Pricing of Discrete Barrier Options

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A thesis submitted in partial fulfilment of the requirements for the MSc in

Mathematical Finance

October 22, 2003
“Pricing of Discrete Barrier Options”

MSc in Mathematical Finance
Trinity 2003

Abstract

This study addresses the pricing of discrete barrier options using analytical methods and numerical simulations. For discrete barrier options, the asset price is only monitored at instants $t_i = iT/m$, where $T$ is the expiration date and $m - 1$ is the number of monitoring points ($i = 1, 2, ..., m - 1$). The analytical solution for discrete barrier options involves $m$-dimensional integrals, which are not analytically tractable for options with a high number of monitoring points ($m > 5$). The use of numerical procedures for pricing discrete barrier options becomes increasingly difficult at even higher numbers of monitoring points, e.g. $m \approx 100$. In this case, it is convenient to perform an asymptotic expansion that becomes exact in the limit as the number of monitoring points goes to infinity. Broadie et al. [1] have derived a continuity correction for discrete barrier options that satisfies this condition. We show that the continuity correction for single barrier options with discrete monitoring can also be derived within the framework of matched asymptotic expansions. This method can be extended to derive an asymptotic expression for double barrier options with discrete monitoring.
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Chapter 1

Introduction

Barrier options are one of the oldest and most popular examples of the so-called "exotic" options that have been proposed and analysed over the last three decades. Unlike many others of these second generation derivatives, barrier options are heavily traded instruments.

The payoff of a barrier option depends on the path of the underlying asset through its option life. The option is activated or extinguished whenever the underlying asset price hits a barrier for the first time. For example, only if the underlying asset price reaches the barrier before maturity a down-and-in call option gives the option holder the payoff of a European call option at maturity. Other types of barrier options are the down-and-out, up-and-in and up-and-out call/put option, whose payoffs are self-explanatory. The extension of single barrier options to double barrier options has also become popular in OTC markets. Alternative barrier option contracts have been proposed by Davydov et al. [2]. They designed the so-called barrier step options, which regularize the knockout of the option by introducing a finite knockout of the option, in which the payoff of the option depends on the time that the underlying asset price is beyond a pre-specified barrier level.

Another class of barrier options involves the nature of the barrier itself. First, we consider the form of the barrier. Partial barrier options belong to this class of barrier options. They are characterized by a changing barrier level from one time period of the option’s life to another. As an example, we mention the front-end and rear-end barrier options, whose barriers only exist at the beginning or at the end of the options life, respectively. Second, we address the monitoring frequency of the barrier, i.e. how frequently the barrier condition is checked. Most models assume continuous monitoring of the barrier. However, in
practice, most of the barrier options are monitored at discrete instants, e.g. daily or weekly. Numerical investigations have shown that discrete and continuous barrier options can have significantly different prices even for daily monitoring (see [3]). The pricing of continuous and discrete barrier options is the subject of this work.

Why should an investor use a barrier option rather than a plain vanilla option? One reason is that barrier options are generally cheaper than standard options, since the asset price has to satisfy an additional condition for the option holder to receive the payoff. The premium for a barrier option can be significantly smaller for high volatile assets. Thus, barrier options prevent investors from paying for scenarios that they feel are unlikely to occur. Barrier options may also match hedging needs more closely than standard options. A drawback in using barrier options is the possibility of triggered barrier events around popular levels as a consequence of a manipulation of the underlying asset price, e.g. by inducing an increased volatility of the asset.

Various approaches for the valuation of continuous barrier options have been developed and published. Merton [4] has first derived a formula for a down-and-out call option. Double barrier options and their extension to curved boundaries are discussed by Kunitomo et al. [5] using a probabilistic approach. Geman et al. [6] and Pellsr [7] used the Laplace transform to solve the Black-Scholes equation for various types of barrier options. Hui et al. [8, 9] has exploited the Fourier formalism in order to obtain an analytical solution for a double barrier option. He has also obtained analytical solutions for the front-end and rear-end barrier options.

We have a completely different situation for discretely monitored barrier options. In general, the price of a discrete barrier option can be expressed in terms of an $m$-dimensional integral over the $m$-dimensional multivariate normal distribution function ($m - 1$ is the number of monitoring points). Unfortunately, this expression is not analytically tractable for options with a high number of monitoring points ($m > 5$). To deal with these difficulties various numerical procedures and asymptotic expansions have been developed [1, 10, 11, 12].

The outline of this work is as follows. In chapter 2, we discuss continuous barrier options and present a method for solving the Black-Scholes equation of a down-and-out call option. The results are compared with a Monte Carlo simulation. In chapter 3, we present a pricing formula for discrete barrier options. The analytical solution is compared with the outcome
of a Monte Carlo simulation for an option with a single barrier at one instant. Furthermore, we consider the continuity correction for discrete barrier options as introduced by Broadie et al. [1]. Again, the analytical solution for the asymptotic behaviour is compared with the results of a Monte Carlo simulation. In chapter 4, we perform an asymptotic expansion of the discrete down-and-out option in the limit of a high number of monitoring events using the method of matched asymptotic expansions. We show that the continuity correction for single barrier options can be derived within this framework. Finally, we apply this method to discrete double barrier options to derive an asymptotic result for large $m$ in chapter 5.
Chapter 2

Barrier options with continuous monitoring

The purpose of this chapter is to review the pricing of a continuously monitored barrier option. First, we derive the price of a simple barrier option and compare the result with a Monte Carlo simulation. Furthermore, we discuss different approaches for the pricing of more complicated barrier options.

2.1 Model

Throughout this work we consider the Black-Scholes framework, in which the price of an option is given by an expectation value of the discounted final payoff of the option with respect to the risk-neutral probability measure. In the Black-Scholes world, the underlying asset price $S_t$ follows a geometric Brownian motion

$$\frac{dS}{S} = r\,dt + \sigma\,dW_t,$$

(2.1)

where $r$ is the risk-free rate, $\sigma$ is the volatility and $W_t$ is a standard Wiener process. The solution of Eq.(2.1) can be written as

$$S_t = S_0 \exp \left\{ (r - \sigma^2/2)t + \sigma W_t \right\}.$$

(2.2)

In the following, we concentrate on single barrier options. The barrier may be constant
over time, i.e. a time-independent barrier, or may vary with time, i.e. a time-dependent
barrier. We mention the continuously monitored barrier option as an example for a time-
independent barrier and the discretely monitored barrier option and the front-end barrier
option as examples for a time-dependent barrier. Let $B_t$ denote the level of the barrier at
time $t$. The first time when the asset price reaches the barrier is called the first exit time or
stopping time of the process $S_t$. The first exit time is defined by

$$
\tau(B_t, S_t) = \inf \{ t \geq 0; S_t \leq B_t \} .
$$

The value of a barrier option can now be expressed as a function of the first exit time. For
example, the price of a continuously monitored down-and-out call option in the risk-neutral
measure $Q$ is given by the expectation value of the discounted payoff of the option

$$
V(S, B_t, t) = E_Q \left( e^{-\tau(T-t)} \max(S_T - K, 0) 1_{\tau(B_t, S_t) > T} \right),
$$

where $S = S_0$. We have also introduced the characteristic function $1_{\tau(B_t, S_t) > T}$ given by

$$
1_{\tau(B_t, S_t) > T} = \begin{cases} 
1 & \tau(B_t, S_t) > T \\
0 & \text{otherwise} 
\end{cases} .
$$

All other types of barrier options, e.g. up-and-in put option etc., can be represented similarly. In the next step, we change the probability measure using the Girsanov theorem.

The expectation of the payoff with respect to the measure $Q$ can be transformed to an ex-
pectation with respect to some measure $Q^*$ by the use of the Radon-Nykodym derivative
$dQ/dQ^*$ [13]. Thus, we have

$$
E_Q \left( e^{\sigma W_T - \sigma^2 T/2} 1_{\tau(B_t, S_t) > T} \right) = E_{Q^*} \left( \frac{dQ}{dQ^*} e^{\sigma W_T - \sigma^2 T/2} 1_{\tau(B_t, S_t) > T} \right) .
$$

Here, we have restricted our consideration to the first term in (2.4) and inserted Eq.(2.2) for
the underlying asset price $S_T$. To simplify the calculations, we change the measure in such
a way that it satisfies the following relation

$$
E_Q \left( e^{\sigma W_T - \sigma^2 T/2} 1_{\tau(B_t, S_t) > T} \right) = E_{Q^*} \left( 1_{\tau(B_t, S_t) > T} \right) .
$$

In order to satisfy this condition, the Radon-Nykodym derivative should equal
According to Girsanov’s theorem, a change of measure leads to a change of the drift in the underlying process. Suppose the underlying process, given in Eq. (2.1), under the measure $Q^*$ reads as

$$\frac{dS}{S} = r^* dt + \sigma dW^*_t. \quad (2.9)$$

Girsanov’s theorem relates the measure $Q$ to the measure $Q^*$. The theorem can be written as

$$\frac{dQ}{dQ^*} = e^{-(r^* - r)\sigma W^*_T + \frac{(r^* - r)^2}{2}\sigma^2 T}. \quad (2.10)$$

Now, we can apply Girsanov’s theorem to obtain the drift $r^*$ of the underlying process under the measure $Q^*$ that satisfies the condition (2.8). A simple analysis shows that the drifts $r$ and $r^*$ are related by

$$r^* = r + \sigma^2. \quad (2.11)$$

Thus, the underlying process $\ln S_t$ has the drift $(r + \sigma^2)$ under the measure $Q^*$ instead of $r$ under the measure $Q$. The price of the down-and-out call option in terms of the measure $Q^*$ can be written as

$$V(S, B, t) = S E_{Q^*} (1_{S_T \geq K, \tau(B, S) > T}) - E_{Q} (e^{-r(T-t)} K 1_{S_T \geq K, \tau(B, S) > T}). \quad (2.12)$$

In terms of the probability, Eq. (2.12) can be rewritten as

$$V(S, B, t) = S P_{Q^*} (S_T \geq K; \tau(B, S) > T) - e^{-r(T-t)} KP_{Q} (S_T \geq K; \tau(B, S) > T). \quad (2.13)$$

Eq. (2.13) represents the Black-Scholes price of a down-and-out call option as a function of the probability that the asset price at expiry is larger than the strike $K$ and that the first
exit time is larger than $T$ with respect to the probability measure $P_Q$, and the risk neutral measure $P_Q$, respectively. In the following, we evaluate these probabilities for a constant barrier $B_t = B$. First, we insert Eq.(2.2) for the underlying asset price $S_t$. We obtain

\[
V(S, B, t) = SP_Q(Se^{(r+\sigma^2/2)T+\sigma W_T} \geq K; \tau(B, S_t) > T) + e^{-r(T-t)}KP_Q(Se^{(r-\sigma^2/2)T+\sigma W_T} \geq K; \tau(B, S_t) > T).
\]

In order to compute these probabilities, we exploit a theorem by Bielecki and Rutkowski (see lemma 3.1.4 in [14]). Suppose that the stochastic process $Y_t$ is related to a standard Wiener process $W_t$ by $Y_t = Y_0 + \nu t + \sigma W_t$, where $\nu$ is the drift of the process. Furthermore, we assume that the first exit time $\tau(0, Y_t)$ is given by $\tau(0, Y_t) = \inf \{t \geq 0; Y_t \leq 0\}$. Then, the theorem reads as follows:

\[
P(Y_T \geq y; \tau \geq T) = N \left( \frac{Y_t - y + \nu(T-t)}{\sigma \sqrt{T-t}} \right) - e^{-2\nu Y_t/\sigma^2} N \left( \frac{-Y_t - y + \nu(T-t)}{\sigma \sqrt{T-t}} \right),
\]

where $N(x)$ is the cumulative probability distribution function for a normally distributed variable with a mean of zero and a standard deviation of 1. In the next step, we compute the probabilities in (2.14). Applying this theorem with $Y_t = \ln(S/B)$, $y = \ln(K/B)$ and $\nu = (r + \sigma^2/2)$ for the first contribution in (2.14) and $\nu = (r - \sigma^2/2)$ for the second contribution in (2.14) yields

\[
V(S, B, t) = SN \left( \ln(S/K) + (r + \sigma^2/2)(T-t) \right) 
- B \left( \frac{S}{B} \right)^{1-2r/\sigma^2} N \left( \ln(B^2/SK) + (r - \sigma^2/2)(T-t) \right) 
+ Ke^{-r(T-t)}N \left( \ln(S/K) + (r - \sigma^2/2)(T-t) \right) 
- Ke^{-r(T-t)} \left( \frac{S}{B} \right)^{1-2r/\sigma^2} N \left( \ln(B^2/SK) + (r - \sigma^2/2)(T-t) \right).
\]

This equation can be rewritten in terms of the value of the vanilla call option. The result reads as follows:
\[ V(S, B, t) = V_C(S, t) - (S/B)^{1-2r/\sigma^2} V_C(B^2/S, t). \] (2.17)

In the following section, we show that the same expression can be derived from the Black-Scholes equation using the method of Green’s function.

### 2.2 Time-independent barrier option

In the following, we restrict ourselves to the study of a down-and-out call option. Other barrier options can be priced in a similar way. The value of the down-and-out call option is obtained within a Black-Scholes pricing model. A formula for pricing a down-and-out call option was first obtained by Merton [4] in 1973. In the following, we derive his expression.

Our starting point is the Black-Scholes equation given in its general form by

\[ \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \] (2.18)

where the parameters have the usual meaning, i.e. \( \sigma \) is the volatility, \( r \) is the risk-free rate and \( S \) denotes the underlying asset price. The final payment at time \( T \) is

\[ V(S, B, T) = \max(S_T - K, 0), \] (2.19)

where \( K \) is the strike of the option. To simplify the calculations, we assume throughout this work that the strike is above the barrier, i.e. \( B < K \). Note that down-and-out barrier options with a barrier above the strike can be reduced to those with a strike above the barrier. One first observes that the payoff of the barrier option is equal to the payoff of a vanilla call option truncated at the barrier \( S = B \) with a value of zero for \( S < B \). This payoff is the same as that of the sum of a vanilla call option with strike \( B \) and \( (B - K) \) times the payoff of a standard digital call option with strike \( B \) that is paying 1$. After subtracting the standard digital call option, one is left with exactly the same problem as if the strike is above the barrier.

Now, let us assume that \( B < K \). The option becomes worthless if the underlying asset price reaches the barrier at any time \( t < T \). The value of the option at \( S = B \) is zero. Thus, the boundary condition of a down-and-out call option is
\[ \text{The Black-Scholes equation can be transformed to the heat equation by introducing the following transformations} \]

\[ S = Be^x; \quad \tau = (T - t)\sigma^2/2; \quad V(S, B, t) = Be^{\alpha_1 x + \alpha_2 \tau} u(x, \tau), \quad (2.21) \]

where

\[ \alpha_1 = (1 - 2r/\sigma^2)/2; \quad \alpha_2 = -(1 + 2r/\sigma^2)^2/4; \quad y = \log(K/B). \quad (2.22) \]

Then, Eq.(2.18) can be rewritten in the well-known form of the heat equation

\[ \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad (2.23) \]

with the initial and boundary condition

\[ u(x, 0) = e^{-\alpha_1 x} \max(e^x - e^y, 0); \quad u(0, \tau) = 0, \quad (2.24) \]

respectively.

The following calculation is based on the Green’s function approach. The Green’s function of the partial differential equation (2.23) is easily found by Fourier transformation. The fundamental solution of Eq.(2.23) is given by (see e.g. [15])

\[ u_0(x, \xi, \tau) = \frac{1}{(4\pi \tau)^{1/2}} e^{-\frac{(x-\xi)^2}{4\tau}}. \quad (2.25) \]

This solution does not satisfy the homogeneous boundary condition at \( x = 0 \) (2.24). To find a solution with the correct boundary condition, we first observe that \( u_0(-x, \xi, \tau) \) is also a solution of the heat equation. Every linear combination of \( u_0(x, \xi, \tau) \) and \( u_0(-x, \xi, \tau) \) satisfies the differential equation, and so does

\[ u_G(x, \xi, \tau) = u_0(x, \xi, \tau) - u_0(-x, \xi, \tau). \quad (2.26) \]

By inspection of Eq.(2.26), we find that \( u_G(x, \xi, \tau) \) satisfies the correct boundary condition. Given the Green’s function (2.26) of the boundary value problem, the Black-Scholes value of a down-and-out call option is obtained from the following integral expression

\[ V(S = B, B, t) = 0. \quad (2.20) \]
\( V(S, B, t) = Be^{\alpha_1 x + \alpha_2 \tau} u_c(x, \tau) = Be^{\alpha_1 x + \alpha_2 \tau} (u^+_c(x, \tau) + u^-_c(x, \tau)) \). \quad (2.27)

with

\[ u^+_c(x, \tau) = \int_{-\infty}^{\infty} d\xi u(\xi, 0) u_0(x, \xi, \tau) \] \quad (2.28)

and

\[ u^-_c(x, \tau) = -\int_{-\infty}^{\infty} d\xi u(\xi, 0) u_0(-x, \xi, \tau) . \] \quad (2.29)

The remaining integral can be expressed in terms of the value of a vanilla call option \( V_C(S, t) \) according to

\[ V(S, B, t) = V_C(S, t) - (S/B)^{1-2r/\sigma^2} V_C(B^2/S, t) . \] \quad (2.30)

The first term in Eq.(2.30) is the price of a vanilla call option. The second term represents the value of the additional condition that the asset price has to satisfy for the option holder to receive the payoff. This contribution is always negative, since the value of a vanilla call option is always positive. Thus, we are left with a lowering of the price of a down-and-out call option if compared with the otherwise identical vanilla call option.

We have performed a Monte Carlo simulation for a down-and-out call option. The underlying asset price \( S \) is simulated in time according to Eq.(2.2). Random numbers are generated using standard routines from the GNU Scientific Library (GSL) available at [16]. The number of paths used in the simulation is \( 10^8 \). The results of the Monte Carlo simulation are presented in Fig. 2.1. The result is shown to be in agreement with the analytical solution in Eq.(2.30). The price of the continuous barrier option converges to the price of an otherwise identical vanilla call option as the spot price moves away from the barrier.
Figure 2.1: Comparison of the Monte Carlo results with the analytical solution for a down-and-out call option with a single, continuously monitored barrier (strike=5, barrier=4, expiry=T=2y, annual volatility=20%, annual rate=5%). We have also included the payoff of the option and the corresponding plain vanilla call option price.

2.3 Time-dependent barrier option

So far, we have considered time-independent barrier options. Next, we discuss time-dependent barrier options. As an example, we mention the front-end single barrier option. This option is characterized by a barrier that exists only from the start of the option to some pre-specified time $t_1 < T$. The option behaves as standard down-and-out option within the first period and becomes an ordinary call option after $t_1$. As usually, the Black-Scholes equation is solved backwards in time. First, we have to find the value of the option at $t_1$ by solving the Black-Scholes equation for the period from $t_1$ to expiry $T$ assuming that the barrier does not exist at $t_1$. Second, the value of the option at $t_1$ is used to construct the initial condition for the barrier period of the option. Third, we can apply the pricing method of a down-and-out option (2.27), explicitly given in the first part of this chapter, by
appropriately replacing the initial condition \(u(x, 0)\) in Eq.(2.24).

To make it more precise, the value of the option at \(t_1\) is equal to the value of an ordinary call option if the asset price at \(t_1\) is above the barrier and zero otherwise

\[
V(S, t_1) = \begin{cases} 
V_C(S, t_1) & S > B \\
0 & S \leq B
\end{cases}.
\]  

(2.31)

The value of the option at \(t_1\) serves now as the initial condition for the barrier option problem. Next, we apply Eq.(2.27) to find the price of the front-end down-and-out option

\[
V(S, t) = e^{\alpha_1 x} \int_{-\infty}^{\infty} d\xi e^{-\alpha_1 \xi} V(Be^{\xi}, t_1) u_G(x, \xi, \tau),
\]  

(2.32)

where \(\tau\) is now given by \(\tau = (t_1 - t)\sigma^2/2\), the initial condition \(V(Be^{\xi}, t_1)\) by Eq.(2.31) and the Green’s function \(u_G(x, \xi, \tau)\) by Eq.(2.26). All other parameters have their usual meaning. The integral expression (2.32) can be rewritten as an integral over the 2-dimensional multivariate normal distribution function [9], but it must be carried out numerically. Note that this option can be reduced to a combination of a compound option, i.e. a call option on a call option, and a standard digital option (see also the discussion in the first part of section 2.2). The value of the front-end down-and-out option is given by the sum of a compound option with a strike given by the value of the underlying vanilla call option at \(t_1\), a standard digital option with the same strike and their reflected solutions.

Applying the same steps as described above, one can also derive an expression for rear-end barrier options or other more complicated time-dependent barrier options [9].

In general, this method may also be applied for the discrete barrier problem. Suppose the barrier is monitored at time \(t_i = i\Delta t, i = 0, 1, \ldots, m\) with \(\Delta t = T/m\). Then, one first solves the Black-Scholes equation for the period from \(t_m = T\) to \(t_{m-1}\). The result is used as the initial condition for the next period. The price of a discrete barrier option is obtained after \(m\) steps. The pricing of discrete barrier options will be discussed in more detail in chapter 3.
2.4 Down-and-Out call option revisited

In this section, we derive two alternative expressions for the Black-Scholes price of a down-and-out call option. Basically, the one-dimensional integral (2.27) for the price of a down-and-out call option is converted into an \( m \)-dimensional integral. As indicated at the end of the previous section, one may solve the continuous barrier option by an iterative procedure. One starts with solution at \( \tau_1 = (t_m - t_{m-1})\sigma^2/2 \), given by

\[
u_G(x, \xi, \tau_1) = u_0(x, \xi, \tau_1) - u_0(-x, \xi, \tau_1).
\](2.33)

Using Eq.(2.31) as the initial condition for the next period, the solution at \( \tau_2 \) is given by

\[
u_G(x, \xi, \tau_2) = \int_0^\infty d\xi_1 \left( u_0(\xi_1, \xi, \tau_1) - u_0(-\xi_1, \xi, \tau_1) \right) \times \left( u_0(x, \xi_1, \tau_2 - \tau_1) - u_0(-x, \xi_1, \tau_2 - \tau_1) \right).
\](2.34)

After \( m \) steps, i.e. at \( \tau_m = \tau \), we have

\[
u_G(x, \xi, \tau_m) = \int_0^\infty d\xi_1 \int_0^\infty d\xi_2 \ldots \int_0^\infty d\xi_{m-1} \times \prod_{i=0}^{m-2} \left( u_0(\xi_{i+1}, \xi_i, \tau_{i+1} - \tau_i) - u_0(-\xi_{i+1}, \xi_i, \tau_{i+1} - \tau_i) \right).
\](2.35)

Here, we have introduced the notation \( \xi_m = x, \xi_0 = \xi \) and \( \tau_m = \tau \). Alternatively, one may exploit the semi-group property of the Green’s function of the heat equation, which reads as

\[
u_0(x, \xi, \tau) = \int_{-\infty}^{\infty} d\xi_1 \nu_0(x, \xi_1, \tau - \tau_1) \nu_0(\xi_1, \xi_1, \tau_1).
\](2.36)

Applying this property \( m \)-times to Eq.(2.31) yields the following integral expression for the Green’s function:

\[
u_G(x, \xi, \tau) = \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \ldots \int_{-\infty}^{\infty} d\xi_{m-1} \prod_{i=0}^{m-2} u_0(\xi_{i+1}, \xi_i, \tau_{i+1} - \tau_i) \times \left( u_0(\xi_m, \xi_{m-1}, \tau_m - \tau_{m-1}) - u_0(-\xi_m, \xi_{m-1}, \tau_m - \tau_{m-1}) \right).
\](2.37)

Clearly, Eq.(2.37) can also be obtained from Eq.(2.35) by a simple change of the region of integration. The alternative integral expressions (2.35,2.37) have a similar structure as
the general solution for discretely monitored down-and-out call option (see section 3.2). Furthermore, one expects that the discrete barrier option converges to its continuous counterpart in the limit as number of monitoring points goes to infinity. It is therefore convenient to use these expressions as a starting point to study the asymptotic behaviour of a discrete barrier option. However, we will not follow these lines. Instead we use the method of matched asymptotic expansion to study the asymptotic behaviour in chapter 4.

2.5 Double Barriers

Next, we briefly focus our attention on double barrier options. These contracts consist of an upper barrier $B_+$ and a lower barrier $B_-$. In general, one distinguishes up-and-down out or knockout and up-and-down in or knockin barrier options. The upper barrier leads to an additional boundary condition. Thus, the boundary conditions read as

$$V(S = B_-, t) = 0; \quad V(S = B_+, t) = 0.$$ (2.38)

Again, we have arrived at a well-known boundary value problem. It can be solved by the Fourier series method. The price of the continuous double barrier knockout option may be obtained from

$$V(S, B_-, B_+, t) = B_- e^{\alpha_1 x + \alpha_2 \tau} u_c(x, \tau)$$ (2.39)

with $u_c(x, \tau)$ given by the following Fourier series:

$$u_c(x, \tau) = \sum_{n=1}^{\infty} C_n \sin \left( \frac{n\pi a}{x} \right) e^{-\frac{x^2}{a^2} \tau}.$$ (2.40)

Here, we have introduced $a = \ln(B_+/B_-)$ and $x = \ln(S/B_-)$. The Fourier coefficients $C_n$ are given by

$$C_n = \frac{2}{a} \int_0^a d\xi u(\xi, 0) \sin \left( \frac{n\pi a}{x} \xi \right).$$ (2.41)

The payoff $u(\xi, 0)$ (2.24) is of course defined only for $B_- < S < B_+$ and vanishes outside this interval. Details of the application of the Fourier method for solving the heat equation
can also be found in e.g. [8] and [15]. The integral expression for the Fourier coefficients $C_n$ (2.41) can be explicitly evaluated for the payoff of a call option. A simple analysis shows that

$$C_n = 2 \left( -1 \right)^{n+1} n \pi e^{(1-\alpha)x} + e^{(1-\alpha)y} \left( n \pi \cos \left( \frac{n \pi}{a} y \right) - (1 - \alpha) a \sin \left( \frac{n \pi}{a} y \right) \right) \frac{n^2 \pi^2 + (1 - \alpha)^2 a^2}{n^2 \pi^2 + \alpha^2 a^2}$$

$$- 2 \left( -1 \right)^{n+1} n \pi e^{y - \alpha a} + e^{(1-\alpha)y} \left( n \pi \cos \left( \frac{n \pi}{a} y \right) + \alpha a \sin \left( \frac{n \pi}{a} y \right) \right) \frac{n^2 \pi^2 + (1 - \alpha)^2 a^2}{n^2 \pi^2 + \alpha^2 a^2}, \quad (2.42)$$

where $y$ is defined by $y = \ln(K/B)$. Alternatively, one may solve the boundary value problem by the Laplace transform method. Then, according to Davydov et al. [2] or Feller [17], the price of a double barrier option is obtained from (2.27) with $u_c(x, \tau)$ given by

$$u_c(x, \tau) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi \ u(\xi, 0) \ (u_0(x + 2ka, \xi, \tau) - u_0(-x + 2ka, \xi, \tau)) \quad (2.43)$$

Note that (2.40) and (2.43) are different representations of the exact solution. Double barrier options with discrete monitoring will be studied in more detail in chapter 5.
Chapter 3

Pricing of discrete barrier options

As already mentioned in the introduction, most of real financial contracts involving barriers are based on discrete monitoring events. In the following, we study the pricing of barrier options with discrete monitoring. First, we discuss the model and the analytical solution of a down-and-out call option. The analytical solution is compared with the results of a Monte Carlo simulation. In the following section, we derive an alternative integral expression for the price of a down-and-out call option. Finally, we present the results, as obtained by Broadie et al. [1, 18] and Kou [19], for the asymptotic behaviour of discrete barrier options in the limit as the number of monitoring points goes to infinity.

3.1 Model and solution

We start our investigation by setting up the model and obtaining a general solution for a discrete barrier option. For simplicity, we concentrate on the down-and-out call option. Unlike in the continuous case, the asset price is only monitored at instants $t_i = i\Delta t = iT/m$, where $T$ is the expiration date and $m - 1$ is the number of monitoring points ($i = 1, 2, ..., m - 1$). The option is knocked out, if at one of these instants the asset price is below the barrier. The barrier levels at these instants are denoted by $B_i$. To simplify the notation, one defines the $m$-th barrier level as the strike of the option, i.e. we have $B_m = K$. The underlying asset price may follow a lognormal random walk. The asset price at the $n$-th monitoring point under the risk-neutral measure is given by
\[ S_n = S_0 \exp \left\{ (r - \sigma^2/2)t_n + \sigma \sqrt{t_n} \sum_{i=1}^{n} W_i \right\}, \quad (3.1) \]

where \( W_i \)'s are independent standard normal random variables. The stopping time \( \tau_d(B_n, S_n) \) is the first time at which \( S_n \) hits the barrier level \( B \)

\[ \tau_d(B_n, S_n) = \inf \{ n \geq 1; S_n \leq B_n \}. \quad (3.2) \]

Then, the value of this option within the Black-Scholes framework is given by the discounted expectation of the payoff under the risk-neutral measure \( Q \)

\[ V_m(S, B_n, t) = E_Q \left[ e^{-r(T-t)} \max(S_m - B_m, 0) 1_{\tau(B_n, S_m) > m} \right], \quad (3.3) \]

where \( S_m \) is the asset price at expiry and \( 1_{\tau(B_n, S_m) > m} \) is the indicator function of the crossing of the barrier level. Following Heynen et al. [20] and Tse et al. [11], the value of a discrete down-and-out call option can be expressed as

\[ V_m(S, t) = SN_m(d_1(S, B_i, t_i), \rho) - B_m e^{-r(t_m-t)} N_m(d_2(S, B_i, t_i), \rho), \quad (3.4) \]

with \( N_m(x, \rho) \) being the \( m \)-dimensional multivariate normal distribution function defined by

\[ N_m(x, \rho) = \frac{1}{((2\pi)^m | \rho |)^{1/2}} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_m} dx e^{-\frac{1}{2} x^T \rho^{-1} x} \quad (3.5) \]

and

\[ d_1(S, B_i, t_i) = \frac{\ln(S/B_i) + (r - \sigma^2/2)(t_i - t)}{\sigma \sqrt{t_i - t}}; \quad d_2 = d_1 + \sigma (t_i - t)^{1/2}. \quad (3.6) \]

The correlation function \( \rho \) is given by \( \rho_{ij} = \min((t_i - t), (t_j - t))/\sqrt{(t_i - t)(t_j - t)} \).

An expression for the inverse matrix \( \rho^{-1} \) can be found in Ref.[11]. As can be seen from Eq.(3.4) and Eq.(3.5), the pricing of a discrete barrier option involves the calculation of an \( m \)-dimensional integral. So far, no simplifications of these integral expressions have been
As mentioned before, the general solution (3.4) for discretely monitored down-and-out options involves the calculation of an $m$-dimensional integral over $m$-dimensional multi-
variate normal distribution function and, thus, becomes numerically intractable for a large number of monitoring points. Therefore, one is interested in alternative numerical pricing procedures. In the following, I mention various numerical approaches for the pricing of discrete barrier options. Gao et al. [10] have investigated the Adaptive Mesh Model, which is a trinomial model with adapted time and price steps, i.e. small time and price steps are used for sections, where high resolution is required and vice versa. A high resolution is required near the monitoring points and for asset prices close to the barrier. This method permits an accurate pricing near the barrier and improves the efficiency of the numerical calculation.

Wei [12] proposed a heuristic interpolation formula that is based on the general solution, e.g. Eq.(3.4), with the highest number of monitoring points that is analytically tractable on the one hand and the analytical solution for the continuous case on the other hand. Tse’s et al. [11] approach is based on the evaluation of the integrals in the exact solution (3.4). They developed an efficient algorithm for the calculation of the multivariate normal distribution function for any given accuracy, which is based on the tridiagonal structure of the correlation matrix. Additionally, they incorporated an error bound in the approximation scheme, which determines the number of evaluation points. Unlike other numerical methods, this technique works well when the asset price is near the barrier.

### 3.2 Discrete down-and-out call option revisited

Again, we derive an alternative integral expression for the price of a discrete down-and-out call option based on the Black-Scholes equation. Using the standard transformation

$$V_m(B, S, t) = Be^{\alpha_1 x + \alpha_2 \tau} u_m(x, t), \quad (3.7)$$

the Black-Scholes equation takes the well-known form of the heat equation

$$\frac{\partial u_m}{\partial \tau} = \frac{\partial^2 u_m}{\partial x^2} \quad (3.8)$$

with the initial and boundary condition

$$u_m(x, 0) = e^{-\alpha_1 x} \max(e^x - e^y, 0); \quad u_m(0, \tau) = 0 \quad (3.9)$$
respectively. As discussed at the end of section 2.4, one may solve the Black-Scholes equation iteratively. The solution of the Black-Scholes equation for the period from $\tau_0$ to $\tau_1$ provides the initial condition for the period from $\tau_1$ to $\tau_2$ etc. After $m$ steps, we arrive at the following expression for the Green’s function:

$$
G(x, \xi, \tau) = \int_0^\infty d\xi_1 \int_0^\infty d\xi_2 \ldots \int_0^\infty d\xi_{m-1} \prod_{i=0}^{m-1} u_0(\xi_{i+1}, \xi_i, \tau_{i+1} - \tau_i)
$$

(3.10)

Inserting Eq.(3.10) into Eq.(2.27) yields the price of a discrete down-and-out call option.

### 3.3 Continuity correction for discrete barrier options

In this section, we present an asymptotic approximation for the pricing of discrete barrier options as derived by Broadie et al. [1, 18] and Kou [19]. They introduced a correction procedure for discrete options, which is based on the observation that in the limit of an infinite number of monitoring points the option price of the discrete barrier option converges to the price of the continuous barrier option. Then, they introduced a continuity correction for discretely monitored barrier options. The resulting approximation is a simple correction to the continuous formula. It is based on some general theorems from sequential analysis [21, 22, 23]. In the following, we present the theorem and sketch its derivation without going into the details of the proof of the theorem.

According to Broadie et al. [1, 18] and Kou [19], the price of a discrete barrier option $V_m(S, B, t)$ can be expressed in terms of its continuous counterpart $V(S, B, t)$ by

$$
V_m(S, B, t) = V\left(S, Be^{\pm \beta \sigma \sqrt{T/m}}, t\right) + o(1/\sqrt{m}),
$$

(3.11)

with $+$ for an up option and $-$ for a down option and the constant

$$
\beta = -\frac{\zeta(1/2)}{\sqrt{2\pi}} \approx 0.5826,
$$

(3.12)

with $\zeta(x)$ the Riemann zeta function. They interpret this result as a shift of the barrier away from the spot price $S$ by a factor $e^{\beta \sigma \sqrt{T/m}}$. We will use this argument in chapter 4 to derive the asymptotic solution (3.11) by using the method of matched asymptotic expansions.

Again, we have performed a Monte Carlo simulation to illustrate the quality of the continuity correction (3.11) as derived by Broadie et al. The results are shown in Fig. 3.2. There,
we have made a comparison of the asymptotic solution with the exact solution as a function of the number of monitoring points. The exact solution is the result of a Monte Carlo simulation using $10^8$ paths. The asymptotic solution converges to the exact solution as the number of monitoring points increase. Remarkably, the continuity correction is a very good approximation to the exact solution even for numbers of monitoring points for which the price of the discrete barrier and the price of the continuous barrier differ significantly, i.e. for a number of monitoring points for which the correction contribution to the continuous barrier option is not a small contribution compared with the continuous barrier option. This is also the reason why most practitioners rely on the continuity correction for a wide range of monitoring points instead of using time consuming Monte Carlo simulations. In Fig. 3.3, we have plotted the relative difference of the exact solution and the asymptotic solution for the numerical results shown in Fig 3.2. For the parameter chosen in this example, the difference between the continuity correction and the exact solution is less than 0.1% for 90 monitoring points.

Broadie et al. [1, 18] and Kou [19] have presented various proofs of equation (3.11). The shortest and most transparent proof has been given by Kou [19]. In the following, we outline the main steps of this proof. Kou’s approach is based on a theorem by Siegmund and Yuh [23] that was later extended by Kou [19]. The theorem connects the first exit time

$$\tau_d(B,U_n) = \inf\{n \geq 1 : U_n \leq B\sqrt{m}\}$$  \hspace{1cm} (3.13)

of a discrete random walk

$$U_n = \sum_{i=1}^n (W_i + \mu/\sqrt{m}),$$  \hspace{1cm} (3.14)

where $W_i$’s are independent standard normal random variables, and the first exit time

$$\tau(B,U_t) = \inf\{t \geq 0 : U_t \leq B\}$$  \hspace{1cm} (3.15)

of a continuous Brownian motion

$$U_t = W_t + \mu t,$$  \hspace{1cm} (3.16)

where $\mu$ is the drift of the process. Here, $U_n$ describes a random walk with a small drift $\mu/\sqrt{m}$ as $m \rightarrow \infty$. The theorem states that the probability of a discrete random walk $U_n$
Figure 3.2: Comparison of the Monte Carlo results with the asymptotic solution for a down-and-out call option (3.11) as a function of the number of monitoring points (spot=strike=100, barrier=90, expiry=T=4y, annual volatility=40%, annual rate=10%). We have also included the exact solution for the continuous down-and-out call option.

that $U_m > y\sqrt{m}$ and $\tau_d(B, U_n) > m$ is equal to the probability of its continuous counterpart $U_t$ that $U_1 > y$ and $\tau(B - \beta/\sqrt{m}, U_t) > 1$ up to the order $o(1/\sqrt{m})$. Mathematically, the theorem reads as follows:

$$P(U_m \geq y\sqrt{m}, \tau_d(B, U_n) > m) = P(U_1 \geq y, \tau(B - \beta/\sqrt{m}, U_t) > 1) + o(1/\sqrt{m})$$

(3.17)

in the limit as $m \to \infty$. In the next step, the Black-Scholes price of a discrete and of a continuous barrier option is expressed in terms of the probabilities as given in Eq.(3.17). For the continuous barrier option, the result is given by Eq.(2.13). Starting from Eq.(3.3), one can also derive an expression of the discrete barrier option in terms of the the probability $P(U_m \geq y\sqrt{m}, \tau_d(B, U_n) > m)$. A comparison of both equations yields the asymptotic
Figure 3.3: Plot of the relative difference of the Monte Carlo results and the asymptotic solution (3.11) for a down-and-out call option as a function of the number of monitoring points (spot=strike=100, barrier=90, expiry=T=4y, annual volatility=40%, annual rate=10%).

result for the discrete barrier option (3.11). Details of the analysis can be found in [19].

Let me make some comments about this theorem. From a mathematical point of view, one is interested in the exact nature of the asymptotic expansion for a large number of monitoring points. By inspection of the results of Broadie and Glasserman (3.11), one expects the asymptotic expansion to be

\[ V_m(S, B, t) = V(S, B, t) + \frac{V_1(S, B, t)}{m^{1/2}} + \frac{V_2(S, B, t)}{m} + o(1/m). \]  

(3.18)

Strictly speaking, Eq.(3.11) is not an asymptotic expansion for large values of \( m \), since they have also included terms of order \( o(1/\sqrt{m}) \) in the leading term. In other words, only the lowest order terms of the asymptotic expansion can be derived from Eq.(3.11). Therefore, Eq.(3.11) is an interpolation formula that exactly reproduces the asymptotic behaviour to
the lowest order in the expansion parameter $1/\sqrt{m}$. 
Chapter 4

Asymptotic expansion for single barrier options

In this chapter, we perform an asymptotic expansion of the discrete down-and-out call option in the limit of a high number of monitoring events. First, we study the asymptotic result for the discrete barrier option as obtained by Broadie et al. (3.11). We calculate its value at the boundary $S = B$, which gives an effective boundary condition. By reversing of the argument, we show that the knowledge of the effective boundary condition at the original boundary $S = B$ yields the asymptotic result for the discrete barrier option. Second, we introduce the method of matched asymptotic expansions. In particular, we present Van Dyke’s matching principle. Finally, we apply the method of matched asymptotic expansions to calculate the effective boundary condition at $S = B$. The result can then be used to calculate the value of the discrete barrier option in the limit of a high number of monitoring points.

4.1 General remarks

In this section, we analyse the asymptotic result for the discrete down-and-out option as derived by Broadie et al. In particular, we pay attention to the argument by Broadie et al. that the price of the continuous barrier option may serve as an approximation to the price of a discrete barrier option, if one shifts the barrier away from the spot price $S$ by a factor $e^{\beta\sigma\sqrt{T/m}}$. We will show that this leads to an effective non-vanishing boundary condition.
at the boundary of the continuous barrier option. Solving the Black-Scholes equation for 
\( S > B \) with the effective boundary condition gives the approximation to the price of the 
discrete barrier option. In other words, the knowledge of the effective boundary condition 
at \( S = B \) provides a simple way of the calculation of price of a discrete barrier option. 
Clearly, the effective boundary condition at \( S = B \) will only be determined in the limit 
of a high number of monitoring points. The effective boundary condition at \( S = B \) in the 
asymptotic limit can be determined by the method of matched asymptotic expansions using 
Van Dyke’s matching principle. This will be discussed in the following sections.

For the following investigations, it is convenient to introduce the small parameter

\[ \epsilon = \sigma \sqrt{T/m}. \]  

The starting point of our investigation is the asymptotic result for the discrete barrier op-
tion \( V_m(S, B, t) \) given in Eq.(3.11). By inspection of this equation, one observes that 
the value of the discrete down-and-out option is approximately the value of the continu-
ous down-and-out option with the barrier shifted to \( Be^{-\beta \epsilon \sqrt{T/m}} \). In terms of the variable 
\( x = \ln(S/B) \) as defined in Eq.(2.21) the boundary is shifted from 0 to \( -\beta \epsilon \). Expanding 
Eq.(3.11) to first order in the small parameter \( \epsilon \) yields

\[ V_m(S, B, t) = V(S, B, t) + \epsilon \frac{\partial V(S, Be^{-\beta \epsilon}, t)}{\partial \epsilon} \bigg|_{\epsilon=0} + \ldots, \]  

where \( V(S, B, t) \) is the solution of the continuous down-and-out call option. Next, we 
compute the value of the discrete option at the boundary \( S = B \) and in the asymptotic limit 
\( m \to \infty \). The result is readily found to be

\[ V_m(B, B, t) = \beta \epsilon B \frac{\partial V(S, B, t)}{\partial S} \bigg|_{S=B} + o(\epsilon). \]  

Alternatively, one may express the boundary condition in terms of the transformed vari-
able \( x \) and function \( u_c(x, \tau) \) that satisfies the heat equation with a homogeneous boundary 
condition at \( x = 0 \) (2.23-2.24). In terms of \( u_c(x, \tau) \), the boundary condition (4.3) can be 
rewritten as

\[ V_m(B, B, t) = \beta \epsilon Be^{\alpha \tau} \gamma(\tau) + o(\epsilon) \]  

(4.4)
with $\gamma(\tau)$ given by

$$
\gamma(\tau) = \frac{\partial u_c(x, \tau)}{\partial x} \Big|_{x=0}.
\tag{4.5}
$$

Thus, we have derived the effective boundary condition at $S = B$ for $V_m(S, B, t)$ or equivalently for $u_m(x, \tau) (3.7)$ at $x = 0$. In other words, the value of the discrete barrier option in the asymptotic limit may be obtained from solving the original initial boundary value problem for the continuous barrier (2.23-2.24) with the effective boundary condition given by $u(0, \tau) = \beta \epsilon \gamma(\tau)$ instead of $u(0, \tau) = 0$.

One may look at this problem from a different perspective by reversing the argument. Assume that we know the boundary condition at $x = 0$ for the discrete barrier option and it is given by

$$
u_{m}(0, \tau) = \beta \epsilon \gamma(\tau).
\tag{4.6}
$$

Then, we solve the corresponding heat equation

$$
\frac{\partial u_{m}}{\partial \tau} = \frac{\partial^{2} u_{m}}{\partial x^{2}}
\tag{4.7}
$$
in the quarter plane $x > 0$ and $\tau > 0$ with the initial and boundary conditions

$$
u_{m}(x, 0) = e^{-\alpha x} \max(e^x - e^y, 0);
u_{m}(0, \tau) = \beta \epsilon \gamma(\tau),
\tag{4.8}
$$
respectively. The general solution to this problem can be written as

$$
u_{m}(x, \tau) = \nu_{c}(x, \tau) + u_{1}(x, \tau),
\tag{4.9}
$$
where $\nu_{c}(x, \tau)$ is the solution of the continuous barrier as defined in Eq.(2.27), i.e. it satisfies the homogeneous boundary condition. Therefore, $u_{1}(x, t)$ is the solution of the heat equation that satisfies the boundary condition (4.8) and vanishes at $\tau = 0$. This problem is solved by Duhamel’s formula

$$
u_{1}(x, \tau) = \beta \epsilon \int_{0}^{\tau} ds \, \gamma(\tau - s) G(x, s)
\tag{4.10}
$$
with

$$
G(x, \tau) = \frac{x}{(4\pi \tau)^{1/2}} e^{-\frac{x^2}{4\tau}}.
\tag{4.11}
$$
The $s$-integration in (4.10) can be carried out exactly. The detailed evaluation of this integral is given in appendix A. Performing the integration, one arrives at the following expression for $u_1(x, t)$

$$u_1(x, t) = \beta \epsilon \frac{1}{(4\pi)^{1/2} \tau^{3/2}} \int_{-\infty}^{\infty} d\xi u(\xi, 0)(x + \xi) e^{-\frac{(x+\xi)^2}{4\tau}}. \quad (4.12)$$

Finally, we use the standard transformation from the Black-Scholes equation to the heat equation (2.21) to derive the asymptotic result for the discrete barrier option as given in (4.2). Thus, we have shown that solving the Black-Scholes equation subject to the effective boundary condition (4.4) at $S = B$ leads to the asymptotic result for the price of the discrete barrier option as obtained by Broadie et al. [1].

Again, one may look at this problem from a different perspective. We are looking for a solution to the boundary problem (4.7-4.8) of the form (4.9). We know that $u_c(x, \tau)$ can be written as the sum of the vanilla call option $u_c^+(x, \tau)$ and its image solution $u_c^-(x, \tau)$ according to Eqs.(2.27-2.29). Thus, we have

$$u_c(x, \tau) = u_c^+(x, \tau) + u_c^-(x, \tau). \quad (4.13)$$

Additionally, we know that $\partial u_c^-(x, \tau) / \partial x$ satisfies the heat equation (4.7) and that its value at $\tau = 0$ for $x > 0$ is zero, i.e. $u_c^-(x, 0) = 0$. To study the boundary condition, we observe that

$$\left. \frac{\partial u_c(x, \tau)}{\partial x} \right|_{x=0} = \left. \frac{\partial u_c^+(x, \tau)}{\partial x} \right|_{x=0} + \left. \frac{\partial u_c^-(x, \tau)}{\partial x} \right|_{x=0} = 2 \left. \frac{\partial u_c^-(x, \tau)}{\partial x} \right|_{x=0} \quad (4.14)$$

since by reflection $u_c^+(x, \tau) = -u_c^-(x, \tau)$. Thus, the following expression for $u_1(x, \tau)$ is a solution of the heat equation and satisfies the initial and boundary condition:

$$u_1(x, \tau) = 2\beta \epsilon \frac{\partial u_c^-(x, \tau)}{\partial x}. \quad (4.15)$$

Inserting Eq.(2.29) into (4.15) yields the integral expression (4.12). Then, we can apply the same argument as discussed above to show that this equation leads to the asymptotic result for the price of the discrete barrier option as obtained by Broadie et al. [1].

We are left with the determination of the effective boundary condition at the barrier. This will be the subject of the following sections. Using the method of matched asymptotic
expansions and Van Dyke’s matching principle, we work out the value of the discrete barrier option at the boundary up to the order $o(\epsilon)$. The result is used to calculate the asymptotic behaviour for the discrete barrier option for all values of $x$ as outlined in this section. In the following section, we briefly explain the method of matched asymptotic expansions and Van Dyke’s matching principle that are used to derive asymptotic expressions for solving partial differential equations.

4.2 Van Dyke’s matching principle

In this section, we briefly present the method of matched asymptotic expansions. The method of matched asymptotic expansions is often used in connection with problems for which the exact solution changes rapidly from one domain to another. In other words, the solution cannot be found as an expansion in terms of a single scale, but of two or more scales each valid in a separate part of the domain. Typical examples of these kind of problems are transition layer problems such as boundary layer, initial layer and internal layer problems. As we will show in the next section, the discrete barrier option is also an example of a layer problem. Here, the solution changes rapidly when the asset price hits the barrier. In the following, we only discuss the general idea of Van Dyke’s matching method. In doing so, we essentially follow the lines of Campell [24]. An introduction to the method of matched asymptotic expansions can also be found in [25, 26, 27, 28, 29].

Assume that a function $y^o(x, \epsilon)$ is the solution of a differential equation in one part of the domain, generally referred to as the outer solution. The solution in the other part of the domain, generally referred to as the inner solution, may be given by $y^i(\xi, \epsilon)$. Here $\xi$ is the inner variable that is obtained from the outer variable $x$ by rescaling according to $\xi = x/\alpha(\epsilon)$ with an appropriately chosen function $\alpha(\epsilon)$. We look for an approximation to the exact solution in the limit as the parameter $\epsilon$ goes to zero. In order to create an approximation to the exact solution that is valid over the entire domain, we need to match the inner to the outer solution. Van Dyke’s method provides a way to fit these solutions together. In the following, we present the three hypotheses on which Van Dyke’s matching principle are based.

First, we introduce the $n$-term inner expansion operator $I_n$ and the $m$-term outer expansion operator $O_m$. When acting on any function $y$, these operators are defined as follows:
\( I_n = \) express the function in terms of the inner variable \( \xi \) and then expand in powers of \( \epsilon \), truncating all but the first \( n \) terms.

\( O_m = \) express the function in terms of the outer variable \( x \) and then expand in powers of \( \epsilon \), truncating all but the first \( m \) terms.

Van Dyke’s hypotheses can now be expressed in terms of the inner and outer expansion operators. The first Van Dyke’s hypothesis concerns the exact solution \( y^e \) and reads as

\[
O_m I_n y^e = I_n O_m y^e .
\] (4.16)

This hypothesis basically states that the inner and outer expansion operators are commutative when applied to the exact solution. In other words, the outcome of the application of both operators on the exact solution is independent of the order of application. The next two hypotheses concern the inner and the outer expansion and are given by the following equations

\[
I_n y^e = y^i ,
\] (4.17)

\[
O_m y^e = y^o .
\] (4.18)

These hypotheses relate the inner expansion of the exact solution with the \( m \)-term inner solution and the outer expansion of the exact solution with the \( n \)-term outer solution. However, the exact solution is in general not available. If the exact solution is known, the search for an approximation to the exact solution is unnecessary. The use of Van Dyke’s hypotheses becomes clear, when one relates the inner expansion of the outer solution to the outer expansion of the inner solution. This can be achieved by substituting (4.17) and (4.18) in (4.16). We get

\[
I_n y^o = O_m y^i .
\] (4.19)

Thus, we have found a way for matching the outer solution with the inner solution. Eq.(4.19) states in words that

The \( n \)-term outer expansion of the \( m \)-term inner solution equals the \( m \)-term inner expansion of the \( n \)-term outer solution
The remaining task is to find a composite solution \( y^c \) that is valid in the entire domain. This can be accomplished by the equation

\[
y^c = y^i + y^o - O_m y^i = y^i + y^o - I_n y^o.
\]  

Unfortunately, Van Dyke’s matching rule does not always work. In particular, the method may fail in problems for which the respective validity domains of the outer and inner solution do not overlap. Then, one can introduce an intermediate expansion that connects the different domains. Details on this subject can be found in [25, 26, 27, 28, 29].

In the next section, we apply Van Dyke’s matching method to compute an approximation to the discrete down-and-out option for which the time between monitoring events becomes small.

### 4.3 Matched asymptotic expansions

In this section, we apply the method of matched asymptotic expansions to derive the effective boundary condition at the barrier. This condition is used to calculate the asymptotic result for the discrete down-and-out option as outlined in section 3.3.

First, let us discuss the behaviour of the solution above the barrier, i.e. for \( x > 0 \), in the limit as the number of monitoring points \( m \) goes to infinity. This domain will be referred to as the outer region. Obviously, the outer solution converges to the price of the continuous barrier option as \( m \to \infty \). Relative to the reset timescale, this is a slowly varying function. For large \( m \), one expects a correction term to the continuous barrier option that vanishes as the number of monitoring points go to infinity. In other words, the detailed nature of the boundary condition at \( x < 0 \) leads to a small correction contribution to the price of the continuous barrier option for large \( m \). Thus, we look for an outer solution of the initial boundary value problem (3.8,3.9) of the form

\[
u^o_2(x, \tau) = u_c(x, \tau) + \delta(\epsilon) u_1(x, \tau), \tag{4.21}
\]

where \( u_c(x, \tau) \) is the solution of the heat equation for the continuous barrier option. The function \( \delta(\epsilon) \) of the small parameter \( \epsilon = \sigma \sqrt{T/m} \) as defined in (4.1) and \( u_1(x, \tau) \) must be
determined.

The behaviour of the solution below the barrier, i.e. for \( x < 0 \), is different. This domain will be referred to as the inner region. The slow scale is not appropriate in the inner domain. As a consequence of the boundary condition, the solution is approximately time periodic in the inner domain. Therefore, it is convenient to introduce faster variables in the inner domain given by

\[
\tau = \tau_i + \epsilon^2 s; \quad x = \epsilon \xi ,
\]

(4.22)

where \( \tau_i \) is an arbitrary reset. The outer solution satisfies the boundary condition at \( x \to \infty \). However, in general, it does not satisfy the boundary condition at \( x < 0 \). Contrary, the inner solution satisfies the boundary condition for \( x < 0 \) and does not satisfy the boundary condition at \( x \to \infty \). In order to find a solution valid in the entire domain, we have to match the inner and the outer solution in a region near \( x = 0 \). This can be achieved by applying Van Dyke’s matching principle as discussed in the previous section.

First, we express the outer solution (4.21) in terms of the inner variables and expand to first order in the small parameter \( \epsilon \). We know that \( u_c(x, \tau) \sim \gamma(\tau)x + o(x) \) for \( x \to 0 \). Next, we study the one-term inner expansion of the one-term outer solution. We obtain

\[
I_1 u_1^o(\xi, \tau) = \epsilon \gamma(\tau) \xi .
\]

(4.23)

As will be shown below, this equation may be used to determine the function \( \delta(\epsilon) \). Now, we compute the one-term inner expansion of the two-term outer solution. We express the outer solution (4.21) in terms of the inner variables and expand to first order in \( \epsilon \). We find that

\[
I_1 u_2^o(\xi, \tau) = \epsilon \gamma(\tau) \xi + \delta(\epsilon) u_1(0, \tau) .
\]

(4.24)

Next, we solve a canonical inner problem. The solution to the inner problem is time-periodic to the lowest order of accuracy in the small parameter \( \epsilon \). We will also change the independent variables from \((x, \tau)\) to \((\xi, s)\) according to Eq.(4.22). The heat equation equation for the inner solution becomes
\[
\frac{\partial u^i_m}{\partial s} = \frac{\partial^2 u^i_m}{\partial \xi^2},
\]  
(4.25)

with the initial condition

\[
u^i_m(\xi, 0) = 0, \quad \xi < 0; \quad u^i_m(\xi, 0) = F(\xi), \quad \xi > 0, 
\]  
(4.26)

and the periodicity condition

\[
u^i_m(\xi, 0) = \nu^i_m(\xi, 1) = F(\xi),
\]  
(4.27)

respectively. A standard way for solving this type of partial differential equation is the method of Green's functions as discussed in the previous sections. The general solution is found by applying the iterative procedure described in sections 2.4 and 3.2. Imposing periodicity on the solution leads to the following integral equation:

\[
F(\xi) = \int_{0}^{\infty} dy F(y) u_0(\xi, y, 1),
\]  
(4.28)

where \(u_0(\xi, y, 1)\) is the heat kernel as defined in Eq.(2.25). This integral equation has been extensively studied by Spitzer [30, 31] in connection with problems in probability theory. Spitzer has shown that there is a unique non-decreasing positive solution to the integral equation. The general solution to the integral equation (4.28) is obtained by the Laplace transform. The Laplace transform of \(F(\xi)\) is given by

\[
\int_{0}^{\infty} e^{-\lambda \xi} F(\xi) d\xi = \exp \left\{ -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\lambda}{\lambda^2 + y^2} \ln \left[ 1 - e^{-y^2/2} \right] dy \right\} - 1.
\]  
(4.29)

Furthermore, Spitzer has shown that the asymptotic behaviour of \(F(\xi)\) at large values of \(\xi\) reads as

\[
\lim_{\xi \to \infty} F(\xi) = \sqrt{2} \left( \xi + \frac{1}{2\pi} \int_{-\infty}^{\infty} y^{-2} \ln \left[ (y^2/2)(1 - e^{-y^2/2})^{-1} \right] dy \right).
\]  
(4.30)

Chernoff [32] has calculated the approximate value 0.5824 for the constant in (4.30). Chang et al. [33] have shown that the constant is given by \(-\zeta(1/2)/\sqrt{2\pi} = 0.5826\). Thus, the asymptotic behaviour of \(F(\xi)\) can be written as
\[
\lim_{\xi \to \infty} F(\xi) = \sqrt{2} \left( \xi - \zeta(1/2)/\sqrt{2\pi} \right). \tag{4.31}
\]

Next, we apply the arguments of Van Dyke’s matching principle. We start with the one-term inner solution. We have found a solution to a canonical inner problem (4.25-4.27) given by \( F(\xi) \). The inner solution is modulated by the time-dependent function \( \gamma(\tau) \) from the outer solution. Thus, the one-term inner solution reads as

\[
u_i^m(\xi, \tau) = \alpha(\epsilon) \gamma(\tau) F(\xi). \tag{4.32}\]

Now, to determine \( \alpha(\epsilon) \), we compute the one-term outer expansion of the one-term inner solution. Therefore, we express the inner solution (4.32) in terms of the outer variables and expand in powers of \( \epsilon \), keeping the first term only. The behaviour of \( F(x/\epsilon) \) for small values of \( \epsilon \), i.e. \( \xi = x/\epsilon \to \infty \), is given by (4.31). Obviously, only the first term in (4.31) contributes to the one-term outer expansion of the one-term inner solution. We find that

\[
O_1 u_1^i(x, \tau) = \sqrt{2} \alpha(\epsilon) \gamma(\tau) \xi = \sqrt{2} \alpha(\epsilon) \gamma(\tau) \frac{x}{\epsilon}. \tag{4.33}\]

This enables us to determine \( \alpha(\epsilon) \) by matching the one-term inner expansion of the one-term outer solution (4.23) with the one-term outer expansion of the one-term inner solution (4.33). Then \( \alpha(\epsilon) \) becomes

\[
\alpha(\epsilon) = \epsilon/\sqrt{2}. \tag{4.34}\]

Now, we apply van Dyke’s matching argument for the two-term outer expansion and the one-term inner expansion. The one-term inner expansion of the two-term outer solution is given in Eq.(4.24). Using (4.34), the two-term outer expansion of the one-term inner solution is given by

\[
O_2 u_1^i(\xi, \tau) = \epsilon \gamma(\tau) \left( \xi - \zeta(1/2)/\sqrt{2\pi} \right). \tag{4.35}\]

Since by matching arguments \( O_2 u_1^i = I_1 u_2^o \) we can see that \( \delta(\epsilon)u_1(0, \tau) \) takes the form

\[
\delta(\epsilon)u_1(0, \tau) = -\epsilon \gamma(\tau) \zeta(1/2)/\sqrt{2\pi} \tag{4.36}\]
Finally, we compute the value of the outer solution at the boundary $x = 0$. Inserting (4.36) in (4.21) and setting $x = 0$ yields

$$u_2^o(0, \tau) = -\epsilon \gamma(\tau) \zeta(1/2)/\sqrt{2\pi}.$$  \hspace{1cm} (4.37)

Thus, we have derived the effective boundary condition seen by the outer problem. This result exactly coincides with the boundary condition as assumed in Eq.(4.6). Following the argument as outlined at the end of section 4.1, the price of the discrete down-and-out call option in the limit of a high number of monitoring points is approximately given by

$$V_m(S, B, t) = V(S, B, t) + \epsilon \frac{\partial V(S, B e^{-\beta \epsilon}, t)}{\partial \epsilon} \bigg|_{\epsilon=0} + o(\epsilon).$$  \hspace{1cm} (4.38)

Finally, we follow the ideas of Broadie et al. [1] and propose an interpolation formula that exactly reproduces the asymptotic behaviour to the lowest order in the expansion parameter $1/\sqrt{m}$ given by

$$V_m(S, B, t) = V \left(S, B e^{-\beta \epsilon} \sqrt{t/m}, t\right) + o(1/\sqrt{m}).$$  \hspace{1cm} (4.39)

This result coincides with the continuity correction (3.11) as derived by Broadie et al. [1]. Note that this derivation can also be used to get an asymptotic expression for the other types of barrier options such as discrete up-and-out or discrete up-and-in options. The discrete down-and-in option is obtained by replacing the first derivative of the price of the continuous down-and-out option with respect to $x$ in (4.23) by the first derivative of the price of the continuous down-and-in option with respect to $x$ and using the same argumentation as discussed in this section. For discrete up-and-out and discrete up-and-in option, one additionally has to replace $\epsilon \gamma(\tau) \xi$ by $-\epsilon \gamma(\tau) \xi$ in (4.23). This leads to a shift of the barrier from $B$ to $B e^{+\beta \epsilon}$ instead of $B e^{-\beta \epsilon}$. 
Chapter 5

Asymptotic expansion for double barrier options

In this chapter, we study discrete double barrier options. We apply the method of matched asymptotic expansions to derive an asymptotic expression for discrete double barrier options. In doing so, we essentially follow the lines as presented for single barrier options in chapter 4.

5.1 Matched asymptotic expansions

Double barrier options with continuous monitoring have been discussed in section 2.5. Now we focus on double barrier options with discrete monitoring. As for single barrier options, the price of discrete double barrier options can be expressed by an \( m \)-dimensional integral using the Green’s function method as discussed in section 2.4. Since the computational effort increases rapidly at large \( m \), one is interested in an asymptotic result for the price of discrete double barrier options that provides a good approximation at large \( m \). In other words, we want to study the asymptotic behaviour of discrete double barrier options in the limit as the number of monitoring points goes to infinity.

The asymptotic behaviour of discrete single barrier options has been the topic of chapter 4. The methods applied to discrete single barrier options can also be used for discrete double barrier options. The basic idea is to work out the efficient boundary condition at \( x = 0 \) and \( x = a \) by using the method of matched asymptotic expansions. The result is used to
formulate a boundary value problem with time-dependent boundary conditions, which can then be solved by standard methods such as the Laplace transform.

Following the same ideas as outlined in section 4.3, we look for an outer solution of the original boundary value problem. Suppose the two-term outer solution of a discrete double barrier knockout option is given by

$$ u_{o2}(x, \tau) = u_c(x, \tau) + \delta(\epsilon)u_1(x, \tau), \quad (5.1) $$

where $u_c(x, \tau)$ is the solution for the continuous double barrier option as given by (2.40). We want to determine the functions $\delta(\epsilon)$ and $u_1(x, \tau)$ by applying the method of matched asymptotic expansions. First, we observe that the behaviour of the solution is different in the domains $x \leq 0$ and $x \geq a$ on the one hand and in the domain $0 < x < a$ on the other hand. In the domain $x \leq 0$ and $x \geq a$, the solution is approximately time-periodic due to the boundary condition. In the domain $0 < x < a$, the solution is slowly varying and converges to the price of the continuous double barrier option as $m \to \infty$. In contrast to single barrier options, there are two regions $x \approx 0$ and $x \approx a$ for double barrier options in which the solution rapidly changes. However, to lowest order, the detailed nature of the boundary condition at $x = a$ does not influence the solution at $x = 0$ and vice versa. In other words, if we work out the outer solution at $x \approx 0$, we assume that the boundary condition at $x = a$ is zero, i.e. we assume that the boundary condition is that of the continuous double barrier option, and vice versa.

Next, we apply the same arguments as for single barrier options. We expand the solution for the continuous double barrier option near $x \approx 0$. The first derivative of $u_c(x, \tau)$ with respect to $x$ at $x = 0$ serves to modulate the solution in the domain $x \leq 0$. Then, we follow exactly the same steps as outlined in section 4.3 to compute the effective boundary condition at $x = 0$. Here, we only present the result without going into the details of the calculation. The effective boundary condition at $x = 0$ is given by

$$ u_{o2}'(0, \tau) = \beta \epsilon \gamma(0, \tau), \quad (5.2) $$

where $\gamma(x, \tau)$ can be expressed in terms of the solution of the continuous double barrier option as

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\[ \gamma(x, \tau) = \frac{\partial u_c(x', \tau)}{\partial x'} \bigg|_{x' = x}. \] (5.3)

Inserting (2.40) for \( u_c(x, \tau) \), one finds

\[ \gamma(0, \tau) = \sum_{n=1}^{\infty} C_n \left( \frac{n\pi}{a} \right) e^{-\frac{n^2}{a^2} \tau}. \] (5.4)

The effective boundary condition at \( x = a \) is obtained similarly. The first derivative of \( u_c(x, \tau) \) with respect to \( x \) at \( x = a \) determines the solution in the domain \( x \geq a \). Again, we can apply the same steps as outlined in section 4.3 to compute the effective boundary condition at \( x = a \). Alternatively, one may consider symmetry arguments. If we shift the boundary from \( a \) to \( 0 \) and reflect the solution, i.e. we introduce \( u_1(-x, \tau) \), we have derived exactly the same problem that has led to the effective boundary condition (5.2) at \( x = 0 \) in terms of the original variable. The transformation can be achieved by introducing the new variable \( x' = -(x - a) \). Both methods give the following expression for the effective boundary condition at \( x = a \):

\[ u_2^o(a, \tau) = -\beta \epsilon \gamma(a, \tau), \] (5.5)

with \( \gamma(a, \tau) \) defined by

\[ \gamma(a, \tau) = \sum_{n=1}^{\infty} (-1)^n C_n \left( \frac{n\pi}{a} \right) e^{-\frac{n^2}{a^2} \tau}. \] (5.6)

So far, we have derived the effective boundary conditions for the outer problem. In the next step, we use this result to solve the following initial boundary value problem:

\[ \frac{\partial u_m}{\partial \tau} = \frac{\partial^2 u_m}{\partial x^2} \] (5.7)

with the initial condition

\[ u_m(x, 0) = e^{-\alpha_1 x} \max(e^x - e^y, 0) \] (5.8)

and the boundary conditions
\[ u_m(0, \tau) = \beta \varepsilon \gamma(0, \tau); \quad u_m(a, \tau) = -\beta \varepsilon \gamma(a, \tau). \] \tag{5.9}

The continuous double barrier option \( u_c(x, \tau) \) satisfies the heat equation with initial condition (5.8) and vanishing boundary condition. The solution to the boundary value problem can be written as

\[ u_m(x, \tau) = u_c(x, \tau) + \beta \varepsilon u_1(x, \tau), \tag{5.10} \]

where \( u_1(x, \tau) \) satisfies the heat equation with vanishing initial condition and with time-dependent boundary conditions (5.9). The general solution to this standard problem is obtained by the Laplace transform. Details of the derivation can be found in [34]. We obtain

\[ u_1(x, \tau) = \frac{2}{a} \sum_{k=1}^{\infty} \int_{0}^{\tau} ds \left\{ \gamma(0, s) \sin \left( \frac{k\pi}{a} x \right) - \gamma(a, s) \sin \left( \frac{k\pi}{a} (a - x) \right) \right\} \times \left( \frac{k\pi}{a} \right) e^{-\frac{k^2 \pi^2}{a^2} (\tau - s)}. \tag{5.11} \]

The remaining \( s \)-integration in this expression can be immediately carried out. The result reads as follows:

\[ u_1(x, \tau) = \frac{2}{a} \sum_{k=1}^{\infty} \left\{ A_k \sin \left( \frac{k\pi}{a} x \right) - B_k \sin \left( \frac{k\pi}{a} (a - x) \right) \right\} e^{-\frac{k^2 \pi^2}{a^2} \tau} \tag{5.12} \]

with the coefficients \( A_k \) and \( B_k \) given by

\[ A_k = C_k \frac{\pi^2 k^2}{a^2} \tau + \sum_{n=1, n \neq k}^{\infty} C_n \frac{n^2}{k^2 - n^2} \left( e^{-\frac{n^2}{\pi^2} (n^2 - k^2) \tau} - 1 \right) \tag{5.13} \]

and

\[ B_k = (-1)^k C_k \frac{\pi^2 k^2}{a^2} \tau + \sum_{n=1, n \neq k}^{\infty} (-1)^n C_n \frac{n^2}{k^2 - n^2} \left( e^{-\frac{n^2}{\pi^2} (n^2 - k^2) \tau} - 1 \right), \tag{5.14} \]
respectively. The coefficients $C_n$ are given in (2.42) for the payoff of a call option. Alternatively, one may insert the payoff of a put option in (2.41) to get the asymptotic result for a double knockout put option.

Figure 5.1: Comparison of the Monte Carlo results with the asymptotic solution for a discrete double knockout call option (5.19) as a function of the number of monitoring points (spot=strike=100, lower barrier=95, upper barrier=105, expiry=T=0.2y, annual volatility=10%, annual rate=10%). We have also included the exact solution for the continuous double knockout call option.

In principle, Eq.(5.12) represents the first order correction of the discrete double knockout option to the continuous double knockout option. However, this series converges very badly due to the occurrence of the term $\sum_{k=1}^{\infty} \sin(k\pi x/a)k/(k^2 - n^2)$. Therefore, we perform the $k$-summation explicitly by using the following relation [35]:

$$\sum_{k=1}^{\infty} \frac{k}{k^2 - n^2} \sin(kx) = \frac{1}{2} (\pi - x) \cos(nx) - \frac{1}{4n} \sin nx. \quad (5.15)$$

Inserting this expression in (5.12) gives an alternative representation for $u_1(x, \tau)$. We get
\[ u_1(x, \tau) = \frac{2}{a} \sum_{k=1}^{\infty} \left\{ a_k \sin \left( \frac{k\pi}{a} x \right) + b_k \cos \left( \frac{k\pi}{a} x \right) \right\} e^{-\frac{k^2 \tau}{a^2}}, \quad (5.16) \]

with the coefficients \( a_k \) and \( b_k \) given by

\[ a_k = 2C_k \frac{\pi^2 k^2}{a^2} \tau - \frac{C_k}{2} - \sum_{n=1}^{\infty} C_n \frac{nk}{k^2 - n^2} \left( 1 + (-1)^{n+k} \right), \quad (5.17) \]

and

\[ b_k = C_k k \frac{\pi}{2a} (a - 2x), \quad (5.18) \]

respectively.

Finally, we summarize the results. The asymptotic result for the discrete double barrier knockout option to lowest order can be written as

\[ V_m(S, B_-, B_+, t) = B_- e^{a_1 x + a_2 \tau} (u_c(x, \tau) + \beta \epsilon u_1(x, \tau) + o(1/\sqrt{m})), \quad (5.19) \]

where \( u_1(x, \tau) \) is given by (5.12) or (5.16). In order to compare the asymptotic result (5.19) with the exact solution, we have performed a Monte Carlo simulation for a discrete double knockout call option. Fig. 5.1 shows a comparison of the Monte Carlo results with the asymptotic result in Eq.(5.19). The number of paths used in the simulation is \( 5 \times 10^7 \). The simulation shows that Eq.(5.19) adequately describes the asymptotic behaviour of a discrete double knockout call option. Note that this method can also be applied to derive an asymptotic result for the discrete double barrier knockin option.
Chapter 6

Conclusion and Outlook

In this thesis, we have studied the pricing of discrete barrier options using analytical methods and numerical simulations. In particular, we have introduced the method of matched asymptotic expansions to study the asymptotic behaviour of discrete barrier options. In a first approach, we have used this method to perform an asymptotic expansion for discrete single barrier options. As a result of our analysis, we have derived the first order correction of the price of a discrete single barrier option to the price of the continuous single barrier option (4.38). This result coincides with the continuity correction (3.11) as derived by Broadie et al. [1] within a probability approach. In the next step, we have applied this method to discrete double barrier options to derive an asymptotic result for large $m$. Again, we have derived the first order correction of the price of a discrete double barrier option to the price of the continuous double barrier option (5.19).

A subject of future investigation might be the extension of the method outlined in this thesis to incorporate a local volatility surface. This is completely inaccessible by probability methods. The general method may also be extended to discrete lookback or hindsight options as discussed in [18]. It may also serve as a starting point to study more exotic barrier options with discrete monitoring such as corridor options or barrier options with curved barriers. In principle, one can also proceed to higher order terms in the asymptotic expansion. The periodic structure of the inner solution will be modulated by a time-dependent function from the outer solution for higher order contributions.
Appendix A

Useful integral

In this appendix, we present the calculation of an integral used in section 4.1. The integral reads as follows

$$\int_0^1 dx \frac{1}{x^{3/2} (1-x)^{3/2}} e^{-\frac{a}{x} - \frac{b}{1-x}},$$

(A.1)

with \(a>0\) and \(b>0\). After the substitution \(x = \frac{1}{x' + 1}\), this equation may be rewritten as

$$= \int_0^\infty dx \frac{x+1}{x^{3/2}} e^{-ax - \frac{b}{x}} e^{(-a-b)},$$

(A.2)

where we have dropped the prime. Now, we may perform the integration (see reference [35], page (344), equation (3)) to obtain the following result

$$= \left( \frac{\pi}{a} e^{-2\sqrt{ab}} - \frac{\pi}{a} \frac{\partial}{\partial b} e^{-2\sqrt{ab}} \right) e^{(-a-b)}.$$

(A.3)

Simple algebra yields the final expression for the integral. Thus, we arrive at the following expression for the integral

$$\int_0^1 dx \frac{1}{x^{3/2} (1-x)^{3/2}} e^{-\frac{a}{x} - \frac{b}{1-x}} = \sqrt{\pi} \left( \frac{\sqrt{a} + \sqrt{b}}{\sqrt{ab}} \right) e^{-(\sqrt{a}+\sqrt{b})^2}.$$

(A.4)

This integral expression is used in section 4.2 to find an approximate result for the discrete barrier option for a high number of monitoring points.
Bibliography


[16] GNU Scientific Library is available at www.gnu.org/software/gsl/gsl.html


