]

(3.11)

Here, the expectation values has to be derived for the risk neutral asset process.

If \( S_k \) is not too far out of the money, formula 3.11 can also be seen as an approximation to the trading strategy in the real world. The expectation values has to be calculated then with the real world asset process. The approximation holds because the first term dominates the right hand side of equation 3.9 since \( \langle \Delta \tilde{S}_k \rangle_k \) is small and the expectation value \( \langle \phi_l^* \Delta \tilde{S}_l \cdot \Delta \tilde{S}_k \rangle_k \) is approximatly equal to \( \langle \phi_l^* \Delta \tilde{S}_l \rangle_k \cdot \langle \Delta \tilde{S}_k \rangle_k \). This argument doesn’t hold far out of the money. Far out of the money, the term \( \langle V(T) \Delta \tilde{S}_k \rangle_k \) itself is small.

\footnote{For \( k > l \) the relation is exact, see appendix A}
3.4 The Minimal Price Approach

3.4.1 Overview

In the Bouchaud-Sornette approach, the global strategy is to minimize the risk. As long as all investors follow this strategy, one can argue that the Bouchaud-Sornette option price is the market price. All option writers will price their options in the same way, so there will be only one common option price. But to minimize the risk is not the only strategy one can think about. Other strategies might give a higher risk, but also another price. If this price is smaller, the option is more competitive on the market. The higher risk don’t have to be a disadvantage because the motivation of trading is not to minimize the risk, but to maximize the profit for a given risk. Therefore, from the writers point of view, the only requirement for a strategy should be that the ratio of profit to risk is equal to the price of risk.

The pricing method presented in this section optimises the hedging strategy to find the cheapest option price. In the following, this method will be called minimal price approach. In general, the risk for the resulting hedging strategy will be larger than for the Bouchaud-Sornette approach, but this is compensated by a higher return.

The method is also based on considering a self-financing hedging portfolio. The concept is shortly described as follows:

- One takes the view of the option writer.
- A hedging portfolio will be set up as described in section 2.2.
- It is required that the profit from the hedging portfolio is related to the risk by equation 3.1.
- The optimal hedging strategy \( \phi^*(S,t) \) is such that the option premium \( V_0 \) of equation 3.3

\[
V_0 \cdot B^{-m} = \lambda \cdot R(T) - \langle K(T) \rangle_0 = \lambda \cdot \sqrt{\langle K^2(T) \rangle_0 - \langle K(T) \rangle_0^2} - \langle K(T) \rangle_0
\]

is minimal.

For this method, it will be required that \( \lambda \geq \lambda_S \). Otherwise, the optimal strategy would be to hold the asset only which gives a better return to risk ratio as to write the option.
Due to the term $-\langle K(T) \rangle_0$ in equation 3.12, minimizing the risk and minimizing the price is not the same. The result of the Bouchaud-Sornette approach is different from the result obtained with the minimal price approach.

### 3.4.2 The Optimal Hedging Strategy

#### 3.4.2.1 Zero Price of Risk

In the special case where the price of risk is zero, equation 3.12 reduces to:

$$V_0 B^{-m} = \langle V(T) \rangle_0 - \sum_{l=0}^{m-1} \langle \phi^*_l \Delta \tilde{S}_l \rangle_0$$  \hspace{1cm} (3.13)

Since $\lambda \geq \lambda_S$ and $\lambda_S \geq 0$, also $\lambda_S$ is zero and the asset price should in average grow with the risk free rate: $\langle \Delta \tilde{S}_l \rangle_l = 0$. From $\langle \phi^*_l \Delta \tilde{S}_l \rangle_0 = \int \phi^*_l \langle \Delta \tilde{S}_l \rangle_l p_{li} \, dS_i$ (see appendix A), it follows that

$$V_0 = B^{-m} \cdot \langle V(T) \rangle_0$$  \hspace{1cm} (3.14)

This shows that if the price of risk is zero, prices are independent of the risk. Hedging is not necessary.

#### 3.4.2.2 Non-Zero Price of Risk

For a non-zero price of risk, the optimal hedging strategy is obtained when the option price is minimal. The minimization of the premium, as given by equation 3.12, with respect to $\phi_k$ yields$^3$

$$0 = \frac{\lambda}{R(T)} \cdot \left( \langle K(T) \frac{\partial K}{\partial \phi_k} \rangle_k - \langle K(T) \rangle_0 \langle \frac{\partial K}{\partial \phi_k} \rangle_k \right) - \langle \frac{\partial K}{\partial \phi_k} \rangle_k$$

$$= \langle K(T) \frac{\partial K}{\partial \phi_k} \rangle_k - \left[ \langle K(T) \rangle_0 + \frac{1}{\lambda} R(T) \right] \langle \frac{\partial K}{\partial \phi_k} \rangle_k$$

$$= \langle K(T) \Delta \tilde{S}_k \rangle_k - \left[ \langle K(T) \rangle_0 + \frac{1}{\lambda} R(T) \right] \langle \Delta \tilde{S}_k \rangle_k$$  \hspace{1cm} (3.15)

$^3$For the case $\lambda = 0$, discussed in the last section, the equation is identical to zero, since then $\langle \frac{\partial K}{\partial \phi_k} \rangle_k = \langle \Delta \tilde{S}_k \rangle_k = 0$. 

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The first term in this equation can be reduced to:

$$\langle K(T) \cdot \Delta \tilde{S}_k \rangle_k = \langle [H(T) - V(T)] \cdot \Delta \tilde{S}_k \rangle_k$$

$$= \sum_{l=0}^{m-1} \langle \phi_l^* \Delta \tilde{S}_l \cdot \Delta \tilde{S}_k \rangle_k - \langle V(T) \Delta \tilde{S}_k \rangle_k$$

$$= \phi_k^* \langle \Delta \tilde{S}_k^2 \rangle_k + \sum_{l=0}^{m-1} \langle \phi_l^* \Delta \tilde{S}_l \cdot \Delta \tilde{S}_k \rangle_k - \langle V(T) \Delta \tilde{S}_k \rangle_k$$

This finally gives an involved equation for the hedging strategy:

$$\phi_k^* = \frac{1}{\langle \Delta \tilde{S}_k^2 \rangle_k} \left\{ \langle V(T) \Delta \tilde{S}_k \rangle_k + \langle \Delta \tilde{S}_k \rangle_k \left[ \frac{1}{\lambda} \mathcal{R}(T) + \sum_{l=0}^{m-1} \langle \phi_l^* \Delta \tilde{S}_l \rangle_0 - \langle V(T) \rangle_0 \right] - \sum_{l=0}^{m-1} \langle \phi_l^* \Delta \tilde{S}_l \cdot \Delta \tilde{S}_k \rangle_k \right\}$$

Compared to the Bouchaud-Sornette method, one extra term appears. For $\lambda \to \infty$, this term disappears and the Bouchaud-Sornette result is obtained. The convergence can be also seen in the price equation 3.12. For $\lambda \to \infty$ the price value is dominated by the risk and a minimization of the price is equivalent to a minimization of the risk.

### 3.5 The Continuous Time Limit

For a log-normal process and continuous hedging, the Bouchaud-Sornette and the minimal price approach should recover the Black-Scholes result. Otherwise, the methods are not arbitrage free.

This is obviously the case for the Bouchaud-Sornette method where the risk is minimized. Since the Black-Scholes hedging eliminates the risk, it fulfils the Bouchaud-Sornette condition of minimal risk.

For the minimal price approach there is no such simple argument. The convergence will be shown explicitly now. We will discuss here the simple case with only one hedge (no rehedge) for a small time interval $\Delta t = T - t_0$. It is shown in appendix B.2 that the general case can be reduced to the single hedge case. The formulas for the risk,
premium and hedging strategy for a single hedge are

\[ B^{-1} V_0 = \lambda R + \langle V \rangle_0 - \phi^*_0 \langle \Delta \hat{S}_0 \rangle \]  

(3.18)

\[ \phi^*_0 = \frac{1}{\langle \Delta \hat{S}_0^2 \rangle - \langle \Delta \hat{S}_0 \rangle^2} \left\{ \langle V \Delta \hat{S}_0 \rangle_0 + \langle \Delta \hat{S}_0 \rangle \left[ \frac{c}{\lambda} R - \langle V \rangle_0 \right] \right\} \]

\[ R = \sqrt{\langle (V - \phi^* \Delta \hat{S}_0)^2 \rangle - \langle V - \phi^* \Delta \hat{S}_0 \rangle^2} \]

with \( c = 1 \) for the minimal price approach, \( c = 0 \) for the Bouchaud-Sornette method and \( V \equiv V(T) \).

In the limit \( \Delta t \to 0 \), only the leading order in \( \Delta t \) has to be considered. Using the results of appendix B, the risk is

\[ R = \sigma S_0 \left( \phi^* - \frac{\partial V}{\partial S} \right) \sqrt{\Delta t} + O(\Delta t) \]  

(3.19)

When deriving the limit for the hedging strategy, one has to take into account that the price of risk depends on \( \Delta t \):

\[ \lambda = a \lambda_S = a \cdot \frac{\langle \Delta \hat{S}_0 \rangle}{\sqrt{\langle \Delta \hat{S}_0^2 \rangle - \langle \Delta \hat{S}_0 \rangle^2}} = a \cdot \frac{\mu - r}{\sigma} \sqrt{\Delta t} + O(\Delta t) \]  

(3.20)

where \( a \geq 1 \) is a constant factor. Together with the result from appendix B.1 for the expectation values, the hedging strategy can be derived to lowest order in \( \Delta t \):

\[ \phi^*_0 = \frac{1}{\sigma^2 S_0^2 \Delta t} \left\{ S_0 \left[ (\mu - r) V + \sigma^2 S_0 \frac{\partial V}{\partial S} \right] \Delta t \right. \]

\[ + S_0 (\mu - r) \Delta t \left[ \frac{\sigma}{a(\mu - r) \sqrt{\Delta t}} R - V \right] \} + O(\Delta t) \]

\[ = \frac{\partial V}{\partial S} + \frac{1}{a} \left( \phi^* - \frac{\partial V}{\partial S} \right) + O(\Delta t) \]

\[ = \frac{\partial V}{\partial S} + O(\Delta t) \]

This is the Black-Scholes hedging strategy.

In the price equation the term \( \lambda R \) vanish since it is of order \( \Delta t \). Using again the results of appendix B.1 for the expectation values, it follows that

\[ 0 = \langle V(T) - V_0 e^{r \Delta t} \rangle_0 - \phi^*_0 \langle \Delta \hat{S}_0 \rangle_0 \]  

(3.22)

\[ = \left\{ \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right\} \Delta t - r V \Delta t - \frac{\partial V}{\partial S} S (\mu - r) \Delta t + O(\Delta t^2) \]

\[ \Delta t \to 0 \]

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - r V \]

\[ \lambda = a \lambda_S \]
This is the Black-Scholes equation.

In conclusion, it has been shown that in the continuous time limit for a log-normal process the Bouchaud-Sornette and the minimal price approach leads to the Black-Scholes equation for the option price. The hedging strategy converges towards the Black-Scholes delta hedging and the risk is eliminated.
Chapter 4
Comparison of Pricing Methods

4.1 Introduction

In the last chapter, the different pricing methods were described in detail. The conceptual differences of the methods are summarized in the following table.

<table>
<thead>
<tr>
<th></th>
<th>Black-Scholes</th>
<th>Bouchaud-Sornette</th>
<th>Minimal Price Approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assumptions</td>
<td>continuous hedging</td>
<td>discrete hedging</td>
<td>i.i.d. process</td>
</tr>
<tr>
<td></td>
<td>log-normal process</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Price Equation</td>
<td>$V_0 \cdot \mathcal{B}^{-m} = \lambda \cdot \mathcal{R}(T) - \langle K(T) \rangle_0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trading Strategy</td>
<td>zero risk</td>
<td>minimal risk</td>
<td>minimal premium</td>
</tr>
</tbody>
</table>

Table 4.1: Conceptual differences of the pricing methods. The limit of continuous hedging can be handled for the Bouchaud-Sornette and minimal price approach according to section 3.5.

Although it is clear, that the different hedging strategies and the resulting option prices are not equal, it is of importance to know how large these differences are and whether they are of relevance for typical options. This question will be addressed in this chapter. Typical options and asset processes will be studied, and the pricing methods are compared, e.g. with respect to the hedging frequency, the maturity, or the strike.
Log-Normal Process

\[
p(S, t | S_0, t_0) = \frac{1}{\sqrt{2\pi\Delta t} \sigma S} e^{-\hat{\mu}^2/(2\sigma^2 \Delta t)}
\]

\[
\hat{\mu} = \log(S/S_0) - (\mu - 1/2 \sigma^2)\Delta t
\]

\[
\Delta t = 1/250
\]

Spot \( S(0) = 50 \)

Drift \( \mu = 0.1 \)

Volatility \( \sigma = 0.2 \)

<table>
<thead>
<tr>
<th>Fat Tail Process</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(S, t</td>
</tr>
<tr>
<td>( \hat{\mu} = \log(S/S_0) - (\mu - 1/2 \sigma^2)\Delta t )</td>
</tr>
<tr>
<td>( \Delta t = 1/250 )</td>
</tr>
<tr>
<td>( \sigma_G = 0.15 )</td>
</tr>
<tr>
<td>Spot ( S(0) = 50 )</td>
</tr>
<tr>
<td>Drift ( \mu = 0.1 )</td>
</tr>
<tr>
<td>Volatility ( \sigma \equiv \sqrt{\text{variance}} \approx 0.2 )</td>
</tr>
</tbody>
</table>

Table 4.2: Asset processes for which the pricing methods are compared.

### 4.2 Asset Processes and Options

#### 4.2.1 The Underlying Process

##### 4.2.1.1 Log-Normal Process

The pricing methods will be compared for two different underlying processes. One is a log-normal process. For this process, analytic Black-Scholes pricing formulas exist for continuous time. Comparing the results for discrete hedging with the Black-Scholes price, reveals the effect of discrete hedging.

The log-normal process for the asset price \( S \) is given by

\[
dS = \mu S \, dt + \sigma S \, dX
\]  

(4.1)

with constant drift \( \mu \) and constant volatility \( \sigma \). \( dX = N(0, dt) \) is a normal random variable with mean 0 and variance \( dt \). By integration, the probability density to observe \( S \) at time \( t \) given a spot price of \( S_0 \) at time \( t_0 \) is obtained [2]:

\[
p(S, t | S_0, t_0) = \frac{1}{\sqrt{2\pi(t-t_0)} \sigma S} e^{-\left(\log(S/S_0) - (\mu - 1/2 \sigma^2)(t-t_0)\right)^2/(2\sigma^2(t-t_0))}
\]  

(4.2)

The factor \( 1/S \) is due to the fact, that the probability density is given for \( S \) instead
Plain Vanilla Option

| Maturity $T$ | 0.2 years |
| Strike $E$ | 50 |
| Payoff call | $\max[0, S(T) - E]$ |
| Payoff put | $\max[0, E - S(T)]$ |

Binary Option

| Maturity $T$ | 0.2 years |
| Strike $E$ | 50 |
| Payoff call | $\begin{cases} 1 & \text{if } S(T) > E \\ 0 & \text{else} \end{cases}$ |
| Payoff put | $\begin{cases} 0 & \text{if } S(T) > E \\ 1 & \text{else} \end{cases}$ |

General Parameters

| Interest rate | 0.05 |
| Price of risk $\lambda$ | $1.1 \cdot \lambda_S$ |

Table 4.3: Options for which the pricing methods are compared.

of log $S$. The values of the parameters $S_0$, $\mu$, and $\sigma$, which are chosen for this study, are shown in table 4.2.

For completeness, the spot price evolution for finite time steps, which is needed for Monte Carlo simulations, is given here as well [2]:

$$S(t) = S(t_0) \, e^{(\mu - 1/2 \sigma^2) \cdot (t-t_0) + \sigma \cdot [X(t) - X(t_0)]} \quad (4.3)$$

4.2.1.2 Fat Tail Process

In section 2.1, it was shown that a log-normal process doesn’t fit to market data. In data tails are more pronounced, which can be described by an exponential slop. The second asset model under consideration is a sum of a log-normal and an exponential function (see table 4.2). The form of the distribution is given by the fit to the IBM data in section 2.1 and scaled such that the square root of the variance is close to the volatility value taken for the log-normal process. This guarantees that the option prices are roughly the same for both processes.

The probability distribution is shown for different time horizons in figure 4.1. Overlaid is a normal distribution. For short time horizons both distributions show large differences in the tails, while for large time horizons the fat tail distribution converges towards the Gaussian distribution, as expected. The convergence is also
Figure 4.1: Probability distributions for the fat tail model for different time horizons. A Gaussian distribution is overlaid to demonstrate the convergence towards a normal distribution for large time horizons.
seen when studying the kurtosis

\[
\kappa = \frac{\langle (\hat{\mu} - \langle \hat{\mu} \rangle)^4 \rangle}{\sigma^4} - 3
\]  

(4.4)

of the fat tail distribution which tends to 0 with increasing time horizon. The variance, on the other hand, scales with \(dt\) which is a property of any i.i.d. variable (see section 2.1.2).

### 4.2.2 Option Characteristics

Since the study focus on the comparison of valuation methods, simple option types were chosen, for which analytic Black-Scholes solutions [2, 12] exist. Four option types were considered: European plain vanilla and digital options with call and put payoff type. The option parameters of table 4.3 are taken as a basis for the different studies.

Another parameter, to fix for the study, is the price of risk the option writer charges for the option. A value of

\[
\lambda = 1.1 \cdot \lambda_S
\]  

(4.5)

was chosen for all studies except for section 3.2. In section 3.2, the dependence of the option price on the price of risk is studied.

### 4.3 Software Implementation

#### 4.3.1 Price and Strategy Calculation

##### 4.3.1.1 Solution Algorithm

The first step to determine the option price is to solve the equation

\[
\phi_k^* = \frac{1}{\langle \Delta \tilde{S}_k^2 \rangle_k} \left\{ \langle V(T)\Delta \tilde{S}_k \rangle_k \right. \\
+ \langle \Delta \tilde{S}_k \rangle_k \left[ \frac{c}{\lambda} \mathcal{R}(T) + \sum_{l=0}^{m-1} \langle \phi_l^* \Delta \tilde{S}_l \rangle_0 - \langle V(T) \rangle_0 \right] - \sum_{l=0}^{n-1} \langle \phi_l^* \Delta \tilde{S}_l \cdot \Delta \tilde{S}_k \rangle_k \right\}
\]  

(4.6)

with

\[
c = 0 \quad \text{for the Bouchaud-Sornette method (see eq. 3.9)},
\]

\[
c = 1 \quad \text{for the minimal price approach (see eq. 3.17)}.
\]  

(4.7)
The right hand side of this equation is dominated by the term \( \langle V(T)\Delta \tilde{S}_k \rangle_k / \langle \Delta \tilde{S}_k^2 \rangle_k \), if the value \( S_k \) is not too far out of the money (see section 3.3.2). This term is independent on the hedging strategy. The equation 4.6 is solved by an iteration procedure with starting values \( \phi_k = \langle V(T)\Delta \tilde{S}_k \rangle_k / \langle \Delta \tilde{S}_k^2 \rangle_k \). In each iteration step, the old values for \( \phi_k \) are substituted on the right hand side of equation 4.6, giving a new set of values \( \phi_k \).

For each iteration step, the option price, given by equation 3.3

\[
V_0 \cdot B^m = \lambda \cdot R(T) - \langle K(T) \rangle_0
\]

\[
= \lambda \cdot \sqrt{\langle K^2(T) \rangle_0 - \langle K(T) \rangle_0^2 - \langle K(T) \rangle_0}
\]

with

\[
K(T) = \sum_{l=0}^{n-1} \langle \phi_l^* \Delta \tilde{S}_l \rangle_0 - V(T)
\]

is calculated. If the option price changes from one iteration step to the next by less than \( 10^{-7} \), the iteration is stopped. For the Bouchaud-Sornette method, the convergence is usually reached after a few steps. For the minimal price approach, typically 10 to 20 iterations are necessary.

4.3.1.2 Calculation of Expectation Values

The main effort, in the iteration procedure, is to calculate the expectation values which are involved in the strategy and price equation. The expectation values are integrals over the asset price as shown explicitly in appendix A. They are solved numerically by a Riemann sum approximation. For this approximation, the asset price is taken to be discrete. Together with the discrete time steps, a grid is defined. It has the following characteristics:

- The number of mesh points in \( t \) is equal to the number of hedges plus one. The hedging intervals were taken to be equidistant between \( t_0 = 0 \), for which the option price is searched, and the option maturity \( T \).

- The asset price is discrete. The mesh has 200 points in \( S \) for each time value.

- The points in \( S \) are equidistant for each time value, but with different spacing for different times. The distance between two points is given by the upper and lower bound for \( S \).

\[1\] Another possibility would be to start with the Black-Scholes delta hedging strategy.
The time dependent upper and lower bounds on the asset price are:

\[ S_{\text{max}}(t) = \left\{ S_0 + \frac{t}{T}(S_2 - S_0) \right\} \cdot e^a \]  \hspace{1cm} (4.10)

\[ S_{\text{min}}(t) = \left\{ S_0 + \frac{t}{T}(S_1 - S_0) \right\} \cdot e^{-a} \]

with

\[ S_0 = S(t_0) \]

\[ S_1 = \min\left( E, S(t_0) \cdot e^{\mu T} \right) \]

\[ S_2 = \max\left( E, S(t_0) \cdot e^{\mu T} \right) \]

At time \( t_0 = 0 \) there is only one mesh point given by the spot value \( S(t_0) \). The value of \( \mu \) is taken from the price process.

The size of the window is mainly determined by the factor \( a \). The fat tail process needs a larger window for short times, which is taken into account by the following definition:

\[ a = 6 \sigma \sqrt{t} \]  \hspace{1cm} \text{for the log-normal process} \hspace{1cm} (4.11)

\[ a = \max\left( 6 \sigma \sqrt{t}, 0.25 \right) \]  \hspace{1cm} \text{for the fat tail process} \hspace{1cm}

\( \sigma \) is the volatility of the process. Studies similar to the error estimate in section 4.3.3 show that this window size is sufficient.

For binary options, it is important for reducing numerical errors that the strike is located in the middle of two grid points. Therefore, the grid points for each time were shifted, such that the point \( S_0 + (E - S_0) \cdot t/T \) lies in the middle of two mesh points.

All calculations, except of the calculation for the integral \( I_{llk} \) and the probabilities, were done in double precision. The integral \( I_{llk} \) is defined in appendix A. Since the integral \( I_{llk} \) and the probabilities are independent on the hedging strategy, they were calculated only once per iteration procedure and stored in single precision. Using single precision reduces the memory consumption considerably, with small effects on the overall precision.

4.3.1.3 Calculation of Probabilities

For the determination of expectation values, probabilities according to table 4.2 have to be calculated for different time horizons (see appendix A). For the log-normal
process, this is trivial since the explicit formula for all time horizons is known. For the fat tail process, a distribution is given for a one day time horizon. For larger time horizons, the distribution has to be convoluted with itself. This autoconvolution was done by calculating the Fourier transformed, and then taking it to a power which corresponds to the time horizon, e.g. a power of 2 for two days. At the end, the inverse Fourier transformation yields the searched probability distribution. This also allows for time horizons which are not a multiple of one day, e.g. two and a half days. The Fourier transformation was done with the routine `four1` out of reference [13] which performs a fast Fourier transformation.

Since always the same probabilities are needed in each iteration step, they are determined only once at the beginning, stored, and reused for the different iteration steps.

To further improve performance and to reduce the memory consumption, the integration for the expectation values is done only over the region in $S$ where the involved probabilities are not negligible. The choice of the grid boundaries in $S$ are motivated by this idea. In addition, performance improvements are obtained in the following way. As described above, the integrals of the expectation values are approximated by Riemann sums. In these sums, only those terms were taken into account for which the involved probabilities are larger than $1.5 \cdot 10^{-6}$. This limit roughly corresponds to five standard deviations. For the study of implied volatilities the limit of $1.5 \cdot 10^{-6}$ was reduced to $10^{-9}$ since here prices are calculated far out or far in the money for which the tails of the probability distribution are more important. The actual values for the limit are justified by the error analysis, described in section 4.3.3. This restriction on the probabilities also reduces the number of integrals $I_{ljk}$ to be stored and therefore reduces the memory consumption.

### 4.3.2 Monte Carlo Simulation

Another way to calculate the expectation values is to use Monte Carlo techniques. The implementation is much simpler, but the calculation is much more time consuming. Nevertheless, routines for Monte Carlo simulation were coded, to be able to check the results from the numerical integration and to estimate errors. In addition, the Monte Carlo simulations were used to generate profit distributions of the hedging portfolio for illustration.

The Monte Carlo implementation will be described in the following. For the log-normal asset process uniform random numbers are produced with the standard
C function `rand()`. These random numbers are then transformed to normal distributed random numbers using the Marsaglia method [2]. For the fat tail process the acceptance-rejection method [13] was used to generate random numbers, distributed according to the fat tail distribution. Two uniform random numbers are taken for each acceptance-rejection test. It turned out, that the random number sequence from the Microsoft Visual C++ generator `rand()` is not free of correlations. An explanation for the correlation was not found, since the generation algorithm is not documented. No correlation effects were observed when using the random number generator `ran1()` from reference [13]. This generator is an implementation of the method by Park and Miller with Bays-Durham shuffle. It was used to generate random numbers for the simulation of the fat tail process. For the log-normal process, the C generator `rand()` is sufficient and faster.

As mentioned, the Monte Carlo method was used for the error estimate. For this purpose, it is essential to know the statistical error of the Monte Carlo result. This error is obtained by sampling the Monte Carlo events and estimating an error from the different results of the samples. The error of the full sample is then obtained by scaling with the square root of the event number. To keep the scaling law, no acceleration methods, like antithetic variables or moment matching, are used.

### 4.3.3 Error Estimate

For some of the calculated prices and implied volatilities, errors are quoted in the following sections. These errors were estimated by studying the variation of results from different calculations. One set of calculations was done to estimate the uncertainty which comes from numerical errors in the hedging strategy. For each of these calculations, the determination of the expectation values was changed in one of the following ways:

- The number of grid points in $S$ was increased from 200 to 300.
- The grid window in equation 4.11 was increased from 6 to 8 standard deviations.
- For the fat tail process, the term $\max(6\sigma\sqrt{t}, 0.25)$ in equation 4.10 was replaced by $\max(6\sigma\sqrt{t}, 0.3)$.
- Probabilities were taken into account down to values of $1.5 \cdot 10^{-8}$, instead of $1.5 \cdot 10^{-6}$, for price calculations and down to $10^{-11}$, instead of $10^{-9}$, for implied volatilities.
• The grid points were shifted in spot direction by 10% of the spacing in $S$. The grid points are then no longer located symmetric around the strike. For binary options, large effects are expected from such a shift. They are mainly due to the calculation of the expectation values which enter in the price equation. Since the focus lies on the error from the hedging strategy, the expected additional uncertainty from the price equation was eliminated by solving the price equation with Monte Carlo techniques. For plain vanilla options, the uncertainty from solving the price equation is small, and therefore the price equation was solved by numerical integration.

• Probabilities for the fat tail process and arbitrary time horizons were calculated by a fast Fourier transformation, as explained in section 4.3.1.3. The number of points used by the fast Fourier transformation was increased from $2^{14}$ to $2^{16}$. For each of these calculations, the deviation of the option price or implied volatility from the standard result was taken and added quadratic. This gives an estimate of the error, which will be referenced to as $error~1$.

In a second step, a Monte Carlo simulation was performed to estimate the error which comes from solving the price equation 3.3. The hedging strategy was taken from the standard calculation, and a Monte Carlo sample of $10^6$ events was generated. From this sample, the option price and the implied volatility was determined. The difference to the standard result is an estimate of the systematic error. This estimate only makes sense unless the difference is larger than the statistical error on the Monte Carlo result. If this condition is not fulfilled, the statistical Monte Carlo error was taken as a conservative estimate of the systematic error. The statistical Monte Carlo error was obtained in the way described above, by sampling the $10^6$ events in 20 sets. The error from the Monte Carlo study will be called $error~2$.

The two estimates, $error~1$ and $error~2$, are combined, by adding them quadratic, to give the final error. The main contribution to this error is

• for plain vanilla options: the statistical Monte Carlo error.

• for plain vanilla options far in the money (strike = 38): the grid window$^2$.

• for binary options: the systematic error from solving the price equation (error 2). The observed difference between Monte Carlo and the numerical integration is up to 18 times larger as the statistical Monte Carlo error. This indicates a

$^2$Errors for binary options far out of the money were not determined.
systematic error. However, this error is not as large as it might be indicated by the factor 18, because the statistical Monte Carlo error itself is small.

The errors are included in different figures in the following sections, e.g. in figure 4.2.

### 4.3.4 Hardware and Performance

The software was coded in C++ and was run on a Pentium III processor with 750 MHz and 256 MB RAM. The available RAM memory restricts the number of hedges to 50. It was not possible to overcome this restriction by using swap memory because in this case the CPU load drops from about hundred to a few percent. The performance of the numerical calculations is given in table 4.4. Listed are typical values for the real time consumption.

### 4.4 Results

#### 4.4.1 Dependence on the Rehedging Frequency

The Bouchaud-Sornette and the minimal price approach are developed to handle discrete hedging and non-Gaussian probability distributions. As seen in section 3.5, the results of the two pricing methods converge towards the Black-Scholes results in the continuous time limit for a log-normal process. The question arises by how much the results differ for a realistic process and typical rehedging frequency. This will be addressed in this section.
Figure 4.2 shows results for a plain vanilla call at the money. The option characteristics are as given in section 4.2.2. The risk free interest rate is 5%. Plotted is the premium, including the risk premium according to equation 3.3, as a function of the rehedging frequency. For comparison, the standard Black-Scholes price without risk premium is included in the figure. The Black-Scholes price was calculated with a volatility equal to the square root of the variance of the asset process.

The other three curves were determined from the price equation 3.3 using the hedging strategies of the Bouchaud-Sornette method, the minimal price approach and the Black-Scholes model. The Bouchaud-Sornette strategy and the minimal price strategy were obtained, as described in chapter 3, by minimizing the risk, resp. the option price. The Black-Scholes strategy is the standard delta hedging which is calculated analytically with a volatility equal to the square root of the process variance. Errors for the numerical precision are estimated for the rehedging frequencies of 0.1, 0.4 and 1. The error analysis was explained in section 4.3.3. Most of the errors are smaller than the marker size and are therefore hidden.

The Black-Scholes and the Bouchaud-Sornette hedging give almost the same results. The minimal price approach leads to cheaper prices. This was expected since the price is minimized in this approach. All three curves lie above the Black-Scholes price without risk premium. For the log-normal asset process, the Black-Scholes price must be a lower limit for any reasonable model. Otherwise, there would be an arbitrage opportunity. For the log-normal process, the convergence towards the risk-free Black-Scholes price with increasing rehedging frequency is visible. The price gap remains larger for the fat tail process.

The reason for the larger gap at high rehedging frequencies is explained by figure 4.3. The upper two plots show the risk of the hedging portfolio which enters in the option price. For the log-normal process, the risk drops much faster with increasing rehedging frequency. The third plot in figure 4.3 show the contribution of the risk premium to the option price. The risk premium is roughly twice as large for the minimal price approach as for the Bouchaud-Sornette method, which is consistent with the risk distributions. The contribution of the risk premium is of the order of a few percent of the total price.

The risk, as shown in figure 4.3, is a measure of the width of the profit distribution at maturity. To get a better understanding of the methods, it is helpful to have a look at the profit distributions. From the numerical integration procedure, these distributions are not available. The profit distributions in figure 4.4 were produced with Monte Carlo simulations. The upper two plots show the profit of the replication
Figure 4.2: Plain vanilla call premium as a function of the rehedging frequency for two different asset processes. The curves without marker show the standard Black-Scholes price. The curves with marker include a risk premium and are calculated for different hedging strategies. Errors are shown for the rehedging frequencies of 0.1, 0.4, and 1. Most of the errors are smaller than the marker size, so that the error bars are hidden.
Figure 4.3: The upper two figures show the risk as a function of the rehedging frequency. The contribution of the risk premium to the option price is visualized below.
Figure 4.4: The upper two figures show the profit of the replication portfolio versus the asset price at expiry, as obtained from Monte Carlo simulation. The lower plot shows the residual distributions, integrated over $S(T)$. 
part of the hedging portfolio (the $\phi$ dependent part on the right hand side of equation 3.3). Overlaid is the option payoff shifted by the fair option price. The lower plot shows the residual of the Monte Carlo distribution and the payoff curve, integrated over the asset price $S$. The two distributions are consistent with figure 4.3 where the risk for the minimal price approach was observed to be roughly twice as large as the risk for the Bouchaud-Sornette approach.

Figure 4.4 also reveals that there is a correlation of the residual with the asset price for the minimal price approach. We will come back to this point in section 4.4.2 when discussing the differences in the hedging strategies.

In figure 4.5 and 4.6 the same comparison as in figure 4.2 was done for other option types. The results are similar as for the plain vanilla call option. For options out of the money, with strike at 55, the difference between the Bouchaud-Sornette price and the premium from the minimal price approach is larger, up to almost 10 percent of the option price.

4.4.2 Comparison of Hedging Strategies

In the last section, it was pointed out that for the minimal price approach, the profit is correlated with the spot at expiry (see figure 4.4). In contrast, the Bouchaud-Sornette method doesn’t show such a correlation. The correlation indicates that more assets are held compared to the Bouchaud-Sornette method. This is confirmed by comparing the hedging strategies directly. In figure 4.7, the hedging strategy $\phi$ is shown as a function of the asset price and time. The figure refers to the plain vanilla call option. The minimal price strategy requires to hold about 0.2 units of the asset more as for the Bouchaud-Sornette strategy. The Black-Scholes delta hedging strategy, not shown in the figure, turns out to be very similar to the Bouchaud-Sornette strategy.

The difference in the strategies can be understood in terms of the portfolio theory. To see this, consider the Bouchaud-Sornette hedging portfolio as a second asset in addition to the underlying. The Bouchaud-Sornette hedging portfolio has a “price” (profit) which is almost uncorrelated with the underlying price $S(T)$. From portfolio theory it is known that an investment in two uncorrelated assets gives a better return to risk ratio as an investment in only one of the two assets. This is, because for a portfolio of two uncorrelated assets, the return of the assets adds linear while the risk adds quadratic. This explains why more assets should be held for the minimal price approach.

But portfolio theory also points to a possible drawback of the minimal price approach. The optimal portfolio, in the sense of portfolio theory, is sensitive to the asset
Figure 4.5: Plain vanilla option premium as a function of the rehedging frequency for two different asset processes. The curves without marker show the standard Black-Scholes price. The curves with marker include a risk premium and are calculated for different hedging strategies. Errors are shown for the rehedging frequencies of 0.1, 0.4, and 1. Most of the errors are smaller than the marker size, so that the error bars are hidden.
Figure 4.6: Binary option premium as a function of the rehedging frequency for two different asset processes. The curves without marker show the standard Black-Scholes price. The curves with marker include a risk premium and are calculated for different hedging strategies. Errors are shown for the rehedging frequencies of 0.1, 0.4, and 1. Most of the errors are smaller than the marker size, so that the error bars are hidden.
Figure 4.7: Comparison of hedging strategies for the plain vanilla call option with the fat tail asset process. The lower plot shows the difference between the Bouchaud-Sornette and the minimal price strategy.
drift, which is more difficult to estimate than the volatility. The methods, described here, assume a constant drift without any uncertainty. This is obviously only an approximation. The effect of this approximation will be studied later in section 4.4.5.

### 4.4.3 Dependence on the Price of Risk

The price of risk $\lambda$, taken for the analysis so far, is the price of risk $\lambda_S$ from the asset process, increased by 10%. As discussed before, the minimal price approach only makes sense if the price of risk, which enters the minimization procedure, is greater than $\lambda_S$. If the price of risk tends to infinity, the Bouchaud-Sornette and the minimal price approach should give the same results (see section 3.4.2.2). Since the Bouchaud-Sornette strategy doesn’t depend on the price of risk, the Bouchaud-Sornette price should increase linear with $\lambda$. These two effects can be seen in figure 4.8. Shown is the option price as a function of $\lambda$. Compared are the price distributions for the Black-Scholes, Bouchaud-Sornette, and the minimal price strategies for the plain vanilla call option. The rehedging frequency is 0.5. The price of risk varies from $1.1 \cdot \lambda_S$ to $4 \cdot \lambda_S$ in steps of $0.1 \cdot \lambda_S$. 

Figure 4.8: Dependence of the option premium on the price of risk. The rehedging frequency is 0.5.
4.4.4 Implied Volatility Matrix

4.4.4.1 Plain Vanilla Options

The results of section 4.4.1 show that the difference between the option prices, including and excluding the risk premium, is larger out of the money than at the money. Hence, the implied volatility is larger out of the money. This indicates that a volatility smile is present. A detailed study of the implied volatility is subject of this section. The implied volatility will be studied in terms of the volatility matrix, that is the implied volatility as a function of strike and maturity.

The implied volatility is nothing but the option price, transformed into a Black-Scholes volatility. Inserting the implied volatility in the analytic Black-Scholes formula will return the option price. Figure 4.9 shows, as example, a volatility matrix for a plain vanilla call. The volatility corresponds to prices calculated with the minimal price approach including the risk premium. The rehedging frequency is taken to be 0.5, this means one hedge every second day. The typical characteristics, as known from implied volatilities from market prices, are present in the distribution. A smile appears, the volatilities increase in and out of the money, and the smile becomes flatter with increasing maturity. For a quantitative discussion, sectional views, as in figure 4.10, are more appropriate to use. Therefore, first the strike dependence (volatility smile) and afterwards the maturity dependence (term structure) will be

Figure 4.9: Volatility matrix for a plain vanilla call and the fat tail process. The volatilities are calculated with the minimal price approach. The rehedging frequency is 0.5 for all maturities.
addressed.

The volatility smile for different hedging strategies is shown in figure 4.10. The Black-Scholes strategy without risk premium gives, as expected, a constant implied volatility for the log-normal process. For the fat tail process a smile appears, because it becomes more likely that the asset price moves from out or in the money into the strike regime. This effect has been described in the literature, e.g. in reference [12]. The implied volatilities for the minimal price approach are smaller as for the Bouchaud-Sornette method. This is obvious since the price is smaller. More surprising
is the fact that the smile is much more pronounced when the risk premium is included. This is due to the following effect. The implied volatility far out or far in the money is mainly determined by the hypothetical asset movements which cross the strike. If the price is far away from the strike, large price movements for these hypothetical paths are necessary between each hedge. These large movements result in larger hedging errors, as for an option at the money. Larger hedging errors give a larger risk, and hence, a larger price and a larger implied volatility. This can be verified by studying the smile as a function of the rehedging frequency, as in figure 4.11. The smile becomes flatter with increasing rehedging frequency.

For stock options, and here especially for index options, the volatility smile from market prices is asymmetric[12, 14]. The implied volatilities of call options are larger in the money than out of the money. One way to model this is by an asymmetric probability distribution, such that a slump in prices is more likely than an increase of prices (see [12]). This effect is reproduced in figure 4.12 by setting the fat tail probability distribution to zero for large positive price changes\(^3\).

We are now coming to the discussion of the volatility term structure. For at the money plain vanilla options, it is shown in figure 4.13. The rehedging frequency is taken to be 0.5. For the log-normal process, the term structure is flat, while for the fat tail process, the volatility drops for short maturities even for the Black-Scholes

\(^3\)The probability distributions are normalized to 1 afterwards and then shifted such that the drift is about 0.1
Figure 4.12: Volatility smiles for asymmetric probability distributions. The probability distributions are based on the fat tail process for which the probability was set to zero above $\hat{\mu} = 0.05$ (upper plot), resp. above $\hat{\mu} = 0.04$ (lower plot). Errors are shown for strike values of 38, 41, 50, and 62. Most of the errors are smaller than the marker size, so that the error bars are hidden.
Figure 4.13: Term structures for implied volatilities at the money, for different processes and hedging strategies. The rehedging frequency is 0.5 for all maturities. The distributions for a plain vanilla put are similar to the distributions for a call.
strategy without risk premium. It can be shown that the effect is much larger than the numerical errors. Further, the effect is not due to the different number of rehedges at different maturities because the same effect is observed when the number of rehedges, and not the rehedging frequency, is taken to be constant for all maturities. The effect can be understood by the properties of the fat tail distribution and the fact that the option price at the money is approximatly given by

\[ V(S = E) \approx \frac{1}{2} \langle |S - E| \rangle_0 \]  

where \( E \) is the strike. For a log-normal process, this is approximatly equal to \( V(E) \approx \sigma_0 E \sqrt{T}/\sqrt{2\pi} \). \( T \) is the maturity. For the fat tail process, the expectation value \( \langle |S - E| \rangle_0 \) increases faster than \( \sqrt{T} \). Therefore, the implied volatility increases with increasing maturity. The term structures for put options, not shown in figure 4.13, are similar as for call options, and the same argument to explaining the increase of volatilities with maturity holds.

So far, the volatility matrix was studied as a function of maturity and strike. For a log-normal process, it is more natural to show the dependence on the variable \( \log(S/E) \cdot \sqrt{T} \) which enters in the exponential of the probability density. Plotting the implied volatility as a function of this variable, as in figure 4.14, shows that the smile is then almost independent on the maturity. For the fat tail process, shown in figure 4.15, a maturity dependence remains. For larger maturities, the maturity dependence becomes smaller. This is due to the fact that the fat tail probability distribution converges towards a Gaussian for large time horizons. At the money, the implied volatility increases with increasing maturity. This has already been discussed in relation to figure 4.13. Far out or in the money, the implied volatility decreases with increasing maturity. In this regime, the option price is sensitive to the tails. The larger the tails, the higher is the option price. Since the tails decrease with increasing maturity, relative to a log-normal process, the implied volatility is decreasing in the tail regions as well. This is somewhat compensated by the risk premium.

### 4.4.4.2 Binary Options

Volatility matrices for binary options has to be studied separately from plain vanilla options. This is because the shape of the implied volatility matrix is absolutely different. Figure 4.16 shows the implied volatility matrix of a binary call for the minimal price approach with the fat tail asset process. There are two interesting effects visible. First, the implied volatility is larger in the money than out of the money. And second, for options where the strike is just below the threshold
Figure 4.14: Volatility smiles at different maturities for a log-normal process. The distributions scale with $\sqrt{T}$. Without risk premium, the distribution for the Bouchaud-Sornette method is flat.
Figure 4.15: Volatility smiles at different maturities for the fat tail process. Deviations from a $\sqrt{T}$ scaling is observed for all pricing methods.
Figure 4.16: Implied volatility matrix of a binary call for the fat tail asset process. The volatilities are calculated with the minimal price approach. The rehedging frequency is 0.5 for all maturities.

\[
d_1 = \left[ \log\left( \frac{S}{E} \right) + (r + \frac{1}{2} \sigma^2) T \right] / \sigma \sqrt{T} = 0, \text{ where vega } \frac{dV}{d\sigma} \text{ is zero, the volatility is infinite. Just above this threshold, the calculated option price is out of the range of Black-Scholes prices. An implied volatility doesn’t exist in this regime.}
\]

These effects can be understood from the dependence of the Black-Scholes price on the volatility. In figure 4.17, the Black-Scholes price for a binary call is given for different volatilities. Supposes that the volatility is 0.2. If one adds a risk premium to the price, the implied volatility will drop in the money and will rise out of the money. For relatively small price increases far in the money or just out of the money no Black-Scholes price exist.

A comparison of implied volatilities from the different pricing models, at a maturity of 0.2, is shown in figure 4.18. This figure corresponds to figure 4.10 and 4.12 for a plain vanilla call. If volatilities are not plotted in the figure, the price is out of the Black-Scholes price regime.

4.4.5 Drift Dependence

As described in section 4.4.2, the strategy of the minimal price approach can be understood in terms of the portfolio theory. The most critical input to portfolio optimisation is the asset drift. Therefore, the effect of an uncertainty in the drift will be studied in this section.
Using a wrong drift to determine the hedging strategy will result in an unexpected averaged loss or gain, when applying this strategy. This effect was studied in the following way. Assume that an option writer has estimated the drift to be 0.1. With this drift, he calculates his hedging strategy and gets an option price. He also gets an expected profit from the hedging portfolio. If the real drift is different from 0.1, the averaged profit will be different as well. This can be quantified by first calculating the hedging strategy with a drift of 0.1 and then calculating the hedging portfolio profit with Monte Carlo using a different, the real drift. This simulates that in the real market the asset process has a drift, different from the one used for the option pricing. The profit, expected by the option writer, was calculated by using a drift of 0.1. The difference of these two profit values as function of the “real” asset drift, is shown in figure 4.19. For the Black-Scholes and Bouchaud-Sornette hedging strategy, almost no drift dependence is observed. The curve for the minimal price approach shows a slope of about 2 which is in agreement with the following simple estimate. Due to the additional amount of assets ($\Delta \phi \approx 0.2$), the slope is expected to be $\Delta \phi \cdot S(t_0) \cdot T = 0.2 \cdot 50 \cdot 0.2 = 2$.

Obviously, this introduces a new risk for the minimal price approach, which has not been taken into account so far. To handle this in a consistent way would require to use an uncertain drift in the probability distribution. Then, the probability distribution would be not only a function of the asset price, but also on the drift\(^4\). The problem

\(^4\)In the same way one would extend the method for uncertain volatilities
Figure 4.18: Implied volatility as a function of the strike for a binary call. Results for different hedging strategies are shown, for the fat tail process, for a log-normal process and for an asymmetric probability distribution.
will become more complex, and the calculation of the expectation values would require additional integrations over the drift. Here only an order of magnitude estimate of the effect on the price will be given. As example, it is assuming that the uncertainty on the drift is 2 percent points, so it could be 0.08 or 0.12 instead of 0.1. According to figure 4.19, this will introduce an uncertainty on the portfolio profit of 0.04. The risk premium for such an uncertainty is $0.04 \cdot 0.12 \approx 0.005$, where the price of risk $\lambda = 0.12$ is the one used in the previous sections. This additional risk premium has to be added to the option price. In Figure 4.2 it can be seen that the additional risk premium reduces the difference to the Bouchaud-Sornette price by roughly 10%. So, the effect is small, but not negligible.

Figure 4.19: The unexpected averaged profit from the hedging portfolio due to a wrong asset drift, taken for the hedging strategy determination. The hedging strategies were all determined for a drift of 0.1. The profit difference is shown as a function of the real drift.
Chapter 5

Conclusions

In this thesis, option pricing was studied with assumptions closer to reality as for the Black-Scholes method. The continuous time condition was replaced by discrete hedging, and the restriction of a log-normal asset process was extended to any i.i.d. process. This pricing problem can be handled in the framework of the Bouchaud-Sornette method. The method considers a hedging portfolio for which the risk is minimized.

The Bouchaud-Sornette method was compared to an alternative pricing approach which was introduced in this thesis. Instead of minimizing the risk, the option price is minimized in the new approach. The ratio of return to risk for the hedging portfolio is the same for both methods, by definition. Since the new method minimizes the price, the hedging strategy of this method allows to write options which are more competitive on the market.

The two pricing methods were compared for a log-normal asset process and for a process with fat tails. The main results of the study are the following:

General results

- For a log-normal process in the continuous time limit, both methods recover the standard Black-Scholes result.
- The option premium of the Bouchaud-Sornette and the minimal price approach converge towards each other with increasing price of risk.
- For the minimal price approach and the Bouchaud-Sornette method, the ratio of return to risk for the hedging portfolio is the same, by definition of the methods.
• The option premium is smaller for the minimal price approach. Especially for options far out or in the money, the difference is large. To be competitive on the market, one should use the minimal price approach.

• If an investor is only interested in hedging the risk, the Bouchaud-Sornette strategy is the optimal choice. In practice, the Black-Scholes delta hedging strategy is more convenient to use, since it is easier to calculate and the differences are marginal.

• For both, the Bouchaud-Sornette and the minimal price approach, the risk is measured in terms of the variance. If an investor has a different view on the risk, e.g. using the kurtosis, the hedging strategy and the price will be different. As long as the risk doesn’t vanish, the models always have to make assumptions on the risk preference.

Results on implied volatilities

• Fat tails in the asset process leads to a volatility smile for plain vanilla options.

• At the money, fat tails give rise to an increasing implied volatility with maturity.

• Including the risk premium in the option price gives a volatility smile for plain vanilla options. For the options studied, the effect is larger as the one due to fat tails.

• Asymmetric probability distributions yield asymmetric implied volatility smiles.

• Implied volatility matrices for binary options have a different shape as for plain vanilla options. Smiles are not present. Implied Black-Scholes volatilities doesn’t exist for all strike values.

Dependence on the asset drift

• The Bouchaud-Sornette method shows almost no dependence on the drift of the asset. The drawback of the minimal price approach is the drift dependence. It introduces an additional risk, but the corresponding risk premium is about an order of magnitude smaller than the price difference between the two method.
It has been seen in the study, that both methods are able to reproduce the key features of implied volatility matrices, as they are observed on the market. The difference in the pricing methods is that the minimal price approach allows to write cheaper options. These options are more competitive on the market which is a strong argument in favour of the minimal price approach.
Appendix A

Formulas for Expectation Values

A.1 Strategy Independent Integrals

This appendix contains full formulas for the expectation values which are part of equation 3.3, 3.9 and 3.17. First, formulas are given for expectation values which are not strategy dependent. These expectation values have to be calculated only once for each iteration procedure described in section 4.3.1.

\[
\langle V^j(T) \rangle_0 = \int V^j(S_m) \prod_{i=1}^m p_{i|i-1} d_m S_i = \int V^j(S_m) p_{m|0} dS_m
\]

\[
I_{l|k} = \int \Delta \tilde{S}_k p_{l|k+1} p_{k+1|k} dS_{k+1}
\]

\[
\langle (\Delta \tilde{S}_k)^j \rangle_k = \frac{1}{p_{k|0}} \int (\Delta \tilde{S}_k)^j \prod_{i=1}^m p_{i|i-1} dS_m \ldots dS_{k+1} dS_k \ldots dS_1
\]

\[
= \int (\Delta \tilde{S}_k)^j p_{k+1|k} dS_{k+1}
\]

\[
\langle V(T)\Delta \tilde{S}_k \rangle_{k<m-1} = \frac{1}{p_{k|0}} \int V(S_m) \Delta \tilde{S}_k \prod_{i=1}^m p_{i|i-1} dS_m \ldots dS_{k+1} dS_k \ldots dS_1
\]

\[
= \int V(S_m) \Delta \tilde{S}_k p_{m|k+1} p_{k+1|k} dS_m dS_{k+1}
\]

\[
= \int V(S_m) I_{m|k} dS_m
\]

\[
\langle V(T)\Delta \tilde{S}_k \rangle_{k=m-1} = \int V(S_m) \Delta \tilde{S}_k p_{k+1|k} dS_m
\]
A.2 Strategy Dependent Integrals

The following expectation values are strategy dependent and have to be recalculated for every step in the iteration procedure.

\[
\begin{align*}
\langle \phi \Delta \tilde{S}_l \rangle_0 &= \int \phi_l(S_l) \left( \tilde{S}_{l+1} - \tilde{S}_l \right) \prod_{i=1}^{m} \pi_{i\|i-1} \, d^m S_i \\
&= \int \phi_l \Delta \tilde{S}_l \, p_{l+1|l} \, p_{l\|0} \, dS_{l+1} \, dS_l = \int \phi_l \langle \Delta \tilde{S}_l \rangle_l \, p_{l\|0} \, dS_l \\
\langle \phi_0 \Delta \tilde{S}_0 \rangle_0 &= \phi_0 \langle \Delta \tilde{S}_0 \rangle_0
\end{align*}
\]

\[
\begin{align*}
\langle V(T) \phi_l \Delta \tilde{S}_l \rangle_0 &= \int V(S_m) \phi_l \Delta \tilde{S}_l \, p_{m|l+1} \, p_{l+1|l} \, p_{l\|0} \, dS_m \, dS_{l+1} \, dS_l \\
&= \int V(S_m) \phi_l \, I_{m|l} \, p_{l\|0} \, dS_m \, dS_l \\
\langle V(T) \phi_0 \Delta \tilde{S}_0 \rangle_0 &= \phi_0 \int V(S_m) \Delta \tilde{S}_0 \, p_{m|1} \, p_{1\|0} \, dS_m \, dS_1 \\
&= \phi_0 \int V(S_m) \, I_{m|0} \, dS_m
\end{align*}
\]

\[
\begin{align*}
\langle V(T) \phi_{l=m-1} \Delta \tilde{S}_l \rangle_0 &= \int V(S_m) \phi_l \Delta \tilde{S}_l \, p_{m|l} \, p_{l\|0} \, dS_m \, dS_l
\end{align*}
\]
For the expectation value \( \langle \phi_k \phi_l \Delta \tilde{S}_k \Delta \tilde{S}_l \rangle \), formulas are only given for \( k \geq l \) since the expectation value is symmetric in \( l, k \):

\[
\begin{align*}
\langle \phi_k \phi_l \Delta \tilde{S}_k \Delta \tilde{S}_l \rangle_0 &= \int \phi_k \phi_l \Delta \tilde{S}_k \Delta \tilde{S}_l \ p_{k+1|k} \ p_{k+1|l} \ p_{l+1|l} \ p_{l+1|0} \ dS_{k+1} \ dS_k \ dS_{l+1} \ dS_l \\
&= \int dS_k \ dS_{l+1} \ dS_l \ \phi_k \phi_l \Delta \tilde{S}_l \ p_{k+1|l} \ p_{l+1|0} \ \int dS_{k+1} \ \Delta \tilde{S}_k \ p_{k+1|k} \\
&= \int dS_k \ dS_{l+1} \ dS_l \ \phi_k \phi_l \Delta \tilde{S}_l \ p_{k+1|l} \ p_{l+1|0} \ \langle \Delta \tilde{S}_k \rangle_k \\
&= \int dS_k \ dS_l \ \phi_k \phi_l \ I_{k||l} \ p_{l|0} \ \langle \Delta \tilde{S}_k \rangle_k \\
\langle \phi_k^2 \Delta \tilde{S}_k^2 \rangle_0 &= \int \phi_k^2 \Delta \tilde{S}_k^2 \ p_{k+1|k} \ p_{k|0} \ \int dS_{k+1} \ \Delta \tilde{S}_k^2 \ dS_{k+1} = \int dS_k \ p_{k|0} \ \phi_k^2 \ \langle \Delta \tilde{S}_k^2 \rangle_k \\
\langle \phi_{k-l+1} \phi_l \Delta \tilde{S}_k \Delta \tilde{S}_l \rangle_0 &= \int \phi_{k-l+1} \phi_l \Delta \tilde{S}_k \Delta \tilde{S}_l \ p_{k+1|k} \ p_{k|l} \ p_{l|0} \ dS_{k+1} \ dS_k \ dS_l \\
&= \int dS_k \ dS_l \ \phi_{k-l+1} \phi_l \Delta \tilde{S}_l \ p_{k+1|l} \ p_{l|0} \ \int dS_{k+1} \ \Delta \tilde{S}_k \ p_{k+1|k} \\
&= \int dS_k \ dS_l \ \phi_{k-l+1} \phi_l \Delta \tilde{S}_l \ p_{k+1|l} \ p_{l|0} \ \langle \Delta \tilde{S}_k \rangle_k \\
\langle \phi_k \phi_0 \Delta \tilde{S}_k \Delta \tilde{S}_0 \rangle_0 &= \int \phi_k \phi_0 \Delta \tilde{S}_k \Delta \tilde{S}_0 \ p_{k+1|k} \ p_{k|0} \ dS_{k+1} \ dS_k \ dS_l \\
&= \int dS_k \ dS_l \ \phi_k \phi_0 \Delta \tilde{S}_0 \ p_{k+1|l} \ p_{l|0} \ \int dS_{k+1} \ \Delta \tilde{S}_k \ p_{k+1|k} \\
&= \phi_0 \ \int dS_k \ dS_l \ \phi_k \phi_0 \Delta \tilde{S}_0 \ p_{k+1|l} \ p_{l|0} \ \langle \Delta \tilde{S}_k \rangle_k \\
&= \phi_0 \ \int dS_k \ \phi_k \ I_{k|0} \ \langle \Delta \tilde{S}_k \rangle_k \\
\langle \phi_1 \phi_0 \Delta \tilde{S}_1 \Delta \tilde{S}_0 \rangle_0 &= \int \phi_1 \phi_0 \Delta \tilde{S}_1 \Delta \tilde{S}_0 \ p_{2|1} \ p_{1|0} \ dS_2 \ dS_1 \\
&= \int dS_1 \ \phi_1 \phi_0 \Delta \tilde{S}_0 \ p_{2|0} \ \int dS_2 \ \Delta \tilde{S}_1 \ p_{2|1} \\
&= \phi_0 \ \int dS_1 \ \phi_1 \phi_0 \Delta \tilde{S}_0 \ p_{1|0} \ \langle \Delta \tilde{S}_1 \rangle_1
\end{align*}
\]
\[ \left\langle \phi_l \Delta \tilde{S}_l \Delta \tilde{S}_k \right\rangle_{k=l-1} = \phi_k \left\langle \Delta \tilde{S}_k^2 \right\rangle_k = \phi_k \int \Delta \tilde{S}_k^2 p_{k+1|k} \, dS_{k+1} \]

\[ \left\langle \phi_l \Delta \tilde{S}_l \Delta \tilde{S}_k \right\rangle_{k=l-1} = \frac{1}{p_{k|0}} \int \phi_{k+1} \Delta \tilde{S}_{k+1} \Delta \tilde{S}_k \prod_{i=1}^{m} p_{i|0} \, dS_{m} \cdots dS_{k+1} dS_{k-1} \cdots dS_{1} \]

\[ = \int \phi_{k+1} \Delta \tilde{S}_{k+1} \Delta \tilde{S}_k \, p_{k+2|k+1} \, p_{k+1|k} \, dS_{k+2} \, dS_{k+1} \]

\[ = \int dS_{k+1} \phi_{k+1} \Delta \tilde{S}_k \, p_{k+1|k} \int dS_{k+2} \Delta \tilde{S}_{k+1} \, p_{k+2|k+1} \]

\[ = \int dS_{k+1} \phi_{k+1} \Delta \tilde{S}_k \, p_{k+1|k} \left\langle \Delta \tilde{S}_l \right\rangle_l \]

\[ \left\langle \phi_l \Delta \tilde{S}_l \Delta \tilde{S}_k \right\rangle_{k=l+1} = \frac{1}{p_{k|0}} \int \phi_{l+1} \Delta \tilde{S}_{l+1} \Delta \tilde{S}_k \, p_{l|0} \, dS_{l+1} \]

\[ = \int \phi_{l+1} \Delta \tilde{S}_{l+1} \Delta \tilde{S}_k \, p_{l+1|l} \, p_{l|0} \, dS_{l+1} \, dS_{l} \]

\[ = \phi_0 \Delta \tilde{S}_0 \int \Delta \tilde{S}_k p_{k+1|k} \, dS_{k+1} = \phi_0 \Delta \tilde{S}_0 \left\langle \Delta \tilde{S}_k \right\rangle_k \]

\[ \left\langle \phi_l \Delta \tilde{S}_l \Delta \tilde{S}_k \right\rangle_{k<l+1} = \frac{1}{p_{k|0}} \int \phi_{l+1} \Delta \tilde{S}_{l+1} \Delta \tilde{S}_k \, p_{l|l-1} \, dS_{l} \cdots dS_{k+1} dS_{k-1} \cdots dS_{1} \]

\[ = \int \phi_{l+1} \Delta \tilde{S}_{l+1} \Delta \tilde{S}_k \, p_{l+1|l} \, p_{l|0} \, dS_{l+1} \, dS_{l} \]

\[ = \int dS_{l+1} \phi_{l+1} \Delta \tilde{S}_k \, p_{l+1|l} \int dS_{l+1} \Delta \tilde{S}_{l+1} \, p_{l+1|l} \]

\[ = \int dS_{l+1} \phi_{l+1} \Delta \tilde{S}_k \, p_{l+1|l} \left\langle \Delta \tilde{S}_l \right\rangle_l = \int dS_{l} \phi_{l} \, I_{l|k} \left\langle \Delta \tilde{S}_l \right\rangle_l \]

\[ \left\langle \phi_l \Delta \tilde{S}_l \Delta \tilde{S}_k \right\rangle_{k>l+1,l>0} = \frac{1}{p_{k|0}} \int \phi_{l+1} \Delta \tilde{S}_{l+1} \Delta \tilde{S}_k \, p_{l+1|l} \, p_{l|0} \, dS_{l+1} \, dS_{l} \]

\[ = \frac{1}{p_{k|0}} \int dS_{l+1} \Delta \tilde{S}_k \, p_{k+1|k} \cdot \int dS_{l+1} \Delta \tilde{S}_{l+1} \, \phi_{l} \Delta \tilde{S}_l \, p_{l+1|l} \, p_{l|0} \]

\[ = \left\langle \Delta \tilde{S}_k \right\rangle_k \frac{1}{p_{k|0}} \int dS_{l} \phi_{l} \, I_{l|l} \, p_{l|0} = \left\langle \Delta \tilde{S}_k \right\rangle_k \left\langle \phi_l \Delta \tilde{S}_l \right\rangle_k \]

\[ \left\langle \phi_l \Delta \tilde{S}_l \Delta \tilde{S}_k \right\rangle_{k>l+1,l=0} = \frac{1}{p_{k|0}} \int dS_{k+1} \Delta \tilde{S}_k \, p_{k+1|k} \cdot \int dS_{l+1} \phi_{l} \Delta \tilde{S}_l \, p_{k|l+1} \, p_{l+1|l} \]

\[ = \left\langle \Delta \tilde{S}_k \right\rangle_k \phi_{0} \left\langle \Delta \tilde{S}_0 \right\rangle_k \]
Appendix B

Continuous Time Limit

B.1 The Limit for Expectation Values

To study the continuous time limit, expectation values

\[ \langle f(S, \tau) \rangle = \int_0^\infty f(S, \tau) p(S|S_0) dS \]  \hspace{1cm} (B.1)

have to be calculated up to first order in \( \Delta t \). \( \Delta t = t - t_0 \) is the time difference between two hedges. The time argument \( \tau \) of \( f(S, \tau) \) is equal to \( t \), but is named differently to avoid confusion when a Taylor expansion of \( f(S, \tau) \) will be done later. Here, the continuous time limit will be studied for a log-normal process

\[ p(S|S_0) = \frac{1}{\sqrt{2\pi\sigma_t}} e^{-1/2[\ln(S/S_0) - \hat{\mu}\Delta t]^2/\sigma_t^2} \]  \hspace{1cm} (B.2)

with \( \sigma_t = \sigma \sqrt{\Delta t} \) and \( \hat{\mu} = \mu - \frac{1}{2}\sigma^2 \). First, the variable transformation

\[ y = \log \left( \frac{S}{S_0} \right) - \hat{\mu}\Delta t \iff S = S_0 e^{y + \hat{\mu}\Delta t} \Rightarrow dS = S dy \]  \hspace{1cm} (B.3)

is applied. With this transformation, it follows that

\[ p(S|S_0) dS = \frac{1}{\sqrt{2\pi\sigma_t}} e^{-1/2y^2/\sigma_t^2} dy = p(y) dy \]

Hence, the expectation value can be written as

\[ \langle f(S, \tau) \rangle_0 = \int_{-\infty}^\infty f(y, \tau) p(y) dy \]  \hspace{1cm} (B.4)

In the next step, \( f(y, \tau) \) is replaced by its tailor expansion around \( y = y_0 = -\hat{\mu}\Delta t \) and \( \tau = t_0 \)

\[ f(y, \tau) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{1}{i! (n-i)!} \frac{\partial^n f}{\partial y^i \partial \tau^{n-i}} \bigg|_{y_0, t_0} (y - y_0)^i (\tau - t_0)^{n-i} \]  \hspace{1cm} (B.5)
which gives

\[
\langle f(y, \tau) \rangle = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{1}{i! (n-i)!} \frac{\partial^n f}{\partial y^i \partial \tau^{n-i}} \bigg|_{y_0, t_0} \langle (y-y_0)^i \rangle (\tau-t_0)^{n-i} \quad (B.6)
\]

For the calculation of the terms \( \langle (y-y_0)^i \rangle \), the known moments of the Gaussian distribution \( < y^{2k} > = (2k-1)!! \sigma_t^{2k} \) and \( < y^{2k-1} > = 0 \) are substituted:

\[
\langle (y-y_0)^i \rangle = \sum_{j=0}^{i} \left( \begin{array}{c} i \\ j \end{array} \right) \langle y^j \rangle (y_0)^{i-j} \quad (B.7)
\]

\[
= \sum_{0 \leq k \leq i/2} \frac{i!}{(2k)! (i-2k)!} (2k-1)!! \sigma_t^{2k} \Delta t^k (\hat{\mu} \Delta t)^{i-2k}
\]

\[
= \sum_{0 \leq k \leq i/2} \frac{i! (2k)! (i-2k)!!}{(2k)! (i-2k)!} \sigma_t^{2k} \hat{\mu}^{i-2k} \Delta t^{i-k} \quad (B.8)
\]

For an expansion up to order \( \Delta t \), only terms up to \( i = 2 \) have to be taken into account:

\[
\langle y-y_0 \rangle = \hat{\mu} \Delta t \quad (B.9)
\]

\[
\langle (y-y_0)^2 \rangle = \sigma^2 \Delta t + \hat{\mu}^2 \Delta t^2
\]

Substituting this in equation B.6, the expectation value up to order \( \Delta t \) is

\[
\langle f(y, \tau) \rangle = f(y_0, t_0) + \left\{ \frac{\partial f}{\partial \tau} + \hat{\mu} \frac{\partial f}{\partial y} + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial y^2} \right\} \Delta t + O(\Delta t^2) \quad (B.10)
\]

The inverse transformation from the variable \( y \) to \( S \), using

\[
\frac{\partial f}{\partial y} \rightarrow S \frac{\partial f}{\partial S} \quad (B.11)
\]

\[
\frac{\partial^2 f}{\partial y^2} \rightarrow S \frac{\partial f}{\partial S} + S^2 \frac{\partial^2 f}{\partial S^2}
\]

and setting \( \tau = t \), gives the final result

\[
\langle f(S, t) \rangle_0 = f(S_0, t_0)
\]

\[
+ \left\{ \frac{\partial f}{\partial t} + \left( \hat{\mu} + \frac{1}{2} \sigma^2 \right) S_0 \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S_0^2 \frac{\partial^2 f}{\partial S^2} \right\} \Delta t + O(\Delta t^2)
\]

\[
= f(S_0, t_0) + \left\{ \frac{\partial f}{\partial t} + \mu S_0 \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S_0^2 \frac{\partial^2 f}{\partial S^2} \right\} \Delta t + O(\Delta t^2)
\]

Here, all derivatives has to be taken at \( S = S_0 \) and \( t = t_0 \). This formula will be applied to different functions \( f(S, t) \) in the following.
Equation B.12 can be also derived by using Itô’s lemma. For a log-normal process
\[dS = \mu S \, dt + \sigma S \, dX,\]
the differential \(df\) is given by
\[df = \frac{\partial f}{\partial t} \, dt + \frac{\partial f}{\partial S} \, dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \, dt.\]  

(B.13)

Taking the expectation value of this equation, directly yields equation B.12 since
\[\langle dS \rangle = \mu S \, dt.\]

All relevant expectation values can now be calculated up to order \(\Delta t\):
\[
\left\langle \Delta \tilde{S}_0 \right\rangle_0 = \langle S - S_0 \, e^{r\Delta t} \rangle_0 \quad \quad \text{(B.14)}
\]
\[
= (S_0 - S_0) - rS_0 \Delta t + \mu S_0 \Delta t + \mathcal{O}(\Delta t^2)
\]
\[
= S_0 (\mu - r) \Delta t + \mathcal{O}(\Delta t^2)
\]

\[
\left\langle \left( \Delta \tilde{S}_0 \right)^2 \right\rangle_0 = \left\langle \langle S - S_0 \, e^{r\Delta t} \rangle^2 \right\rangle_0 \quad \quad \text{(B.15)}
\]
\[
= (S_0 - S_0)^2
\]
\[
+ \{ 2S_0 r (S_0 - S_0 e^{r0}) + 2 \mu S_0 (S_0 - S_0 e^{r0}) + \sigma^2 S_0^2 \} \Delta t
\]
\[
+ \mathcal{O}(\Delta t^2)
\]
\[
= S_0^2 \sigma^2 \Delta t + \mathcal{O}(\Delta t^2)
\]

\[
\langle V(S, t) \rangle_0 = V(S_0, t_0) \quad \quad \quad \quad \text{(B.16)}
\]
\[
+ \left\{ \frac{\partial V}{\partial t} + \mu S_0 \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S_0^2 \frac{\partial^2 V}{\partial S^2} \right\} \bigg|_{S_0, t_0} \Delta t + \mathcal{O}(\Delta t^2)
\]

\[
\langle V^2(S, t) \rangle_0 = V^2(S_0, t_0) \quad \quad \quad \quad \text{(B.17)}
\]
\[
+ 2 \left\{ \frac{\partial V}{\partial t} + \mu S_0 V \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S_0^2 \left[ \left( \frac{\partial^2 V}{\partial S^2} \right)^2 + V \frac{\partial^2 V}{\partial S^2} \right] \right\} \bigg|_{S_0, t_0} \Delta t + \mathcal{O}(\Delta t^2)
\]

\[
\langle V(S, t) \Delta \tilde{S}_0 \rangle_0 = \langle V(S, t) \right\rangle_0 \langle S - S_0 \, e^{r\Delta t} \rangle_0 \quad \quad \text{(B.18)}
\]
\[
= V(S_0, t_0) (S_0 - S_0 e^{r0})
\]
\[
+ \left\{ - r S_0 V(S_0, t_0) + \mu S_0 V(S_0, t_0) + \sigma^2 S_0^2 \frac{\partial V}{\partial S} \right\} \bigg|_{S_0, t_0} \Delta t + \mathcal{O}(\Delta t^2)
\]
\[
= S_0 \left\{ (\mu - r) V(S_0, t_0) + \sigma^2 S_0 \frac{\partial V}{\partial S} \right\} \bigg|_{S_0, t_0} \Delta t
\]
B.2 The Limit for the Hedging Strategy

In this section, it will be shown that the Black-Scholes hedging strategy is a solution of equation 3.17

\[
\phi_k^* = \frac{1}{\left< \Delta \tilde{S}_k^2 \right>_k} \left\{ \left< V(T) \Delta \tilde{S}_k \right>_k + \left< \Delta \tilde{S}_k \right>_k \left[ \frac{1}{\lambda} R(T) + \left< K(T) \right>_0 \right] - \sum_{l=0}^{m-1} \left< \phi_l^* \Delta \tilde{S}_l \cdot \Delta \tilde{S}_k \right>_k \right\}
\]  

in the continuous time limit for a log-normal process.

If one substitutes the Black-Scholes strategy in equation B.19, the term proportional to \( R \) will vanish in the continuous time limit. This is, because the Black-Scholes hedging replicates the option in a risk-free way, \( \lambda \) is a constant and \( \left< \Delta \tilde{S}_k \right>_k / \left< \Delta \tilde{S}_k^2 \right>_k \) is finite since the numerator and denominator both are of order \( \Delta t \).

Since the Black-Scholes price is uniquely defined for all times, the following relation can be deduced from equation 2.20:

\[
\tilde{V}(t_i) = V(t_j \triangleright i) - \sum_{l=i}^{j-1} \phi_l \Delta \tilde{S}_l
\]  

with \( \tilde{V}(t_i) \equiv V(t_i) B^{t-i} \). To see this, one has to take into account that the portfolio value, \( \Pi(t) = e^{rt} \cdot \Pi(t_0) = 0 \), increases with the risk-free rate in the Black-Scholes world. Hence, it follows for the expectation values

\[
\left< V(T) \Delta \tilde{S}_k \right>_k = \left< \tilde{V}(t_{k+1}) \Delta \tilde{S}_k \right>_k + \sum_{l=k+1}^{m-1} \left< \phi_l \Delta \tilde{S}_l \Delta \tilde{S}_k \right>_k
\]  

and

\[
\sum_{l=0}^{m-1} \left< \phi_l \Delta \tilde{S}_l \Delta \tilde{S}_k \right>_k = \sum_{l=k+1}^{m-1} \left< \phi_l \Delta \tilde{S}_l \Delta \tilde{S}_k \right>_k + \sum_{l=0}^{k-1} \left< \phi_l \Delta \tilde{S}_l \Delta \tilde{S}_k \right>_k
\]  

\[
= \sum_{l=k+1}^{m-1} \left< \phi_l \Delta \tilde{S}_l \Delta \tilde{S}_k \right>_k + \left( \tilde{V}(t_k) - \tilde{V}_0 \right) \left< \Delta \tilde{S}_k \right>_k
\]

Using this and taking into account that \( \tilde{V}_0 = \lambda R - \left< K(T) \right>_0 \), equation B.19 can be written as:

\[
\phi_k^* = \frac{1}{\left< \Delta \tilde{S}_k^2 \right>_k} \left\{ \left< \tilde{V}(t_{k+1}) \Delta \tilde{S}_k \right>_k - \left< \Delta \tilde{S}_k \right>_k \tilde{V}(t_k) \right\}
\]  

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This equation has the same form as equation 3.18. In the same way as in section 3.5, it can be shown that the right side of equation B.23 is equal to $\partial V / \partial S$ in the continuous time limit. The Black-Scholes hedging strategy is therefore a solution of equation B.19.
Bibliography


