Valuation of Swing Options and Examination of Exercise Strategies by Monte Carlo Techniques

Dr. Uwe Dörr
Kellogg College
University of Oxford

A thesis submitted in partial fulfillment for the MSc in Mathematical Finance
September 29, 2003
Monte–Carlo simulation techniques are used to investigate (standardized) Swing options. In a first approach, this is done by an algorithm which is based on the Longstaff Schwartz method for American and Bermudan options. This algorithm yields the value of the Swing option under the assumption that the optimal exercise strategy is applied. Furthermore the optimal strategy can be extracted from the algorithm. Various examples including Swing options with upswings, downswings and penalties are valued numerically, and an upper boundary for Swing options is found in the computer experiment. In a second approach, the exercise strategy is used as input parameter and the expected payoff with respect to this strategy is calculated by strictly forward evolving Monte Carlo. For these simulations, a one factor log-normal mean-reverting process is used to describe the behaviour of the underlying spot price. The success of several sample strategies is discussed in terms of process properties like mean-reversion speed and volatility.
## Contents

1 Introduction .................................................. 1  
   1.1 Swing Options in the Real World .......................... 1  
   1.2 Valuation of Swing Options ............................... 2  
      1.2.1 Modelling the Price Process ......................... 3  
      1.2.2 Numerical Methods for the Early Exercise Problem .... 4  
   1.3 The Aim of This Thesis .................................. 5  
   1.4 How this Thesis is Organized ............................ 6  

2 Stochastic Processes for the Electricity Spot Price ............. 8  
   2.1 One Factor Mean-Reverting Process ....................... 9  
      2.1.1 The Process ........................................ 9  
      2.1.2 Solution for $S(t)$ ................................ 10  
      2.1.3 Mean and Variance of $S$ ........................... 11  
      2.1.4 Vanilla Call Option ................................. 11  
      2.1.5 Summary ............................................ 12  
   2.2 Two Factor Mean–Reverting Process ....................... 13  
      2.2.1 The Process ........................................ 13  
      2.2.2 Solution for log $S(t)$ ............................. 14  
      2.2.3 Forward Price and Vanilla Call Option ............... 17  

3 Least Squares Monte Carlo for Swing Options .................... 18  
   3.1 The Longstaff–Schwartz Algorithm for American and Bermudan Options 19  
      3.1.1 Theory ........................................... 19  
      3.1.2 The Algorithm .................................... 21  
   3.2 Extension of Longstaff-Schwartz to Swing Options ........... 23  
      3.2.1 Illustrative Example ............................... 23  
      3.2.2 General Case: Upswings, Downswings and Penalty Functions 27  
      3.2.3 Implementation .................................... 29
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3</td>
<td>Computational Results</td>
<td>31</td>
</tr>
<tr>
<td>3.3.1</td>
<td>Tests</td>
<td>32</td>
</tr>
<tr>
<td>3.3.2</td>
<td>Results for the One-Factor Process</td>
<td>32</td>
</tr>
<tr>
<td>3.3.3</td>
<td>Results for the Two Factor Process</td>
<td>35</td>
</tr>
<tr>
<td>3.3.4</td>
<td>Upper and Lower Boundaries</td>
<td>36</td>
</tr>
<tr>
<td>4</td>
<td>Exercise Strategies</td>
<td>45</td>
</tr>
<tr>
<td>4.1</td>
<td>Illustration of Early Exercise for a Bermudan Option with two Exercise Opportunities</td>
<td>46</td>
</tr>
<tr>
<td>4.1.1</td>
<td>The Threshold for Early Exercise</td>
<td>46</td>
</tr>
<tr>
<td>4.1.2</td>
<td>Dependence of the Threshold on the Process Parameters</td>
<td>48</td>
</tr>
<tr>
<td>4.1.3</td>
<td>Summary of the Early Exercise Problem</td>
<td>50</td>
</tr>
<tr>
<td>4.1.4</td>
<td>Interplay between Early Exercise and Option Value</td>
<td>51</td>
</tr>
<tr>
<td>4.2</td>
<td>Valuation of Swing Options in Terms of Exercise Strategies</td>
<td>52</td>
</tr>
<tr>
<td>4.2.1</td>
<td>Optimal Strategy for Swing Options</td>
<td>53</td>
</tr>
<tr>
<td>4.2.2</td>
<td>Comparison of Different Exercise Strategies</td>
<td>58</td>
</tr>
<tr>
<td>5</td>
<td>Summary and Outlook</td>
<td>65</td>
</tr>
<tr>
<td>5.1</td>
<td>Summary</td>
<td>65</td>
</tr>
<tr>
<td>5.2</td>
<td>Outlook</td>
<td>66</td>
</tr>
<tr>
<td>A</td>
<td>Proof of the Statement: Threshold for Early Exercise &gt; Mean Reversion Level</td>
<td>68</td>
</tr>
<tr>
<td>B</td>
<td>The MATLAB Routines</td>
<td>70</td>
</tr>
<tr>
<td></td>
<td>Bibliography</td>
<td>77</td>
</tr>
</tbody>
</table>
Chapter 1
Introduction

1.1 Swing Options in the Real World

In order to hedge themselves against extreme price fluctuations of certain commodities, many consumers enter into forward contracts which give them the right and the obligation to purchase a fixed amount of the commodity for a predetermined price.

However, for some market participants this reduction of risk is not sufficient, since they do not know their exact future need of the commodity. In particular, this is a serious problem with commodities that cannot be stored or for which storage is very expensive.

Therefore so-called Swing contracts have been developed in order to give the holder a certain flexibility with respect to the amount purchased in the future. These sort of contracts are mainly found in energy markets, since energy is difficult (or expensive) to store and exhibits extreme price fluctuations. This refers especially to electricity, but Swing contracts appear also in coal (see [11, 12]) and gas markets (see [4]), for example.

In the following we concentrate mainly on Swing options on electricity. The main characteristic properties of Swing options however, i.e. the multiple early exercise features, are the same for all underlying commodities. Only the choice of suitable stochastic processes depends strongly on the type of underlying asset.

According to [10], typical Swing contracts contain a so called base load agreement. The base load agreement is a set of forward contracts with different expiry dates $t_j$, $j = 1, \ldots, N$. Each forward contract $F_j$ is based on a fixed amount of electricity (or, in general, any commodity), $b_j$. At each expiry date the holder has the option to purchase an excess amount or decrease the base load volume. This means the amount of electricity purchased (for a predetermined price, i.e. the strike price) by the holder
of the Swing option can “swing” within a certain range \((b_j + \Delta_j)\) where

\[
\Delta_j \in (l^1_j, l^2_j) \cup (l^3_j, l^4_j)
\]

\(l^1_j \leq l^2_j \leq 0 \leq l^3_j \leq l^4_j\) \hspace{1cm} (1.1)

If \(\Delta_j\) is positive (negative) the option exercised by the holder at opportunity \(t_j\) is called upswing (downswing).

In the case when \(l^1_j \neq l^2_j\) (or \(l^3_j \neq l^4_j\)) the holder can choose the nominal of the option (that means the amount to purchase for upswings or to sell for downswings) within the respective range.

For typical contracts there a further restrictions:

- the total number of upswings, \(U\), and downswings, \(D\), is limited, i.e., \(U \leq N\), \(D \leq N\), \(U + D \leq N\)
- there may be a penalty payment which depends on the total volume purchased at the end of the Swing contract period (which usually corresponds to the end of the base load contract \(t_N\))

For valuation purposes, the base load contract on the one side and the up- and downswings on the other side can be separated from each other. For the remainder of this thesis we concentrate on the latter part and consider the Swing option as a set of \(U\) upswings and \(D\) downswings, including a penalty agreement.

### 1.2 Valuation of Swing Options

The valuation of electricity Swing options involves two main issues:

- Modelling the underlying electricity price process;
- Solving the complex early exercise problem.

These two aspects are – in principle – independent of each other. The first has been widely discussed in the context of futures price modelling in electricity markets, for example. The second can be considered as an extension of Bermudan exercise features which occur frequently in the context of interest rate derivatives like swaptions.
1.2.1 Modelling the Price Process

As a consequence of liberalization of electricity markets which has taken place in the U.S. and many European countries, the energy price is mainly driven by supply and demand. Together with some characteristic properties of electricity, this leads to large short-term volatility of the electricity spot price. These characteristic properties can be summarized as follows:

- Power as a commodity exhibits a heterogenous nature, both with respect to time and location of its generation (see [13]);

- Electricity cannot be stored efficiently;

- Arbitrage processes are difficult to set up because of technical constraints (see [5]) and thus electricity markets are not completely efficient.

In addition to pronounced short-term volatility electricity prices exhibit further typical properties which result from the peculiarities of supply and demand:

- **Mean reversion**: volatility decreases with increasing time horizon. There is a long-term equilibrium (“fair price”) which is much less volatile than the spot price. The mean-reversion speed is determined by how quickly supply can react on sudden demand changes (see [16]);

- **Cyclical variations**: these occur on different time-scales (time of day, day of week, seasons) and are driven by cyclical demand changes. This aspect of the price process can be considered as deterministic and therefore easily be separated from the stochastic time dependence (see [1] where this is discussed for a temperature process in the context of weather derivatives);

- **Occasional spikes**: in addition to the large short-term volatility extreme changes occur occasionally which last only for a very short time. Positive spikes can, e.g., be caused by outages in the generation or transmission process (see [7]) or by extreme events (politics, weather, etc.). Negative spikes can occur when it is difficult to reduce generation capacity in periods of low demand.

From these observations it is clear that simple geometric Brownian motion which is very popular for the modelling of equity prices is not well suited to model the electricity price process.
Several log-normal mean-reversion processes have been proposed in the literature. These processes can be driven by one stochastic factor, as discussed in [17] or [15], or by two or more factors (see, for example, [8]).

These processes, however, do not cover occasional spikes completely. Therefore jump diffusion processes are frequently used. For example, a discrete jump diffusion component can be added to a log-normal model. In [9] and [16] two-factor mean-reverting processes with jumps are suggested, and among practitioners three-factor models seem to be quite popular [19].

1.2.2 Numerical Methods for the Early Exercise Problem

Because of their complicated early exercise features, Swing options can be considered as a generalization of Bermudan-type options which are common in interest rate markets and thus frequently discussed in literature. A holder of a Bermudan-type option can exercise his right only once. Further, Bermudan-type options can be regarded as discretized American options which are very popular in equity markets, for example.

In general, there is no analytical solution for the early exercise problem and thus numerical methods have to be applied.

For Swing options, tree-based methods as suggested in [10] seem to be the most popular approach so far, but there are other possibilities for tackling the problem.

For example, in a recent thesis (see [18]), valuation of Swing options by finite-differences has been demonstrated.

In the case of Bermudan options in interest markets, Monte-Carlo methods have become popular during the last few years. There are mainly two important approaches:

- Parametrization of the early exercise decision and optimization of the strategy as proposed in [2];
- Finding the optimal strategy by estimating the conditional continuation as demonstrated in [14].

In the latter case, conditional continuation values are estimated via a series of regressions. This approach is pursued in the present thesis.
1.3 The Aim of This Thesis

In summary, there are three main goals tracked by this thesis:

- Implementation of a tool for the valuation of Swing options which is flexible enough to allow for different underlying stochastic processes – this valuation corresponds to the usual optimal strategy approach;

- Monitoring expected payoffs of Swing options for different (suboptimal) exercise strategies and comparison with the optimal strategy;

- Investigating the dependence of Swing option values, expected payoffs and optimal early exercise boundaries on process parameters like mean-reversion speed and volatility.

For these purposes, it seems to be appropriate to simplify the object under consideration in two ways.

First, we restrict ourselves to standardized simplified Swing contracts. The main part of the present thesis concentrates to Swing contracts solely with upswings and no penalties. At $N$ exercise opportunities these contracts give the holder the right (but not the obligation) to purchase a certain amount of the underlying for a specified (strike) price, but the holder can exercise this right at most $m$ times (where $m \leq N$). At each single opportunity, only one right may be exercised.

Note that the fixed exercise amount implied by this contract is no further restriction in the absence of penalty agreements. If there is no penalty, the holder will always choose the maximum amount when he exercises an up- or downswing.

Valuation of Swing contracts with upswings, downswings and penalties however is performed as well, but without detailed discussion of the results.

Second, we consider only two different stochastic processes for the underlying. The first is a one factor mean-reverting process which is well suited to demonstrate the impact of mean-reversion and volatility. Virtually all systematic investigations of the interplay between process parameters on the one side and Swing option values, expected payoffs or early exercise boundaries on the other side refer to this process.

In all practical computations the process components which depend deterministically on time are set constant. Since deterministic effects like seasonality of prices are beyond the scope of the present work, this means no restriction. For a discussion of the impact of seasonality, refer to [18].
The second stochastic process is a two factor mean–reverting process. For this process, valuation of Swing options is demonstrated but no discussion of the impact of the process parameters is given.

Neither of the two processes seem to be first choice for the calibration of real market data, since they do not cover extreme price spikes which are frequently observed in electricity markets, for example (see above). The one factor process covers some very important aspects of real processes like mean-reversion while the number of parameters is small enough to allow for systematic investigations. A further advantage is that for this process the valuation of Swing options by finite-differences has been carried out in a former thesis [18] and thus the results can be compared.

Monte Carlo simulations are chosen in this thesis, although these methods exhibit some disadvantages like slow convergence and difficult handling of early exercise features. However, the following arguments make it clear why Monte Carlo an appropriate approach to the numerical valuation of Swing options:

- In this thesis it is demonstrated that the Least-Squares Monte-Carlo method for American options presented by Longstaff and Schwartz [14] works for Swing options as well;
- The numerical valuation tool presented in this thesis can easily be expanded to any stochastic process;
- Monte Carlo methods allow for monitoring the numerical uncertainty, both in the value of the Swing option and in the early exercise thresholds;
- In addition to the Least-Squares Monte Carlo approach, expected payoffs for different strategies can be determined very easily by Monte Carlo simulations.

1.4 How this Thesis is Organized

After this introductory chapter, the rest of the thesis is organized as follows.

In Chapter 2 the two stochastic processes under consideration are discussed with respect to analytic solutions for the stochastic spot price, its expected value (forward price) and the value of vanilla call options.

The Least-Squares Monte-Carlo method for the valuation of American options is described in the first part of Chapter 3. Its extension to Swing options is introduced in the second part where both the algorithm and its implementation are discussed. The last part of the chapter contains numerical results obtained for both stochastic
processes, including the valuation of Swing options with upswings, downswings and penalties, and the comparison with finite-differences calculations.

Exercise strategies are discussed in Chapter 4. The whole chapter is restricted to the one factor mean-reverting process. In the first part, the early exercise problem including the influence of the process parameters on the threshold for early exercise are discussed in detail for the special case of a Bermudan option with two exercise opportunities. The extension of the early exercise problem to Swing options is given in the second part of the chapter. This includes the determination of the optimal strategy and its convergence from Least-Squares Monte-Carlo and quantitative results for the success of several (suboptimal) sample strategies and its dependence on the process parameters.
Chapter 2

Stochastic Processes for the Electricity Spot Price

In this thesis two different stochastic processes for the spot price of electricity are discussed explicitly. The one factor mean-reverting process described first has already been used in a previous thesis [18]. In that thesis Swing options have been valued by finite differences. This yields the important advantage that option values obtained by the two different methods can be compared with each other. A further advantage of the one factor process is the fact that there are only a few process parameters. Systematic investigation of characteristic process properties like mean–reversion and volatility is therefore much easier than in the case of processes with many parameters.

The second process under investigation is a two factor mean-reverting process. Although significantly more complicated than the one factor process this process can be integrated as well. This is advantegous since the Monte–Carlo simulation does not need to be performed for differential timesteps.

Both processes are treated in the following manner. First, the characteristic parameters are discussed. Then the stochastic differential equation for the spot price is solved. This yields the spot price as a function of a stochastic variable. From this function the expected value of the spot price, i.e. the forward price, is calculated analytically. Finally the price of a vanilla call option is obtained by calculating the expected value of its payoff.
2.1 One Factor Mean-Reverting Process

2.1.1 The Process

According to [15] we consider a process

\[ S_t = F(t)e^{Y_t} \]  

(2.1)

for the electricity spot price \( S \) where \( F(t) \) denotes the deterministic part and \( Y_t \) the stochastic part driven by the process

\[ dY_t = -\alpha Y_t dt + \sigma(t) dW_t \]  

(2.2)

Equation (2.2) describes a mean-reverting process with mean-reversion level zero and mean-reversion speed \( \alpha > 0 \). In the stochastic term of (2.2), \( \sigma(t) \) denotes the time-dependent (but deterministic) volatility and \( dW_t \) the increment of a standard Brownian motion. The stochastic function \( Y \) describes the deviation from the deterministic (but time-dependent) equilibrium level \( F \).

As usually done for stock markets, exponential price changes are assumed in this model. In particular, this assumption prevents negative prices. Furthermore, the process (2.1) has the important advantage that it is integrable for any positive, continuously differentiable function \( F(t) \). This allows for a wide range of models for the deterministic part of the price. The deterministic part of the price includes the effect of seasonality which is an important property of electricity prices.

Using Itô’s lemma we obtain

\[ dS = \left( \frac{d \log F(t)}{dt} + \alpha \log F(t) + \frac{1}{2} \sigma^2(t) - \alpha \log S \right) Sdt + \sigma(t) SdW_t \]  

(2.3)

\[ \rho(t) = \frac{1}{\alpha} \left( \frac{d \log F(t)}{dt} + \frac{1}{2} \sigma^2(t) \right) + \log F(t) \]  

(2.5)

Note that the process above is assumed to be the real-world process. Since no efficient hedging strategy exists for this contract, the Monte-Carlo simulations are performed with the real-world process. There is no market price of risk and therefore no transformation into the risk-neutral world.
2.1.2 Solution for $S(t)$

Substituting
\[ r = \log S \]  
and applying Itô’s lemma once again yields
\[ dr = \left( \alpha(\rho(t) - r) - \frac{1}{2}\sigma^2(t) \right) dt + \sigma(t)dW_t \]  

For simplicity of notation we introduce
\[ \beta(t) = \alpha\rho(t) - \frac{1}{2}\sigma^2(t) = \frac{d\log F(t)}{dt} + \alpha\log F(t) \]  
and obtain
\[ dr + \alpha rd dt = \beta(t)dt + \sigma(t)dW_t \]  
Keeping in mind that
\[ d(e^{\alpha t}) = e^{\alpha t}dr + \alpha e^{\alpha t}dt \]  
we can integrate (2.9) from the start point $t_0$ to the end point $t$ and we get
\[ r(t) = r(t_0)e^{\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-s)}\beta(s)ds + \int_{t_0}^{t} e^{-\alpha(t-s)}\sigma(s)dW_s \]  
With the definition (2.8) the first integral in (2.11) can be calculated explicitly. Substituting
\[ \bar{r} = \log F \]  
and using (2.8) yields
\[ e^{-\alpha(t-s)}\beta(s) = e^{-\alpha(t-s)}\left( \frac{d\bar{r}}{ds} + \alpha\bar{r}(s) \right) = \frac{d\left( \bar{r}(s)e^{-\alpha(t-s)} \right)}{ds} \]  
and therefore we can rewrite (2.11)
\[ r(t) - \bar{r}(t) = (r(t_0) - \bar{r}(t_0))e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-s)}\sigma(s)dW_s \]  
Resubstituting $r$ and $\bar{r}$ according to (2.6) and (2.12), respectively, we finally obtain
\[ S(t) = F(t) \left( \frac{S(t_0)}{F(t_0)} \right) e^{-\alpha(t-t_0)}e^{\int_{t_0}^{t} e^{-\alpha(t-s)}\sigma(s)dW_s} \]  
The integral in the exponent of the last term of (2.15) is stochastic. Since the increment $dW_s$ is normally distributed with zero mean and variance $ds$
\[ dW_s \sim N(0, ds) \]
the integral is also normally distributed with zero mean and variance

$$v(t, t_0) := \text{var} \left[ \int_{t_0}^{t} e^{-\alpha(t-s)} \sigma(s) dW_s \right] = \int_{t_0}^{t} e^{-2\alpha(t-s)} \sigma^2(s) ds \quad (2.17)$$

Thus we can write (2.15) as

$$S(t, t_0) = A(t, t_0) e^X \quad (2.18)$$

where

$$A(t, t_0) = F(t) \left( \frac{S(t_0)}{F(t_0)} \right) e^{-\alpha(t-t_0)} \quad (2.19)$$

and

$$X \sim N(0, v(t, t_0)) \quad (2.20)$$

### 2.1.3 Mean and Variance of $S$

At this stage, the calculation of the expected value and the variance of $S$ is very easy

$$E[S(t)|S(t_0)] = A(t, t_0) E[e^X]$$

$$= A(t, t_0) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v(t, t_0)}} e^{-\frac{X^2}{2v(t, t_0)}} e^X dX$$

$$= A(t, t_0) e^{v(t, t_0)} \quad (2.21)$$

and the variance is given by

$$\text{var}[S(t)|S(t_0)] = E[S^2(t)|S(t_0)] - (E[S(t)|S(t_0)])^2$$

where

$$E[S^2(t)|S(t_0)] = A^2(t, t_0) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v(t, t_0)}} e^{-\frac{X^2}{2v(t, t_0)}} e^{2X} dX$$

$$= A^2(t, t_0) e^{2v(t, t_0)} \quad (2.22)$$

and thus

$$\text{var}[S(t)|S(t_0)] = A^2(t, t_0) \left( e^{2v(t, t_0)} - e^{v(t, t_0)} \right) \quad (2.23)$$

### 2.1.4 Vanilla Call Option

In the next step we consider a vanilla call option with strike $K$ and expiry time $t$. The value of this option at time $t_0$ is given by the discounted expected payoff at expiry

$$c(S, t_0, t) = e^{-r(t-t_0)} E[\max(S(t) - K, 0)|S(t_0)]$$

$$= e^{-r(t-t_0)} E[S(t)|S(t_0)] P(S(t) > K|S(t_0))$$

where

$$P(S(t) > K|S(t_0)) = \frac{1}{\sqrt{2\pi v(t, t_0)}} \int_{K}^{\infty} e^{-\frac{X^2}{2v(t, t_0)}} dX$$
For simplicity, we set the risk–free interest rate to zero. This has been done in [18] with the explanation that a risk–free portfolio containing a physically settled derivative remains constant and does not grow with the risk–free rate.

However, typical time–horizons discussed in this thesis are in the order of magnitude of a couple of days. This means that discounting has virtually no effect.\(^1\)

With zero interest rate we obtain

\[
c(S, t_0, t) = E[\max(A(t, t_0)e^X - K, 0)]S(t_0)]
\]

(2.29)

\[
= A(t, t_0)E[\max(e^X - K/A(t, t_0), 0)]S(t_0)]
\]

(2.30)

\[
= A(t, t_0)\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v(t, t_0)}} \max(e^X - K/A(t, t_0), 0)e^{-\frac{X^2}{2v(t, t_0)}} dX
\]

(2.31)

\[
= A(t, t_0)\cdot e^{v(t, t_0)}N_{v(t, t_0)}(d_1) - K\cdot N_{v(t, t_0)}(d_2)
\]

(2.33)

where

\[
N_{a,b}^a(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi b}} e^{-\frac{(x-a)^2}{2b^2}} dx
\]

(2.34)

is the normal cumulative function with mean \(a\) and variance \(b\) and

\[
d_1 = \log \left( \frac{A(t, t_0)}{K} \right) + v(t, t_0)
\]

(2.35)

\[
d_2 = d_1 - v(t, t_0)
\]

(2.36)

2.1.5 Summary

In summary, using the notation

\[
A(t, t_0) = F(t) \left( \frac{S(t_0)}{F(t_0)} \right) e^{-\alpha(t-t_0)}
\]

(2.37)

\[
v(t, t_0) = \int_{t_0}^{t} e^{-2\alpha(t-s)}\sigma^2(s)ds
\]

(2.38)

\[
d_1 = \log \left( \frac{A(t, t_0)}{K} \right) + v(t, t_0)
\]

(2.39)

\[
d_2 = d_1 - v(t, t_0)
\]

(2.40)

we have found the following expressions for the spot price, the forward price and the value of a vanilla call option.

\(^1\)Note that typical values for the daily volatility are in the order of 40%. This will completely cover up the discounting effect.
Spot Price

The spot price at time $t$ given the spot price at time $t_0$ is

$$S(t, t_0) = A(t, t_0)e^{X}$$  \hspace{1cm} (2.41)

$$X \sim N(0, v(t, t_0))$$  \hspace{1cm} (2.42)

Forward Price

The forward price of $S$ at time $t_0$ with respect to expiry $t$ is given by

$$\mathcal{F}(S, t_0, t) = A(t, t_0)e^{\frac{v(t, t_0)}{2}}$$  \hspace{1cm} (2.43)

Vanilla Call Option

The value at time $t_0$ of a vanilla call option with strike $K$ and expiry $t$ is given by

$$c(S, t_0, t) = A(t, t_0) \cdot e^{\frac{v(t, t_0)}{2}}N_0^0(d_1) - K \cdot N_0^0(d_2)$$  \hspace{1cm} (2.44)

In the special case of constant $F$ and constant volatility, i.e.

$$F(t) \equiv F = \text{const}$$  \hspace{1cm} (2.45)

$$\sigma(t) \equiv \sigma = \text{const}$$  \hspace{1cm} (2.46)

we obtain

$$A(t, t_0) = S(t_0)e^{-\alpha(t-t_0)}F^{1-e^{-\alpha(t-t_0)}}$$  \hspace{1cm} (2.47)

$$v(t, t_0) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha(t-t_0)})$$  \hspace{1cm} (2.48)

2.2 Two Factor Mean–Reverting Process

2.2.1 The Process

According to [8] we consider the following process:

$$dS_t = (r - \delta_t)S_t dt + \sigma_S S_t dW^S_t$$  \hspace{1cm} (2.49)

where $r$ is the interest rate and $\sigma_S$ the variance of the geometric Brownian motion followed by $S$. The convenience yield $\delta_t$ is stochastic and follows a mean–reverting process

$$d\delta_t = \alpha(\kappa_\delta - \delta_t)dt + \sigma_\delta dW^\delta_t$$  \hspace{1cm} (2.50)
The variance of the change of the convenience yield is represented by $\sigma_\delta$, $\alpha_\delta$ is the speed of adjustment and $\kappa_\delta$ the long-run mean yield. Furthermore, the two Wiener processes $W_t^S$ and $W_t^\delta$ are correlated with correlation coefficient $\rho$.

Since the concept of convenience yield is usually only applied to storable commodities, this model is more appropriate for gas or oil prices than for electricity. Nevertheless, it is relevant for Swing options in general.

In this approach the net convenience yield can be interpreted as a theoretical construction to incorporate the special effects of supply, demand and other particularities of the power market into one variable. Those effects are usually stochastic, implying a model for the stochastic behaviour of convenience yield. As a key assumption in the model, electricity is therefore modelled as an asset with stochastic (positive or negative) dividend yield $\delta_t$ which itself follows a mean-reversion process of the Ornstein–Uhlenbeck type.

For simplicity, the market price of convenience yield risk is assumed to be zero.

\section{Solution for $\log S(t)$}

If we consider $B(S, \delta, t)$ to be a twice continuously differentiable function, we obtain from Itô’s lemma

$$ dB = B_SdS + B_\delta d\delta + B_t dt + \frac{1}{2} B_{SS} (dS)^2 + \frac{1}{2} B_{\delta\delta} (d\delta)^2 + B_{S\delta} dSd\delta $$

(2.51)

where the subscripts denote the respective partial derivatives (see [8]).

With $B(S, \delta, t) = \log S$ and (2.49) we obtain immediately

$$ d\log S_t = (r - \delta_t - \frac{1}{2} \sigma_S^2) dt + \sigma_S dW_t^S $$

(2.52)

Integration from $t_0$ to $t$ yields

$$ \log S_t = \log S_{t_0} + \left( r - \frac{1}{2} \sigma_S^2 \right) (t - t_0) - \int_{t_0}^t \delta(y) dy + \sigma_S \int_{t_0}^t dW_s^S $$

(2.53)

Equation (2.50) can be written as

$$ d\delta_t + \alpha\delta_t dt = \alpha\kappa_\delta dt + \sigma_\delta dW_t^\delta $$

(2.54)

Using (2.10) this can be integrated and we obtain

$$ \delta(y) = \kappa_\delta + (\delta_{t_0} - \kappa_\delta) e^{-\alpha(y-t_0)} + \sigma_\delta \int_{t_0}^y e^{-\alpha(y-s)} dW_s^\delta $$

(2.55)

\footnote{Note that in [8] the two-factor model was applied to oil prices.}
and thus
\[ \int_{t_0}^{t} \delta(y) dy = \kappa \delta(t - t_0) + (\delta_0 - \kappa \delta) \frac{1 - e^{-\alpha(t - t_0)}}{\alpha} + \sigma \delta \int_{t_0}^{t} e^{-\alpha(y - s)} dy dW_s^{\delta} \] (2.56)

The integral on the right hand side of (2.56) can be written as
\[ \int_{t_0}^{t} \left[ e^{-\alpha y} \int_{t_0}^{y} e^{\alpha s} dW_s^{\delta} ds \right] dy \] (2.57)

and therefore it can be evaluated using integration by parts. As final result for the integral over \( \delta \) we obtain
\[ \int_{t_0}^{t} \delta(y) dy = \kappa \delta \tau + (\delta_0 - \kappa \delta) \frac{1 - e^{-\alpha \tau}}{\alpha} + \sigma \delta \int_{t_0}^{t} 1 - e^{-\alpha(t - s)} dW_s^{\delta} \] (2.58)

where \( \tau = t - t_0 \). Equation (2.53) can now be rewritten as
\[ \log S_t = \log S_{t_0} + \left( r - \frac{1}{2} \sigma_S^2 - \kappa \delta \right) \tau - (\delta_0 - \kappa \delta) \frac{1 - e^{-\alpha \tau}}{\alpha} + X - Y \] (2.59)

where the stochastic variables \( X \) and \( Y \) are given by
\[ X = \sigma_S \int_{t_0}^{t} dW_s^S = \sigma_S (W_t^S - W_{t_0}^S) \] (2.60)
\[ Y = \sigma \delta \int_{t_0}^{t} 1 - e^{-\alpha(t - s)} dW_s^{\delta} \] (2.61)

Since the increments \( dW^S \) and \( dW^{\delta} \) are correlated, \( X \) and \( Y \) are correlated as well. Provided that \( X \) and \( Y \) are not perfectly correlated or anticorrelated\(^3\) we can decorrelate the random variables by performing the transformation
\[ \left( \begin{array}{c} dW^S \\ dW^{\delta} \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right) \left( \begin{array}{c} dX' \\ dY' \end{array} \right) \] (2.62)

The transformed increments \( dX' \) and \( dY' \) are uncorrelated and their variances are given by
\[ E[dX'dX'] = (1 + \rho) dt \] (2.63)
\[ E[dY'dY'] = (1 - \rho) dt \] (2.64)

With the new increments we obtain
\[ X' = \frac{1}{\sqrt{2}} \int_{t_0}^{t} \left( \sigma_S - \frac{\sigma \delta}{\alpha} (1 - e^{-\alpha(t - s)}) \right) dX_s' \] (2.65)
\[ Y' = \frac{1}{\sqrt{2}} \int_{t_0}^{t} \left( \sigma_S + \frac{\sigma \delta}{\alpha} (1 - e^{-\alpha(t - s)}) \right) dY_s' \] (2.66)

\(^3\)In this case the process could be regarded as a one–factor process.
Equation (2.55) can be written as

$$\delta_t = \kappa\delta + (\delta_{t_0} - \kappa\delta)e^{-\alpha\tau} + Z$$  \hspace{1cm} (2.67)

where the random variable $Z$ is given by

$$Z = \sigma\delta \int_{t_0}^{t} e^{-\alpha(t-s)} dW_s$$  \hspace{1cm} (2.68)

$$= \frac{\sigma\delta}{\sqrt{2}} \int_{t_0}^{t} e^{-\alpha(t-s)} dX_s' + \frac{\sigma\delta}{\sqrt{2}} \int_{t_0}^{t} e^{-\alpha(t-s)} dY_s'$$  \hspace{1cm} (2.69)

$$= Z'_{X} + Z'_{Y}$$  \hspace{1cm} (2.70)

There is full correlation between $X'$ and $Z'_X$, and between $Y'$ and $Z'_Y$. The variances of the random variables can be calculated by evaluating the following integrals:

$$\text{var}[X'] = \frac{1 + \rho^2}{2} \int_{t_0}^{t} \left[ \sigma_S - \frac{\sigma\delta}{\alpha} (1 - e^{-\alpha(t-s)}) \right]^2 ds$$  \hspace{1cm} (2.71)

$$\text{var}[Y'] = \frac{1 - \rho^2}{2} \int_{t_0}^{t} \left[ \sigma_S + \frac{\sigma\delta}{\alpha} (1 - e^{-\alpha(t-s)}) \right]^2 ds$$  \hspace{1cm} (2.72)

$$\text{var}[Z'_{X}] = \frac{1 + \rho^2}{2} \int_{t_0}^{t} \sigma^2_{\delta} e^{-2\alpha(t-s)} ds$$  \hspace{1cm} (2.73)

$$\text{var}[Z'_{Y}] = \frac{1 - \rho^2}{2} \int_{t_0}^{t} \sigma^2_{\delta} e^{-2\alpha(t-s)} ds$$  \hspace{1cm} (2.74)

In summary we obtain the following results:

$$\log S_t = \log S_{t_0} + \beta(\tau) + X' - Y'$$  \hspace{1cm} (2.75)

$$\delta_t = \eta(\tau) + Z'_X + Z'_Y$$  \hspace{1cm} (2.76)

where

$$\beta(\tau) = (r - \frac{1}{2} \sigma^2_S - \kappa\delta)\tau - (\delta_{t_0} - \kappa\delta) \frac{1 - e^{-\alpha\tau}}{\alpha}$$  \hspace{1cm} (2.77)

$$\eta(\tau) = \kappa\delta + (\delta_{t_0} - \kappa\delta)e^{-\alpha\tau}$$  \hspace{1cm} (2.78)

$$\tau = t - t_0$$  \hspace{1cm} (2.79)

The random variables $X'$, $Y'$, $Z'_X$ and $Z'_Y$ are normally distributed with mean zero and their variances are given by

$$\text{var}[X'] = \frac{1 + \rho^2}{2} \left[ \gamma^2_+ \tau + 2 \gamma_+ \frac{\sigma\delta}{\alpha^2} (1 - e^{-\alpha\tau}) + \frac{\sigma^2_{\delta}}{2\alpha^3} (1 - e^{-2\alpha\tau}) \right]$$  \hspace{1cm} (2.80)

$$\text{var}[Y'] = \frac{1 - \rho^2}{2} \left[ \gamma^2_- \tau - 2 \gamma_- \frac{\sigma\delta}{\alpha^2} (1 - e^{-\alpha\tau}) + \frac{\sigma^2_{\delta}}{2\alpha^3} (1 - e^{-2\alpha\tau}) \right]$$  \hspace{1cm} (2.81)

$$\text{var}[Z'_X] = \frac{1 + \rho^2}{4\alpha} \sigma^2_{\delta} (1 - e^{-\alpha\tau})$$  \hspace{1cm} (2.82)

$$\text{var}[Z'_Y] = \frac{1 - \rho^2}{4\alpha} \sigma^2_{\delta} (1 - e^{-\alpha\tau})$$  \hspace{1cm} (2.83)
where

\[ \gamma_{\pm} = \sigma_S \pm \frac{\sigma_d}{\alpha} \]  

(2.84)

### 2.2.3 Forward Price and Vanilla Call Option

We can rewrite (2.75) as

\[ S(t) = S(t_0)e^{\beta(\tau)}e^\chi \]  

(2.85)

where the new random variable

\[ \chi = X' - Y' \]  

(2.86)

is normally distributed with mean zero and variance

\[ \text{var}[\chi] \equiv v_{\chi} = \text{var}[X'] + \text{var}[Y'] \]  

(2.87)

where \( \text{var}[X'] \) and \( \text{var}[Y'] \) are given by (2.80) and (2.81), respectively. Thus we can calculate the forward price and the price of a vanilla call option according to (2.23) and (2.33), respectively, and obtain the following results:

**Forward Price**

The forward price of \( S \) at time \( t_0 \) with respect to expiry \( t \) is given by

\[ F(S, t_0, t) = S(t_0)e^{\beta(\tau)}e^{\frac{v_{\chi}}{2}} \]  

(2.88)

where \( \beta(\tau) \) and \( v_{\chi} \) are given by (2.77) and (2.87), respectively.

**Vanilla Call Option**

The value at time \( t_0 \) of a vanilla call option with strike \( K \) and expiry \( t \) is given by

\[ c(S, t_0, t) = S(t_0)e^{\beta(\tau)}e^{\frac{v_{\chi}}{2}}N_{\epsilon\chi}(d_1) - KN_{\epsilon\chi}(d_2) \]  

(2.89)

where

\[ d_1 = \beta(\tau) + \log \left( \frac{S(t_0)}{K} \right) + v_{\chi} \]  

(2.90)

\[ d_2 = \beta(\tau) + \log \left( \frac{S(t_0)}{K} \right) \]  

(2.91)

and \( N_{\epsilon\chi}(\cdot) \) is given by (2.34).
Chapter 3

Least Squares Monte Carlo for Swing Options

Monte–Carlo methods are very popular in practical finance, since they are – in general – easy to implement and allow the treatment of problems with high dimensionality. In particular, when there are multiple stochastic factors, numerical methods like finite–differences or binomial techniques become impractical while Monte Carlo is still appropriate. Since the convergence of Monte Carlo is fairly slow, other methods are preferred as long as the underlying stochastic process is simple and the number of risk factors is small.

In the case of energy derivatives appropriate stochastic processes for the underlying, e.g. the electricity price, are in general considerably complicated, since they have to account for seasonality, mean reversion, spikes etc (see §1.2.1). Calibration of a model for the energy price to real market data usually requires at least two stochastic factors, frequently in conjunction with jump diffusion. Therefore Monte–Carlo methods may be the best choice if the stochastic processes under consideration are expected to be close to reality.

However, the treatment of early exercise features is a great challenge for Monte–Carlo methods. For American and Bermudan options several approaches to this problem have been discussed in the literature. Some authors use stratification or parametrization techniques to approximate the transitional density function or the early exercise boundary (see references in [14]). Others treat the problem in a different way. They focus directly on the conditional expectation function involved in the iterations of dynamic programming and use least squares regression to estimate the conditional expectations. These conditional expectations, i.e. the continuation values under the assumption that the option is not exercised at a particular opportunity (iteration step) are estimated by least squares regression.
One example for these Least Squares Monte Carlo (LSM) methods is the algorithm proposed by Longstaff and Schwartz [14] which seems to have become more and more popular among practitioners [6]. In the first part of the present chapter the Longstaff Schwartz algorithm for American or Bermudan options is described.

The extension of this algorithm to Swing options with more than one exercise right is discussed in the second part, including the presence of upswings, downswings and penalty functions. As a central issue of the present thesis the extended algorithm has been implemented in MATLAB and the routines involved are sketched in this part.

The third part addresses some computational results which have been obtained using the routines outlined in the second part of this chapter. These results include simulations for both the one factor and the two factor processes described in Chapter 2. In particular, an upper bound for Swing options is obtained from the numerical experiment.

3.1 The Longstaff–Schwartz Algorithm for American and Bermudan Options

3.1.1 Theory

The basic idea of the Longstaff Schwartz algorithm, described in detail in [14] (and similar approaches like those reported in [6]), is to use least squares regression on a finite set of functions as a proxy for conditional expectation estimates.

In a first step, the time axis has to be discretized, i.e. if the American option is alive within the time horizon \([0, T]\) early exercise is only allowed at discrete times \(0 < t_1 < t_2 < \ldots < t_J = T\). The American option is thus approximated by a Bermudan option.

For a particular exercise date \(t_k\), early exercise is performed if the payoff from immediate exercise exceeds the \textit{continuation} value, i.e. the value of the (remaining) option if it is not exercised at \(t_k\). This continuation value can be expressed as conditional expectation of the option payoff with respect to the risk neutral pricing measure \(Q\). The expectation is taken conditional on the information set \(\mathcal{F}_{t_k}\) which is available at \(t_k\). Representing the continuation value for a particular sample path \(\omega\) by \(F(\omega, t_k)\) we can write

\[
F(\omega, t_k) = E_Q \left[ \sum_{j=k+1}^{K} D(t_k, t_j) C(\omega, t_j, t_k, T) | \mathcal{F}_{t_k} \right] \tag{3.1}
\]
where $D(t_k, t_j)$ is the discount factor from $t_k$ to $t_j$ and $C(\omega, t_j, t_k, T)$ denotes the path of cashflows generated by the option, conditional on the option not being exercised at or prior to time $t_k$ and the holder following the optimal exercise strategy for all remaining opportunities $t_j$ between $t_k$ and $T$. Note that for each path $\omega$ there is at most one $j$ with $C(\omega, t_j, t_k, T) > 0$, since the Bermudan option has only one exercise right.

Starting at $t_J$ we work backwards through time. The early exercise decision at time $t_{J-1}$ is made by comparing $F(\omega, t_{K-1})$ with the immediate payoff $P(X)$ where $X$ is the value of the underlying at time $t_{J-1}$ in path $\omega$. While $P(X)$ is known the functional form of $F(\omega, t_{J-1})$ is unknown.

Assuming some technical details, the conditional expectation can be represented by a set of basis functions $B_j$ as

$$F(\omega, t_{K-1}) = \sum_{j=0}^{\infty} a_j(t_{K-1})B_j(X)$$

(3.2)

Since we consider a Markov process for the state variable $X$, only current realizations of it can be included in the basis functions $B_j$. For practical purposes $F(\omega, t_{J-1})$ has to be approximated by $F_M(\omega, t_{J-1})$ using the first $M < \infty$ basis functions.

At this stage the crucial step is to be done, namely estimating $F_M(\omega, t_{J-1})$ by regressing the discounted values of $C(\omega, s, t_{J-1}, T)$, i.e. the cashflows which occur at $t_J$ – onto the basis functions. Since the early exercise decision is only relevant for those paths where the option is in the money at $t_{J-1}$ the regression is restricted to these paths. This regression yields $\hat{F}_M(\omega, t_{J-1})$ as an (unbiased) estimator for $F_M(\omega, t_{J-1})$.

Now the exercise decision is made by comparing $P(X)$ with $\hat{F}_M(\omega, t_{J-1})$ and with these new cashflows at $t_{J-1}$ the iteration steps further backwards in time (see next section). At the end the result obtained by Least–Squares Monte Carlo, $V_{\text{LSM}}$, is the average over the cashflows from each path (note that there is at most one cashflow per path),

$$V_{\text{LSM}}^{N} = \frac{1}{N} \sum_{i=1}^{N} C_{\text{LSM}}(\omega_i)$$

(3.3)

where $C_{\text{LSM}}(\omega_i)$ denotes the discounted cashflows which result from following the LSM strategy.

Since the LSM method represents one particular strategy the “real” option value $V$ (which represents the optimal strategy) must be greater than or equal to $V_{\text{LSM}}^{N}$. The convergence of $V_{\text{LSM}}^{N}$, i.e. the proposition that for any $\epsilon > 0$ there exists an $M < \infty$
such that
\[ \lim_{N \to \infty} \Pr \left[ |V - V_{LSM}^N| > \epsilon \right] = 0 \]  
(3.4)
is proved mathematically by the authors of [14] for the case where there are only two exercise opportunities. A general proof of the convergence of the Longstaff Schwartz algorithm has been given in [6].

### 3.1.2 The Algorithm

In this section the Longstaff Schwartz algorithm is described explicitly. For the sake of simplicity it is assumed that all interest rates are zero and therefore discounting can be omitted. We will use this assumption throughout the rest of the thesis (see §2.1).

Before starting the actual algorithm the paths for the underlying spot prices have to be sampled. For \( N \) paths and \( J \) exercise opportunities (timesteps), this yields an \( N \times J \) matrix \( S \) where the matrix element \( S_{ij} \) is the spot price in the \( i \)-th path at time \( t_j \).

As a second preparation step the set of basis functions \( \{ B_j \}_{j=0}^M \) for the regression has to be chosen from a great variety of possibilities such as Hermite, Legendre, Chebyshev, Gegenbauer or Jacobi polynomials, for example. However, Longstaff and Schwartz emphasize that their numerical tests indicate that Fourier or trigonometric series and even simple powers of the state variables also give accurate results.

The initial step of the actual algorithm is to determine the cashflow vector \( C^J \) at the last timestep \( t_J \). These cashflows are easy to get since the continuation values are then zero, i.e.
\[ C_i^J = P(S_{iJ}) \]  
(3.5)
where \( P \) is the payoff function. In the following we focus on the payoff of a vanilla call option
\[ P(S_{ij}) = \max(S_{ij} - K_j, 0). \]  
(3.6)
where the strike prices \( K_j \) can vary from timestep to timestep.

Second, we consider the spot prices at timestep \( t_{J-1} \) and select those for which \( P(S_{iJ-1}) > 0 \). This yields the \( L_{J-1} \times 1 \)-vector \( \hat{S}_{J-1} \) where \( L_{J-1} \) is number of in-the-money paths at timestep \( t_{J-1} \). The least squares regression of \( C^J \) onto the basis functions \( B_j \) is now performed by minimizing the expression
\[ ||B^{J-1}a^{J-1} - C^J|| \]  
(3.7)
where \( a^{J-1} \) is the \((M + 1) \times 1\)–vector of regression coefficients for timestep \( t_{J-1} \) and the matrix \( B^{J-1} \) is given by

\[
B^{J-1} = \begin{pmatrix}
B_0(\hat{S}_{1J-1}) & \cdots & B_M(\hat{S}_{1J-1}) \\
\vdots & \ddots & \vdots \\
B_0(\hat{S}_{LJ-1J-1}) & \cdots & B_M(\hat{S}_{LJ-1J-1})
\end{pmatrix}
\] (3.8)

The solution of the minimization is given by

\[
a^{J-1} = \left( (B^{J-1})^T B^{J-1} \right)^{-1} (B^{J-1})^T C^{J-1}
\] (3.9)

With that we obtain the vector of continuation values \( \text{Cont}^{J-1} \) by

\[
\text{Cont}^{J-1}_i = \sum_{k=0}^{M} a^{J-1}_k B^{J-1}_k
\] (3.10)

Once we have the continuation values we perform early exercise whenever

\[
P(\hat{S}_{iJ-1}) > \text{Cont}^{J-1}_i
\] (3.11)

The elements \( C^{J-1}_i \) of the cashflow vector \( C^{J-1} \) are then given by

\begin{itemize}
  \item \( P(\hat{S}_{iJ-1}) \) if the early exercise condition (3.11) is true
  \item 0 else
\end{itemize}

Subsequently the elements of the cashflow vector \( C^J \) have to be set to zero for those paths where (3.11) is true.

We then step backwards through time until we reach the first timestep. At each timestep early exercise is performed as described above. Note that whenever a cashflow at timestep \( t_k \) is generated by early exercise in path \( i \) all cashflows which occur in this path later than \( t_k \) (this is at most one) have to be removed.

At the end we can build the cashflow matrix \( C \) from the cashflow vectors \( C^k \) by concatenating the cashflow vectors \( C^k, k = 1, \ldots, J \) and the option value is given by the arithmetic average of the row sums.\(^1\)

In [14] a very helpful illustration of the algorithm is given by a numerical example.

\(^1\)These sums have at most one non-zero addend.
3.2 Extension of Longstaff-Schwartz to Swing Options

For the valuation of Swing options the basic concepts of Least-Squares Monte Carlo can be directly adopted. Since we now have more than one exercise right, however, we have to deal with an additional “dimension”, i.e. the number of exercises left. In order to avoid overformalization the extension of LSM to Swing options is first described using a particular example, i.e. a Swing option with 5 opportunities and 3 exercise rights (upswings). After this the algorithm is generalized by introducing downswings and penalty functions. In the last part of this section the MATLAB implementations of the algorithms are sketched.

3.2.1 Illustrative Example

We consider a Swing option with exercise opportunities at times $t_1$, $t_2$, $t_3$, $t_4$ and $t_5$. There are 3 upswings, and the strike price at each opportunity is $K$. Sampling $N$ paths yields the $N \times 5$-spot price matrix $S$.

The main difficulties arising from the presence of more than one exercise rights are the following:

- The benefit from immediate exercise is not only the payoff, but the payoff plus the value of the remaining Swing option (which has one upswing fewer than the original one);
- When early exercise is performed at $t_k$, rearranging the cashflows at later opportunities requires the cashflow matrix of the Swing option with one upswing less than the original one.

The generalized cashflow matrix of our algorithm must therefore have three dimensions:

- First dimension: number of paths
- Second dimension: number of timesteps (exercise opportunity)
- Third dimension: number of exercise rights (upswings) left

We denote the cashflow matrix for $e$ upswings\(^2\) left as $C^e$. In our example, there are thus three $N \times 5$ matrices $C^1$, $C^2$ and $C^3$.

\(^2\)Note that $e$ is here an integer and not the base of natural logarithms
After the initial step the cashflow matrices look as follows:

\[
C^3 = \begin{pmatrix}
% & % & P(S_{13}) & P(S_{14}) & P(S_{15}) \\
% & % & P(S_{23}) & P(S_{24}) & P(S_{25}) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
% & % & P(S_{N3}) & P(S_{N4}) & P(S_{N5})
\end{pmatrix} \quad (3.12)
\]

\[
C^2 = \begin{pmatrix}
% & % & % & P(S_{14}) & P(S_{15}) \\
% & % & % & P(S_{24}) & P(S_{25}) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
% & % & % & P(S_{N4}) & P(S_{N5})
\end{pmatrix} \quad (3.13)
\]

\[
C^1 = \begin{pmatrix}
% & % & % & % & P(S_{15}) \\
% & % & % & % & P(S_{25}) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
% & % & % & % & P(S_{N5})
\end{pmatrix} \quad (3.14)
\]

where

\[
P(S) = \max(S - K, 0) \quad (3.15)
\]

is the payoff of the upswing. The \%-signs mean that these cashflows are undefined at this stage.

For \(C^3\) we can combine the last three timesteps in the initial step of the algorithm since it is obvious that early exercise takes place at \(t_3\) whenever the payoff at this timestep is positive. Note that this is the third timestep from the last.

Similarly, when two upswings are left immediate early exercise is performed at \(t_4\) and thus we can combine the last two timesteps for \(C^2\).

The matrix \(C^1\) corresponds to the cashflow matrix in the Longstaff Schwartz algorithm for Bermudan options.

As an example, if we have only six paths, after the initial step the matrices might look as follows:

\[
C^3 = \begin{pmatrix}
% & % & P_{13} & 0 & P_{15} \\
% & % & P_{23} & P_{24} & 0 \\
% & % & P_{33} & 0 & 0 \\
% & % & P_{43} & P_{44} & P_{45} \\
% & % & P_{53} & 0 & P_{55} \\
% & % & P_{63} & P_{64} & P_{65}
\end{pmatrix} \quad (3.16)
\]
\[ C^2 = \begin{pmatrix} \% \% \% 0 & P_{15} \\ \% \% \% P_{24} & 0 \\ \% \% \% 0 & 0 \\ \% \% \% P_{44} & P_{45} \\ \% \% \% 0 & P_{55} \\ \% \% \% P_{64} & P_{65} \end{pmatrix} \]  
(3.17)

\[ C^1 = \begin{pmatrix} \% \% \% \% P_{15} \\ \% \% \% \% 0 \\ \% \% \% \% 0 \\ \% \% \% \% P_{45} \\ \% \% \% \% P_{55} \\ \% \% \% \% P_{65} \end{pmatrix} \]  
(3.18)

Here all \( P_{ij} = P(S_{ij}) \) are non zero.

We now start stepping backwards in time. For \( t_4 \) we calculate the continuation values for one upswing left by least squares regression of the cashflow vector \( \hat{C}^1_{15} \) onto the basis functions as described in §3.1.2. In \( \hat{C}^1_{15} \) only the paths where the payoff at \( t_4 \) is positive are considered (see also §3.1.2). We denote the vector of continuation values at timestep four with one upswing left as \( \text{Cont}^1_4 \).

With the continuation values we can perform early exercise and \( C^1 \) may look like

\[ C^1 = \begin{pmatrix} \% \% \% 0 & P_{15} \\ \% \% \% P_{24} & 0 \\ \% \% \% 0 & 0 \\ \% \% \% P_{44} & 0 \\ \% \% \% 0 & P_{55} \\ \% \% \% P_{64} & 0 \end{pmatrix} \]  
(3.19)

In our example, early exercise at \( t_4 \) was carried out for all possible paths, i.e. paths 2, 4 and 6. Note that for paths 4 and 6 the cashflows at \( t_5 \) had been removed. The cashflow matrices \( C^2 \) and \( C^3 \) remain unchanged in this step.

Now we move on to \( t_3 \). In order to get \( \text{Cont}^2_3 \) we first have to add the cashflow vectors \( C^2_4 \) and \( C^2_5 \). Denoting the sum vector as \( C^2_{4+5} \) we obtain the relevant sum vector \( \hat{C}^2_{4+5} \) by omitting all paths where \( P(S_{i3}) \) is zero. The continuation vector \( \text{Cont}^2_3 \) is then obtained by linear regression of \( \hat{C}^2_{4+5} \) on the basis functions.

The early exercise condition now reads

\[ P(S_{i3}) + \text{Cont}^1_3(i) > \text{Cont}^2_3(i) \]  
(3.20)

That means we have to calculate \( \text{Cont}^1_3 \) before we can perform early exercise in \( C^2 \). This calculation is easily done according to the usual Longstaff Schwartz algorithm.
For those paths where condition (3.20) is fulfilled early exercise is performed. This means that for each corresponding path \( i \), \( C_3^2(i) \) is set equal to the payoff \( P(S_{i3}) \) and the cashflows \( C_3^2(i) \) and \( C_5^2(i) \) are replaced by \( C_1^1(i) \) and \( C_6^1(i) \), respectively.

After this early exercise for \( t_3 \) is performed in \( C_1^1 \). While \( C_3^1 \) still remains unchanged in this step the other cashflow matrices in our example might look like

\[
C^2 = \begin{pmatrix}
  % & % & P_{13} & 0 & P_{15} \\
  % & % & P_{23} & P_{24} & 0 \\
  % & % & P_{33} & 0 & 0 \\
  % & % & 0 & P_{44} & P_{45} \\
  % & % & 0 & 0 & P_{55} \\
  % & % & P_{63} & P_{64} & 0
\end{pmatrix}
\] (3.21)

\[
C^1 = \begin{pmatrix}
  % & % & 0 & 0 & P_{15} \\
  % & % & P_{23} & 0 & 0 \\
  % & % & 0 & 0 & 0 \\
  % & % & 0 & P_{44} & 0 \\
  % & % & 0 & 0 & P_{55} \\
  % & % & P_{63} & 0 & 0
\end{pmatrix}
\] (3.22)

For \( C^2 \) early exercise was performed in paths 1, 2, 3 and 6. Note that in path 6 the cashflows after \( t_3 \) had to be modified according to \( C^1 \) at the iteration step before, i.e. \( t_4 \) (see §3.19). For \( C^1 \) early exercise occurs only in paths 2 and 6.

Moving on to \( t_2 \) early exercise must be performed for all three cashflow matrices. First, the continuation vectors are calculated by regressing \( \hat{C}_{e,3+4+5}^e, e = 1, 2, 3 \) onto the basis functions as described above. Then early exercise is carried out starting with \( C^3 \) by evaluating the condition

\[
P(S_{i2}) + \text{Cont}_3^2(i) > \text{Cont}_3^3(i). \tag{3.23}
\]

The condition for \( C^2 \) reads

\[
P(S_{i2}) + \text{Cont}_1^1(i) > \text{Cont}_2^2(i) \tag{3.24}
\]

and for \( C^1 \) we obtain

\[
P(S_{i2}) > \text{Cont}_2^1(i) \tag{3.25}
\]

Since early exercise includes rearranging the cashflows after \( t_2 \) according to the cashflow matrix (with one upswing fewer) at the preceding iteration step, it is important that the procedure is first done with \( C^3 \), then with \( C^2 \) and finally with \( C^1 \).

Repeating the same procedure for \( t_1 \), we end up with the final cashflow matrices \( C^1, C^2 \) and \( C^3 \). From these matrices we obtain the value of the corresponding Swing options by taking the average of the row sums.
Note that although we were only interested in the option with 3 upswings at the beginning, we now have valued simultaneously the options with one and two exercise rights.\footnote{It would not have been necessary to carry out early exercise for $C^2$ and $C^1$ at $t_1$ in order to get $C^3$, but omitting these steps implies no big improvement of computational efficiency.} This is very helpful with respect to the investigation of upper and lower boundaries (see §3.3.4)

### 3.2.2 General Case: Upswings, Downswings and Penalty Functions

The general form of Swing options as described in §1.1 is more complicated since Swing options can include, for example,

- not only call (upswings) but also put features (downswings)
- penalty functions which depend on the total number of exercises (upswings and downswings)

This has two main consequences for the algorithm. First, in the initial step only the last timestep can be treated since continuation values might be negative because of penalty and thus it could be sensible to let one or more exercise rights expire worthless. Second, we have an additional dimension, i.e. the number of downswings left.

Let us first introduce some notation:

- $u$ and $d$ are the numbers of upswings and downswings exercised,\footnote{Note that in the previous Section we have used the number of exercise rights left} respectively (in a particular iteration step)
- $u_{\text{max}}$ and $d_{\text{max}}$ are the total numbers of upswings and downswings, respectively
- $J$ is the number of timesteps (exercise opportunities)

The generalized cashflow tensor has now four dimensions (path, timestep, upswings exercised, downswings exercised) and consists of $[(u_{\text{max}} + 1) \cdot (d_{\text{max}} + 1) - 1]$ cashflow matrices\footnote{We do not need to calculate $C^{u_{\text{max}},d_{\text{max}}}$ since this matrix is zero} $C^{u,d}$. Each of these matrices has dimension $(N \times J)$.

In the initial step, we have to evaluate the cashflows at the last exercise opportunity. With $\phi(u,d)$ denoting the penalty function for $u$ upswings and $d$ downswings
exercised we obtain the following cashflows in path $i$ at the final timestep $t_J$:

for $0 \leq u < u_{\text{max}}$, $0 \leq d < d_{\text{max}}$:

$$C^u_d(i) = \max [P_u(S_{iJ}) - \phi(u + 1, d), P_d(S_{iJ}) - \phi(u, d + 1), 0]$$ (3.26)

for $0 \leq d < d_{\text{max}}$:

$$C^{u, \text{max}}_d(i) = \max [P_d(S_{iJ}) - \phi(u_{\text{max}}, d + 1), 0]$$ (3.27)

for $0 \leq u < u_{\text{max}}$:

$$C^{u, \text{max}}(i) = \max [P_u(S_{iJ}) - \phi(u + 1, d_{\text{max}}), 0]$$ (3.28)

where $P_{u,d}$ is the payoff of the up-, downswing.

Stepping backwards in time we have to do the following in each step:

- calculate the continuation values by least squares regression;
- perform early exercise.

When performing early exercise we have to step forward from $u, d = 0$ to $u, d = u_{\text{max}}, d_{\text{max}}$. It does not matter whether we start with $u$ or $d$, however. Explicitly, in timestep $j$ (and thus iteration step $J + 1 - j$) the early exercise conditions for the upswings are

for $0 \leq u < u_{\text{max}}$, $0 \leq d \leq d_{\text{max}}$, $u + d < j$:

$$P_u(S_{ij}) + \text{Cont}^{u+1,d}_j(i) > \text{Cont}^{u,d}_j(i)$$ (3.29)

For the downswings we obtain

for $0 \leq u \leq u_{\text{max}}$, $0 \leq d < d_{\text{max}}$, $u + d < j$:

$$P_d(S_{ij}) + \text{Cont}^{u,d+1}_j(i) > \text{Cont}^{u,d}_j(i)$$ (3.30)

Eventually the value of the Swing option is obtained by calculating the average of the row sums of $C^{0,0}$ after the final iteration step. Note that in each row there are at most $u_{\text{max}} + d_{\text{max}}$ non-zero cashflows.
3.2.3 Implementation

The implementation was performed in MATLAB. Since this software is matrix-based, matrix operations are optimized with respect to computational speed. In the particular case of Monte–Carlo simulations, one therefore has to avoid any loop over the paths but rather handle the sampled paths as matrices. A further advantage of MATLAB is the fact that it provides a great variety of pre-implemented numerical operations, in particular linear (least-squares) regression.

The implementation of the extended Longstaff-Schwartz (LS) algorithm consists of two main parts:

- the sampling of the paths;
- the least-squares algorithm.

These two parts are independent of each other and hence it is very easy to extend the tools presented below to any stochastic process desired. In this case only the sampling routine has to be modified. For the present thesis I have implemented two sampling routines, one for the one factor mean-reverting process described in §2.1 and the other for the two factor-process discussed in §2.2.

I have implemented two different routines for the LS algorithm with and without penalty functions. The reason for this is that some iteration steps are redundant if there is no penalty (see §3.2.1) and this can be used to improve computational speed. As we will see later, most of the results presented in this thesis are restricted to the absence of penalty functions.

In both cases simple powers up to $M = 2$ are used as basis functions, i.e.

\begin{align*}
B_0(X) &= 1 \\
B_1(X) &= X \\
B_2(X) &= X^2
\end{align*}

(3.31) (3.32) (3.33)

In the following the most important routines are sketched very briefly. An overview over all relevant routines which have been implemented in the context of this thesis is given in Appendix A.

Path Sampling for the One Factor Process

Since the effects of seasonality are not in the focus of the present thesis, the deterministic part $F(t)$ in (2.1) and the volatility in (2.2) are assumed to be constant, according to (2.45) and (2.46), respectively.
For the one-factor process antithetic sampling is performed. If there are $N$ paths and $J$ exercise opportunities, $N/2 \times J$ random numbers $x_{ij}$ are drawn independently from a standard normal distribution. This yields a $N/2 \times J$–matrix which is concatenated with $-x_{ij}$ along the first dimension (rows), i.e. we obtain an $N \times J$–matrix of standard normally distributed random numbers.

Multiplication of each column vector $(x)_j$ with $\sqrt{v(t_j, t_{j-1})}$ where $v(t, t_0)$ is given by (2.48) yields a matrix $X_{ij}$ of normally distributed random numbers with mean zero and variance $v(t_j, t_{j-1})$. According to (2.41), the spot-price matrix is then obtained from this matrix by element-wise taking the exponential and subsequent multiplication of each column vector with $A(t_j, t_{j-1})$ as given by (2.47).

Path Sampling for the Two Factor Process

As above we consider $N$ paths and $J$ exercise opportunities. In the case of the two factor process we have to sample two matrices $x'$ and $y'$ where the random numbers $x'_{ij}$ and $y'_{ij}$ are uncorrelated, i.e. for each $1 \leq i, k \leq N$ and $1 \leq j, l \leq J$ we have

$$E[x'_{ij}y'_{kl}] = 0$$

From $x'$ and $y'$ we obtain the matrices $X'$, $Y'$, $Z'_X$, and $Z'_Y$ by multiplication of the column vectors according to

$$(X')_j = \text{var}[X'](t_j) \cdot x'_j$$

$$(Y')_j = \text{var}[Y'](t_j) \cdot y'_j$$

$$(Z'_X)_j = \text{var}[Z'_X](t_j) \cdot x'_j$$

$$(Z'_Y)_j = \text{var}[Z'_Y](t_j) \cdot y'_j$$

where the variances are given by (2.80) to (2.83), respectively.

For each timestep $t_j$, the vectors $S_{t_j}$ and $\delta_{t_j}$ are then calculated according to (2.75) and (2.55), respectively, where the start values $\delta_{t_0}$ and log $S_{t_0}$ are used to calculate the $\delta_{t_1}$ and log $S_{t_1}$, and the $\delta_{t_{j-1}}$ and log $S_{t_{j-1}}$ are used to compute the $\delta_{t_j}$ and log $S_{t_j}$.

In the last step the spot price matrix $S$ is obtained by taking element-wise the exponential of log $S$.

\[\text{Note that this is an iterative process, i.e. } S_{t_j} \text{ cannot be calculated until } S_{t_{j-1}} \text{ is known.}\]
LS Algorithm for Upswings without Penalty

As explained in §3.2.1 early exercise is performed over as many timesteps as possible in the initial iteration step. If the total number of upswings is $U$ all cashflow matrices from 1 to $U$ exercises left are calculated in the subsequent iteration steps. For each timestep a different strike can be chosen.

LS Algorithm for Upswings and Downswings with Penalty

While upswings correspond to call option features, downswings are the put option equivalent. The LS algorithm for downswings is completely analogous to that for upswings, except the fact that the payoff function is the payoff of a put option. In the actual implementation the strikes for the downswings can be chosen independently of the strikes for the upswings.

The presence of a penalty function implies that the initial iteration steps can only cover one (i.e. the last) timestep. This means that there are more linear regressions to be performed and therefore the algorithm is slightly slower. In contrast to the case of no penalty only the result for the total number of up- and downswings is returned by the routine.

Empirical Error

In both routines the extended Longstaff Schwartz (LS) algorithm is carried out 10 times. The mean and the standard deviation of the 10 results are returned as the option’s value and as a measure for its statistical error, respectively.

3.3 Computational Results

This section contains some results obtained by the routines described above. In order to make sure that the routines yield correct results a number of numerical tests have been applied to them. This is outlined in the first part of this section.

In the second and third parts some numerical results for the one and two factor processes are presented and the influence of penalty functions is discussed.

A discussion of upper and lower boundaries for Swing options is given in the last part of this section. In particular, an upper boundary has been found in the computer experiment. This upper boundary is smaller than the set of Bermudan options which is frequently referred to in the literature.
Table 3.1: One factor mean–reverting process parameters for the convergence check.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>spot value ( S )</td>
<td>20</td>
</tr>
<tr>
<td>timesteps</td>
<td>10</td>
</tr>
<tr>
<td>time between two timesteps ( \delta t )</td>
<td>1</td>
</tr>
<tr>
<td>exercise rights (upswings)</td>
<td>6</td>
</tr>
<tr>
<td>strike at each timestep ( K )</td>
<td>20</td>
</tr>
<tr>
<td>mean reversion speed ( \alpha )</td>
<td>0.5</td>
</tr>
<tr>
<td>mean reversion level ( F )</td>
<td>20.7387</td>
</tr>
<tr>
<td>volatility ( \sigma )</td>
<td>0.392</td>
</tr>
</tbody>
</table>

3.3.1 Tests

The first step of the testing process concentrates on the sampling routines. A large number of paths (usually 1000000) was sampled and the mean and variance of the sampled spot prices have been compared with the theoretical values.

In the next step the LS routines have been applied to a vanilla call option, i.e. one timestep (exercise opportunity) and one upswing. The result was compared with the theoretical values as given by (2.44) and (2.89).

Up to this stage there have been no least squares regressions, i.e. the actual Longstaff Schwartz algorithm has not proved its accuracy yet. In order to achieve this, numerical results for the one factor process have been compared with those obtained by finite-differences as described in [18] (see §3.3.2 below). Note that the LS routine is independent of the sampling. This means that if the algorithm works for the one factor process, it works for the two factor process as well.

The two different LS routines have been tested against each other, i.e. the routine for up- and downswings with penalty has been applied to the special case of zero penalty and zero downswings.

All tests have shown positive results.

3.3.2 Results for the One-Factor Process

Convergence

In order to check the convergence of the LS algorithm the empirical error was investigated as a function of the number of paths \( N \). As a typical property of Monte Carlo methods the variance of the result (and thus the empirical error) is expected to decrease like \( N^{-\frac{1}{2}} \). The process parameters of the example chosen are given in Table 3.1.
For small numbers of paths (up to 800) the empirical error was calculated from 100 simulations, for large numbers of paths the error was obtained by calculating the variance of 10 simulations.

The result is depicted in Figure 3.1. In the logarithmic representation we approximately obtain a straight line with \( \frac{\Delta \log E}{\Delta \log N} \approx -\frac{1}{2} \) which is expected from the square root behaviour mentioned above. For \( \approx 100000 \) paths we end up with an empirical error of about 0.1%.

![Figure 3.1: Empirical error as a function of the number of paths. In this logarithmic representation we approximately obtain a straight line.](image.png)

**Numerical Results**

As a first numerical experiment, the Swing options value has been calculated for various spot prices and two different values of the mean reversion speed \( \alpha \) (see Figure 3.2). All other process parameters were kept constant according to Table 3.1. The impact of mean reversion is clearly discernible. For low spot prices and large \( \alpha \) mean reversion tends to pull the prices up to the mean reversion level which lies slightly above the strike. Therefore the option value is quite high even for a spot price as small as 0.01. The opposite is true for large spot prices when mean reversion pulls the price down and thus the options value does not increase very fast with increasing spot
price. In the case of weak mean reversion ($\alpha = 0.05$) these effects are much weaker and therefore the option behaves more like a vanilla call option with a geometric Brownian motion as underlying process.

Introducing four downswings in addition to the six upswings (and keeping all other parameters constant\(^7\)) must lead to an overall increase of the Swing options value since the new option has more exercise rights while keeping all rights of the old option. This is shown in the left part of Figure 3.3. Since the downswings are in the money for low spot prices the option’s value as a function of the spot price now exhibits a minimum for both values of $\alpha$.

As final step we introduce a penalty function $\phi$:

$$
\phi(v) = \begin{cases} 
50 & \text{for } v \leq -1 \\
30(v - 2) & \text{for } v > 2 \\
0 & \text{otherwise}
\end{cases}
$$

where

$$v = u - d$$  \hspace{1cm} (3.40)

\(^7\)the strikes of the downswings were set equal to the strikes of the upswings
and $u(d)$ is the total number of upswings (downswings) exercised. It is evident that the penalty must lead to an overall decrease of the option value. This is shown in the right part of Figure 3.3.

### Comparison with Finite Differences

Figure 3.4 shows a typical result of the comparison between the two methods. The relative deviations are significantly smaller than one percent and thus lie within the numerical accuracy. For the Monte Carlo results, the accuracy is about 0.3%. However, the accuracy for the finite–difference calculations is not known exactly.

#### 3.3.3 Results for the Two Factor Process

### Convergence

The convergence check for the two factor process has been performed in the same way as for the one factor process (see §3.3.2). Table 3.3.3 shows the process parameters for the calculated example.

As for the one factor process we obtain an approximately linear curve for the empirical error as a function of the number of paths with $\frac{\Delta \log E}{\Delta \log N} \approx -\frac{1}{2}$ which indicates the $\sqrt{N}^{-1}$ law for Monte Carlo simulations. However, as expected the factor of proportionality is larger than for the one factor process. For example, we end up with an empirical error of $\approx 0.5 \%$ for 100000 paths which is significantly larger
Figure 3.4: Relative difference between the Swing option values obtained by Least Squares Monte Carlo and Finite Differences. The calculations have been performed for a Swing option with 10 opportunities and mean-reversion speeds of 0.05 (left, no downswings) and 0.5 (right, 4 downswings). All other parameters are the same as in Table 3.1

than \( \approx 0.1 \% \) in the case of the one factor process. A further reason for the slower convergence is that unlike the one factor sampling routine, no antithetic sampling is done in the two factor routine.

**Numerical Results**

With all other parameters given in Table 3.3 Swing option values have been calculated for:

- six opportunities, four upswings, no downswings and no penalty;
- six opportunities, four upswings, two downswings and no penalty;
- six opportunities, four upswings, two downswings and penalty according to (3.39)

The results are similar to those for the one factor process (see Figure 3.6).

**3.3.4 Upper and Lower Boundaries**

Since the numerical valuation of Swing options is – in general – quite costly, one tries to find approximation methods which are as simple as possible. In this context it is important to find upper and lower bounds which can be considered as a first approximation step.
From simple considerations one can deduce the following boundaries for a swing option with $m$ exercise rights and $N$ opportunities:

- **Upper boundary**: $m$ Bermudan options (each of them with $N$ opportunities according to the opportunities of the Swing option);

- **Lower boundary**: Callstrip, i.e. the sum of the $m$ most valuable in the set of the $N$ vanilla call options which expire at the $N$ opportunities of the Swing option.

These boundaries are frequently discussed in the literature (see [10], for example) and can be explained in the following way:

- **Upper boundary**: A holder of $m$ Bermudan options can exercise in the same way a Swing option’s holder can. Furthermore he has the right of exercising more than one option at the same opportunity. This means that he has more possibilities than the holder of the Swing option and thus a set of $m$ Bermudan options is an upper boundary for the Swing option.

- **Lower boundary**: The holder of a Swing option can exercise at each opportunity while the holder of the callstrip is restricted to $m$ exercise dates which are fixed at the beginning. The Swing option’s holder can exercise at these $m$ dates, but he doesn’t have to. Thus the callstrip must be a lower boundary.

In the next step we want to investigate where the Swing option’s value is situated between the two boundaries. We therefore introduce the position $p$:

$$p = \frac{\text{value of Swing option} - \text{lower boundary}}{\text{upper boundary} - \text{lower boundary}}$$

### Table 3.2: Two factor mean–reverting process parameters for the convergence check.

For the definitions of the parameters see §2.2.1

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot value $S$</td>
<td>20</td>
</tr>
<tr>
<td>Timesteps</td>
<td>6</td>
</tr>
<tr>
<td>Time between two timesteps $\delta t$</td>
<td>1</td>
</tr>
<tr>
<td>Exercise rights (upswings)</td>
<td>4</td>
</tr>
<tr>
<td>Strike at each timestep $K$</td>
<td>20</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.5</td>
</tr>
<tr>
<td>$\delta_0$</td>
<td>0.05</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.03</td>
</tr>
<tr>
<td>$\sigma_S$</td>
<td>0.4</td>
</tr>
<tr>
<td>$\sigma_\delta$</td>
<td>0.05</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.5</td>
</tr>
<tr>
<td>$r$</td>
<td>0.04</td>
</tr>
</tbody>
</table>
Figure 3.5: Empirical error as a function of the number of paths for the two factor process. As for the one factor process we obtain approximately a straight line in this logarithmic representation.

It is obvious that \( p \) lies between 0 and 1. For fixed \( N \) we now consider \( p \) as a function of \( m \) and immediately find the two trivial cases

\[
\begin{align*}
p(1) &= 1 \\
p(N) &= 0
\end{align*}
\]

(3.42) (3.43)

since for \( m=1(N) \) the Swing option is the same as the Bermudan option (callstrip).

As a computer experiment, \( p(m) \) has been determined for both processes and various sets of process parameters.

Before going into the numerical results we first have a look on the straight line between the two limiting cases (3.42) and (3.43). This line \( u \) is given by

\[
u(m) = \frac{N - m}{N - 1}
\]

(3.44)

and the position \( u(m) \) corresponds to a Swing option value \( V_u(m) \) of

\[
V_u(m) = \frac{N - m}{N - 1} m \cdot \text{Bermudan} + \frac{m - 1}{N - 1} \cdot \text{Callstrip}
\]

(3.45)
Figure 3.6: Swing option value as function of the spot price for the two factor process. As for the one factor process, adding downswings leads to an overall increase and the occurrence of a minimum (middle) and subsequent introduction of a penalty leads to an overall decrease of the option value (right).

<table>
<thead>
<tr>
<th>opportunities</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>time between timesteps</td>
<td>1</td>
</tr>
<tr>
<td>strike at each opportunity</td>
<td>20</td>
</tr>
<tr>
<td>volatility</td>
<td>0.392</td>
</tr>
<tr>
<td>mean reversion level</td>
<td>20.7387</td>
</tr>
</tbody>
</table>

Table 3.3: Constant process parameters for the position calculation in the case of the one factor process.

**One Factor Process**

For two different values of the mean reversion speed $\alpha$ and four different spot prices the position $p(m)$ was determined. The other process parameters were kept constant as given in Table 3.3.

Figure 3.7 shows the results for $\alpha = 0.15$. For each spot price the value of the vanilla call option which expires at opportunity $m$ is depicted below the position $p(m)$. With increasing spot price the position first approaches the straight line from below but subsequently moves back towards lower values. It is important to notice that the straight line is not crossed but touched at best.

If we look at the vanilla call options we find that when the positions are close to the straight line, all call options have more or less the same value. With increasing difference between most expensive and cheapest call option the positions move toward lower values.

Similar results are found for $\alpha = 0.5$ (see Figure 3.8). However, the spot value for which the maximum positions are reached lies around 23 which is slightly smaller than for $\alpha = 0.15$ where it lies around 30.
<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>opportunities</td>
<td>10</td>
</tr>
<tr>
<td>time between timesteps</td>
<td>1</td>
</tr>
<tr>
<td>strike at each opportunity</td>
<td>20</td>
</tr>
<tr>
<td>$\sigma_S$</td>
<td>0.4</td>
</tr>
<tr>
<td>$\sigma_\delta$</td>
<td>0.05</td>
</tr>
<tr>
<td>$\delta_0$</td>
<td>0.05</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.03</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.5</td>
</tr>
<tr>
<td>$r$</td>
<td>0</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 3.4: Constant process parameters for the position calculation in the case of the two factor process.

**Two Factor Process**

For both processes the value of the vanilla call option exhibits typical behaviour. As a consequence of mean reversion it does not necessarily increase with increasing time to expiry. Monotonically increasing option value with increasing time to expiry is only observed for small spot prices. For spot prices larger than the mean reversion level the option value decreases monotonically with increasing time to expiry since mean reversion pulls the spot price back towards the mean reversion level.

At the crossing point between increasing and decreasing behaviour the call option value exhibits a maximum, but the dependence on the time to expiry is very weak in this regime. With the process parameters given in Table 3.4 this crossing point occurs for $S \approx 26$. As shown in Figure 3.9 the positions are then very close to the straight line while they move towards lower values when the spot price increases or decreases.

Again, the straight line is not crossed and it should be emphasized that – in all computer experiments performed for this thesis – no set of process parameters has been found for which the position was above the straight line.

**Conclusion**

In computer experiments carried out for both the one factor and the two factor mean-reverting processes, the straight line (3.45) has turned out to be an upper boundary for Swing options. For all $m$, this boundary is smaller than or equal to

$$V^B_a(m) := m \cdot \text{Bermudan}$$

(3.46)
which is frequently discussed as upper boundary in the literature. On the other hand, with suitable process parameters, i.e. those which make all corresponding call options worth more or less the same, the values of Swing the options are very close to the straight line.

Unlike the boundary (3.46) there is no simple explanation why the straight line (3.45) is indeed a theoretical boundary independent of the underlying process. In a heuristic way, it can be interpreted as follows.

If we consider a Bermudan option \((m=1)\) with \(N\) opportunities, we can regard the problem as having \(N-1\) “degrees of freedom”, since for all opportunities except the last one the holder can decide between early exercise and continuation. At the last opportunity the Bermudan option is always exercised if it is in the money.

For a Swing option, there are only \(N-m\) degrees of freedom, since it has turned into a set of \(m\) vanilla call options if no early exercise has occurred before opportunity \(N-m\). Thus we can interpret the Swing option as consisting of Bermudan parts and vanilla call parts. The weight of the Bermudan part is – according to the number of 'degrees of freedom' – given by \(\frac{N-m}{N-1}\) and, consequently, since the weights must sum up to 1, the weight of the vanilla call part is \(\frac{m-1}{N-1}\). Since the vanilla call part can – by definition – not be more valuable than the corresponding callstrip, (3.45) is an upper boundary.

Assuming that the straight line is indeed an upper boundary for all underlying processes, one can find an approximation formula for Swing options. This formula yields some value between the callstrip and the straight line. The position between those two boundaries must then depend on the spread between cheapest and most expensive corresponding vanilla call option.

However, the explicit derivation of this approximation formula is beyond the scope of this thesis, since for this purpose the investigation of a great variety of underlying processes seems to be necessary.
Figure 3.7: Position between upper and lower boundary as a function of the number of exercise rights. Beneath each position graph the values of the vanilla call options expiring at each opportunity are depicted. The positions are close to the straight line when the call option values are more or less the same for all expiry dates.
Figure 3.8: Position between upper and lower boundary as a function of the number of exercise rights for $\alpha = 0.5$ (analogous to Figure 3.7)
Figure 3.9: Position between upper and lower boundary as a function of the number of exercise rights for the two factor process. In contrast to Figure 3.7 and 3.8, the ordinates in the plots of the vanilla call options do not all have the same scale.
Chapter 4
Exercise Strategies

In the previous chapter, the valuation of Swing option was (implicitly) based on the following assumptions:

- the holder applies the **optimal** exercise strategy;
- the payoff from early exercise can be realized immediately.

However, in reality these assumptions are not necessarily fulfilled.

First, applying the optimal strategy requires knowledge at each opportunity as to whether it is better to exercise or not.

Second, if physical delivery is settled the holder may not be able to benefit from early exercise since he has to sell the electricity delivered.

In this chapter we focus on the valuation of Swing options **in terms of different exercise strategies**, i.e. we determine the expected payoff under the condition that a particular exercise strategy is applied. In particular, we try to find an approach to the value of a Swing option for a holder who

- does not know the optimal exercise strategy explicitly or;
- cannot decide by himself when to exercise since he is exposed to external constraints

In the first case, the aim is to find a simple strategy which yields an option value close to the optimal exercise value. This strategy is to be found by simple considerations about the process parameters.

An example for the second case could be the following. The holder has bought the Swing option in order to protect himself from extremely high spot prices, but he does not know in advance when his need for electricity will occur. When there is no need for electricity he cannot realize the payoff from an early exercise since he is not able
to sell the electricity delivered. Under these circumstances this holder is interested in knowing the difference between his expected payoff and the market price of the option which is assumed to be based on optimal exercise.

Throughout this chapter, we restrict ourselves to the one-factor process discussed in §2.1. In this way we keep the number of process parameters small. In particular, this allows us to investigate the interplay of mean reversion and volatility systematically. Furthermore, some aspects of the early exercise problem can be treated analytically.

In the first part of this chapter, some basic concepts of early exercise are illustrated by means of a Bermudan option with two exercise opportunities. The relationship between process parameters and early exercise is investigated in detail.

Although the example discussed in the first section is a very crude simplification compared to the complicated exercise structure of a Swing option, it is shown in the second part that some basic results can be extended to Swing options with virtually no restriction. In this section the values of Swing options in terms of various strategies are compared with each other.

4.1 Illustration of Early Exercise for a Bermudan Option with two Exercise Opportunities

In this section we consider the simplest possible case of a swing option with early exercise, i.e. a Bermudan option with two exercise opportunities at times $t_0$ and $t$. The advantage of this example is that some basic aspects of the early exercise problem can be treated analytically.

4.1.1 The Threshold for Early Exercise

At time $t_0$ the holder has to decide whether to exercise or not. If the holder decides not to exercise, the option turns into a vanilla call option. Therefore, early exercise is optimal if the payoff from realization is greater than the value of the call option, i.e.

$$c(S, t_0, t) > S - K$$

(4.1)

where $S$ and $K$ denote the spot and strike prices, respectively. In the following we keep the strike price $K$, the mean reversion level $F$ and the time to maturity $t - t_0$ constant and consider the value of the vanilla call option as a function of the spot price $S$, the mean reversion speed $\alpha$ and the volatility $\sigma$. Omitting the time arguments
\( c(S, \alpha, \sigma) = A e^{\frac{v}{2}} N_0^0(d_1) - K N_0^0(d_2) \) \hspace{1cm} (4.2)

where \( A \) and \( v \) are given in (2.47) and (2.48), respectively. With these parameters, \( d_1 \) and \( d_2 \) can be written as

\[ d_1 = \log \left( \frac{A}{K} \right) + v = d_2 + v \] \hspace{1cm} (4.3)

and \( N_0^0(.) \) is defined by (2.34). Differentiating (4.2) with respect to \( S \) yields the delta of the call option:

\[ \frac{\partial c}{\partial S} = e^{\frac{v}{2}} \frac{\partial A}{\partial S} \left( N_0^0(d_1) + \frac{1}{\sqrt{2\pi v}} e^{-\frac{d_1^2}{2}} \right) - \frac{K}{A} \frac{\partial A}{\partial S} \frac{1}{\sqrt{2\pi v}} e^{-\frac{d_2^2}{2}} \] \hspace{1cm} (4.4)

where

\[ \frac{\partial A}{\partial S} = x S^{x-1} F^{1-x} \] \hspace{1cm} (4.5)

\[ x = e^{-\alpha (t-t_0)} \] \hspace{1cm} (4.6)

Since \( 0 < x < 1 \), we obtain the following limits for \( S \to \infty \):

\[ A \to \infty \] \hspace{1cm} (4.7)

\[ \frac{\partial A}{\partial S} \to 0 \] \hspace{1cm} (4.8)

\[ d_1 \to \infty \] \hspace{1cm} (4.9)

\[ d_2 \to \infty \] \hspace{1cm} (4.10)

From that it follows directly that

\[ \lim_{S \to \infty} \frac{\partial c}{\partial S} = 0 \] \hspace{1cm} (4.11)

Figure 4.1 shows a typical plot of a call option against spot price together with the payoff from immediate exercise. Since both the call and the delta of the call are always greater than zero\(^1\) and delta approaches zero for \( S \to \infty \) the equation

\[ c(X, t_0, t) = \max(X - K, 0) \] \hspace{1cm} (4.12)

always has a solution for \( X \), i.e. there is a (unique) threshold for early exercise.\(^2\)

One can show that this threshold is always greater than the mean reversion level \( F \) (see Appendix A). This is intuitively clear – why should one exercise early if one knows that the spot price is “drawn up” towards \( F \)?

\(^1\)This is intuitively obvious, but can also easily be seen from Eq. (2.29)

\(^2\)However, the convergence \( S \to \infty \) is very slow. For example, using the same parameters as in Figure 4.1, i.e. \((\alpha, \sigma) = (0.5, 0.4)\), we still obtain \( \delta = 0.1388 \) for a spot price of 1000.
4.1.2 Dependence of the Threshold on the Process Parameters

In the next step we want to find out in which way $X$ depends on the process parameters, in particular,

- the mean reversion speed $\alpha$ and
- the volatility $\sigma$.

The early exercise condition

$$c(X, \alpha, \sigma) - X + K = 0$$  \hspace{1cm} (4.13)

can be regarded as an implicit definition of a function $f$ which depends on $(\alpha, \sigma)$ and fulfills

$$c(f(\alpha, \sigma), \alpha, \sigma) - f(\alpha, \sigma) + K = 0$$  \hspace{1cm} (4.14)
Applying the implicit value theorem to $X = f(\alpha, \sigma)$ we find explicit expressions for the partial derivatives $\frac{\partial X}{\partial \alpha}$ and $\frac{\partial X}{\partial \sigma}$ in the vicinity of some $(X_0, \alpha_0, \sigma_0)$ where $X_0 = f(\alpha_0, \sigma_0)$:

$$\frac{\partial X}{\partial \alpha}(X, \alpha, \sigma) = \frac{1}{\frac{\partial c}{\partial S}} - 1 \cdot \left( -\frac{\partial c}{\partial \alpha} \right)$$  (4.15)

$$\frac{\partial X}{\partial \sigma}(X, \alpha, \sigma) = \frac{1}{\frac{\partial c}{\partial S}} - 1 \cdot \left( -\frac{\partial c}{\partial \sigma} \right)$$  (4.16)

Since (with $t_0=0$)

$$\frac{\partial v}{\partial \alpha} = \frac{\partial}{\partial \alpha} \left( \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha t}) \right)$$  (4.17)

$$= -\frac{\sigma^2}{2\alpha^2} \left( 1 - e^{-2\alpha \wedge} <1 \right) - \frac{\sigma^2 t}{\alpha} e^{-2\alpha t}$$  (4.18)

$$< 0$$  (4.19)

and (with $x = e^{-\alpha t}$)

$$\frac{\partial A}{\partial \alpha} = -\alpha x \cdot F \log \frac{S}{F} e^{x \log \frac{S}{F}}$$

$$< 0$$  (4.20)

the expected value

$$E[S(t)|S(0)] = Ae^{\frac{v}{2}}$$  (4.22)

decreases with increasing $\alpha$ and thus the same is true for the value of the call option, i.e.

$$\frac{\partial c}{\partial \alpha} < 0$$  (4.23)

Note that we are still in the vicinity of the threshold, and therefore we can assume $S > F$. In some vicinity of the threshold delta must be smaller than one (see Figure 4.1) and thus it follows from (4.15) that

$$\frac{\partial X}{\partial \alpha} < 0$$  (4.24)

In the limit $\alpha \to \infty$ we obtain

$$A \to F$$  (4.25)

$$v \to 0$$  (4.26)

$$d_{1,2} \to \ln \frac{S}{K}$$  (4.27)

$$\mathcal{N}_v(d_{1,2}) = \int_{-\infty}^{d_{1,2}} \frac{1}{\sqrt{2\pi v}} e^{-\frac{x^2}{2v}} dx = \int_{-\infty}^{d_{1,2} \wedge} \frac{1}{2\pi} e^{-\frac{x^2}{2\wedge}} \to 1$$  (4.28)
Note that in (4.28) we have used that $S > K$ and therefore $\lim_{\alpha \to \infty} d_{1,2} > 0$. This is certainly true since we are only interested in $S = X$, and the threshold must lie above the strike.

From (4.2) it follows therefore that

$$\lim_{\alpha \to \infty} c = F - K \Rightarrow \lim_{\alpha \to \infty} X = F$$  \hspace{1cm} (4.29)

On the other hand, since $v$ increases with increasing $\sigma$, 

$$\frac{\partial v}{\partial \sigma} = \frac{\sigma}{\alpha} \left(1 - e^{-2\alpha t}\right) > 0$$  \hspace{1cm} (4.30)

and $A$ does not depend on $\sigma$, it follows from Eq. (4.16) that the expected value (4.22) increases with increasing $\sigma$, i.e.

$$\frac{\partial c}{\partial \sigma} > 0$$  \hspace{1cm} (4.31)

Therefore it follows from (4.16) that

$$\frac{\partial X}{\partial \sigma} > 0$$  \hspace{1cm} (4.32)

### 4.1.3 Summary of the Early Exercise Problem

To summarize the results of §4.1.1 and §4.1.2, we have found that the threshold for early exercise:

- always exists if $\alpha > 0$
- is always greater than the mean reversion level $F$;
- decreases with increasing $\alpha$ and approaches $F$ in the limit $\alpha \to \infty$;
- increases with increasing $\sigma$.

These results are illustrated in Figure 4.2. Intuitively it is not surprising that the threshold (which lies always above the mean reversion level) decreases with increasing mean reversion speed. If $\alpha$ is large one should immediately make use of early exercise, since the spot price is expected to be pulled down towards the mean reversion level. However, if $\sigma$ is large, there is still some hope that the spot price will rise again towards values significantly above the mean reversion level. Therefore the need for early exercise is relaxed.
Figure 4.2: Dependence of the threshold on $\alpha$ (top) and $\sigma$ (bottom). While $X$ decreases with increasing $\alpha$ (and constant $\sigma$) it increases with increasing $\sigma$ (and constant $\alpha$).

4.1.4 Interplay between Early Exercise and Option Value

For the Bermudan option described in the previous section, the holder can maximize his expected payoff by early exercise. The optimal exercise strategy is very simple: if the spot price at time $t_0$ exceeds the threshold $X$, he decides to exercise. However, applying the optimal strategy requires exact knowledge of $X$. In the simple case described above it is quite easy to determine $X$ numerically, but for more realistic situations (e.g., more than one opportunity) this may not be the case. Therefore it is important to investigate the benefit from early exercise in more detail. This means that one should get a feeling for the sensitivity of the expected option payoff to the actual strategy.

This shall now be achieved in an intuitive way. As can be seen in Figure 4.2 the cutting angle between the payoff from early exercise and the continuation function) increases with increasing $\alpha$. In Figure 4.3 this is illustrated in an even more drastic manner. For $\alpha = 0.05$, the curves are so close to each other that a suboptimal early exercise decision is expected to have virtually no impact on the expected option payoff. However, the opposite is right for $\alpha = 0.5$. In this case the sensitivity of the expected payoff with respect to the actual strategy must be significant.

---

3 That means the value of the remaining option in the case of no early exercise as a function of $S$.

4 This means exercise below the threshold or no exercise above the threshold.
The cutting angle $\phi$ is given by

$$
\phi(\alpha, \sigma) = \frac{\pi}{4} - \arctan \left( \frac{\partial c}{\partial S} \bigg|_{S=X} \right)
$$

(4.33)

In the following we use $\phi$ as an indicator for the sensitivity of the expected option payoff on the strategy. For the simple case described above it is obvious that the sensitivity increases and with increasing $\phi$. The question whether this is still true for swing options with much more complex exercise structures is investigated in the next section.

But first we should consider how $\phi$ depends on the process parameters $\alpha$ and $\sigma$. Since an analytic treatment of $\phi$ is very difficult (if possible) this has been done numerically. It has turned out that $\phi$ increases with increasing $\alpha$ while it decreases with increasing volatility.\(^5\) However, the dependence on $\alpha$ seems to be much stronger than the dependence on $\sigma$.

### 4.2 Valuation of Swing Options in Terms of Exercise Strategies

Valuation of swing options in terms of exercise strategies means calculation of the option’s expected payoff under the assumption that a particular strategy is applied. For the simple Bermudan option discussed in §4.1 a strategy is characterized by one single threshold of early exercise. In general, the description of a strategy for a swing option requires several threshold values since there are more than one opportunity

\(^5\)Again, $\alpha$ and $\sigma$ have been identified as ‘rivals’ in the one-factor mean reverting process
and more than one exercise right. These thresholds can be expressed by a threshold matrix. The matrix which corresponds to optimal exercise cannot be calculated analytically, but it can be extracted from the LSM algorithm.

Instead of extracting the (optimal) strategy from the LSM algorithm one can directly apply an arbitrary strategy to a particular path of the spot price. In this way the expected payoff under the strategy applied is determined by simply calculating the average of the path payoffs.

This is done in the second part of this section for three different strategies. The first part of the section concentrates on optimal exercise. It starts with the introduction of the strategy matrix. In a second step, it is shown how the optimal strategy is extracted from the LSM algorithm and in which way knowledge of the optimal strategy can be used to reduce computing time in special situations. The focus of the third part is on the convergence of the strategy matrix in the LSM framework. In particular, the convergence is investigated in terms of its dependence on the process parameters.

4.2.1 Optimal Strategy for Swing Options

Definition of the Strategy Matrix

In contrast to the simple case described in §4.1, swing options have in general more than one exercise opportunity and more than one exercise right. Thus a certain exercise strategy cannot be described by one single threshold. In order to define a certain strategy we introduce the strategy matrix \( \Sigma \). For \( N \) opportunities and \( m \) exercise rights this matrix has \( m \) rows and \( N \) columns, and the matrix element \( \Sigma_{ij} \) denotes the threshold for the \( i \)-th exercise at the \( j \)-th opportunity.

Obviously, not all of the \( m \times N \) matrix elements depend on the process parameters. First, for \( i > j \) the definition of a threshold does not make sense, e.g. you cannot perform the second exercise in the first timestep. Second, for \( j > N - m - 1 + i \) the optimal threshold must be the strike, since the respective exercise right expires worthless if the holder does not exercise (see §3.2.1).

For the rest of this chapter we concentrate on the case \( N=6, m=3 \). With \( K \) denoting the strike price the strategy matrix looks then as follows:

\[
\Sigma = \begin{pmatrix} X_{11} & X_{12} & X_{13} & K & K & K \\ \% & X_{22} & X_{23} & X_{24} & K & K \\ \% & \% & X_{33} & X_{34} & X_{35} & K \end{pmatrix}
\] (4.34)

Thus there are 9 threshold values which depend on the process parameters (the \% signs indicate that the corresponding matrix elements are not defined). However, if
Table 4.1: Parameters which are kept constant in the whole Chapter.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean reversion level</td>
<td>20.7387</td>
</tr>
<tr>
<td>time between two timesteps</td>
<td>1</td>
</tr>
<tr>
<td>strike at each timestep</td>
<td>20</td>
</tr>
</tbody>
</table>

Determination of the Optimal Strategy from Least-Squares Monte Carlo

As described in §3.2 the (extended) Longstaff-Schwartz algorithm calculates the cash-flow matrix from the simulated spot price matrix. If the number of paths (i.e. number of rows in the spot price and cashflow matrices) is sufficient, the threshold values $X_{ij}$ can easily be determined by just searching the smallest spot prices which imply early exercise. This has to be done for each timestep and, furthermore, in each timestep one has to distinguish between first, second and third exercise.

In the following we consider a variety of examples. In order to keep this tractable we focus our attention on the impact of $\alpha$ and $\sigma$ and keep all other parameters constant. Explicitly, Table 4.1 shows the values assumed for these parameters. Since $\sigma$ denotes the daily volatility the time unit is 1 day.\footnote{All values used here are in the same order of magnitude as those which were found in [18] as result of a parameter estimation for real market data.}

For $\alpha=0.5$ and $\sigma=0.4$ we obtain

$$
\Sigma = \begin{pmatrix}
24.98 \pm 0.04 & 24.02 \pm 0.04 & 22.75 \pm 0.05 & 20 & 20 & 20 \\
26.65 \pm 0.03 & 25.52 \pm 0.05 & 23.94 \pm 0.04 & 20 & 20 \\
29.49 \pm 0.07 & 28.13 \pm 0.08 & 26.15 \pm 0.03 & 20 \\
\end{pmatrix}
$$

(4.35)

where the errors refer to the empirical error estimated from 10 Monte Carlo simulations as described in §3.2.3. As one would expect by intuition the threshold for $j$-th exercise decreases from timestep to timestep. This decrease is relatively small up to timestep $N - m - 1 + i$ and subsequently the threshold immediately falls down to the strike.

The strategy matrix (4.35) was calculated with a start value for the spot price of $S(t_0)=30$. However, the optimal strategy must be independent of the actual start value since we consider a Markov process. The decision for or against early exercise must not depend on the value of $S$ at some earlier timestep but only on the actual value.
Table 4.2: Optimal Strategy matrices obtained from LSM with different start values. For all threshold values the empirical error is shown in addition.

This is confirmed in a striking manner by simulations performed with different start values (see Table 4.2). The simulations have been carried out with 100000 paths (50000 plus 50000 antithetic).

As can be seen in Table 4.2 the empirical errors for the threshold values are very small, for $S(t_0) > 5$ they scatter around 0.3%. For $S(t_0)=5$, the empirical errors are larger, at least for the first exercise in each timestep. The reason for this is that there are not enough paths for which the spot price exceeds the threshold and therefore the statistics for early exercise is bad. Increasing the number of paths should lead to a decrease of the empirical error even for small start values like $S(t_0)=5$.

If we look back to the simple case described in §4.1, a small empirical error of the threshold value would be expected if the cutting angle $\phi$ was quite large. Explicitly, for $\alpha=0.5$ and $\sigma=0.4$ Eq. (4.13) and (4.33) yield the following values for the threshold and the cutting angle, respectively:

$$X = 26.04 \quad (4.36)$$

$$\phi = 0.346 \quad (4.37)$$

Indeed, the cutting angle for the simple Bermudan option is large (see also Figure 4.3, right part).

At this stage it is worth emphasizing that the result from the simple Bermudan option is transferrable to the more complex swing option. Furthermore the thresholds $X_{ij}$ are of the same order of magnitude as $X$. 

<table>
<thead>
<tr>
<th>$S(t_0)$</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{11}$</td>
<td>24.78 ± 0.69</td>
<td>25.05 ± 0.06</td>
<td>24.99 ± 0.04</td>
<td>24.98 ± 0.04</td>
<td>25.05 ± 0.05</td>
</tr>
<tr>
<td>$X_{12}$</td>
<td>23.97 ± 0.10</td>
<td>24.02 ± 0.06</td>
<td>24.03 ± 0.04</td>
<td>24.02 ± 0.03</td>
<td>24.07 ± 0.04</td>
</tr>
<tr>
<td>$X_{13}$</td>
<td>22.71 ± 0.06</td>
<td>22.74 ± 0.04</td>
<td>22.77 ± 0.05</td>
<td>22.75 ± 0.04</td>
<td>22.79 ± 0.04</td>
</tr>
<tr>
<td>$X_{22}$</td>
<td>27.00 ± 0.56</td>
<td>26.69 ± 0.07</td>
<td>26.67 ± 0.04</td>
<td>26.65 ± 0.03</td>
<td>26.68 ± 0.03</td>
</tr>
<tr>
<td>$X_{23}$</td>
<td>25.49 ± 0.11</td>
<td>25.53 ± 0.04</td>
<td>25.55 ± 0.05</td>
<td>25.52 ± 0.05</td>
<td>25.53 ± 0.04</td>
</tr>
<tr>
<td>$X_{24}$</td>
<td>23.92 ± 0.03</td>
<td>23.99 ± 0.04</td>
<td>23.98 ± 0.05</td>
<td>23.94 ± 0.04</td>
<td>23.98 ± 0.04</td>
</tr>
<tr>
<td>$X_{33}$</td>
<td>30.74 ± 1.55</td>
<td>29.49 ± 0.09</td>
<td>29.49 ± 0.11</td>
<td>29.46 ± 0.07</td>
<td>29.48 ± 0.04</td>
</tr>
<tr>
<td>$X_{34}$</td>
<td>28.04 ± 0.14</td>
<td>28.10 ± 0.06</td>
<td>28.17 ± 0.05</td>
<td>28.13 ± 0.08</td>
<td>28.15 ± 0.03</td>
</tr>
<tr>
<td>$X_{35}$</td>
<td>26.18 ± 0.08</td>
<td>26.22 ± 0.06</td>
<td>26.17 ± 0.09</td>
<td>26.15 ± 0.03</td>
<td>26.18 ± 0.07</td>
</tr>
</tbody>
</table>
As explained in §4.1.4 a large cutting angle implies a large sensitivity of the expected option payoff with respect to the exercise strategy. This means that deviating from the optimal strategy should lead to a significant decrease of the expected option payoff.

In the next two sections we will investigate systematically in which way this can be transferred to the sensitivity of the Swing option’s expected payoff with respect to the actual strategy.

First however we stay with our example for a moment and consider the following: Since the optimal strategy depends only on the process parameters and not on the start value for the spot price, simulating the strategy should yield the same results as the Longstaff Schwartz algorithm. Simulating the strategy means that we use the (optimal) strategy matrix as input parameter and apply the respective strategy to the simulated paths. This yields the cashflow matrix from which the expected payoff and thus the option value can be calculated. It should be stressed that this simulation is much faster than the Longstaff Schwartz algorithm, since there are no linear regressions to be carried out.\(^7\)

Table 4.3 shows the calculated values of the swing option for different spot prices (i.e. start values in the simulation). For each spot price the value obtained by Least Squares Monte Carlo is compared with the value obtained by directly simulating the exercise strategy. As thresholds the mean values of the strategy matrix for \(S(t_0) = 30\) were used, i.e.

\[
\Sigma = \begin{pmatrix}
0.97 & 24.98 & 24.02 & 22.75 & 20 & 20 & 20 \\
13.38 & 26.56 & 25.52 & 23.94 & 20 & 20 & 20 \\
19.74 & 28.46 & 27.15 & 26.13 & 20 & 20 & 20
\end{pmatrix}
\]

\text{(4.38)}

\(^7\)Furthermore, unlike the Longstaff Schwartz algorithm the spot price matrix has to be kept in the memory only once. Early exercise is performed by comparing the spot prices with fixed threshold values as given by the strategy matrix and no continuation values have to be considered.
Table 4.4: Empirical error of the LSM swing value and the thresholds for various sets of process parameters. Beside the process parameters $\alpha$ and $\sigma$ the table contains the cutting angle, the empirical error of the swing option’s value ($\Delta$ LSM), the threshold for the simple Bermudan case ($X$), the average threshold of the strategy matrix obtained by LSM ($\bar{X}_{ij}$) and the average of the empirical error of these thresholds ($\Delta \bar{X}_{ij}$).

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>0.05</th>
<th>0.15</th>
<th>0.15</th>
<th>0.3</th>
<th>0.5</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td>0.4</td>
<td>0.6</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.025</td>
<td>0.080</td>
<td>0.099</td>
<td>0.221</td>
<td>0.346</td>
<td>0.538</td>
</tr>
<tr>
<td>$\Delta$LSM(%)</td>
<td>0.55</td>
<td>0.28</td>
<td>0.23</td>
<td>0.2</td>
<td>0.21</td>
<td>0.14</td>
</tr>
<tr>
<td>$X$</td>
<td>98.79</td>
<td>64.26</td>
<td>35.86</td>
<td>28.79</td>
<td>26.04</td>
<td>23.74</td>
</tr>
<tr>
<td>$X_{ij}$</td>
<td>67.60</td>
<td>59.78</td>
<td>36.45</td>
<td>28.74</td>
<td>25.76</td>
<td>23.42</td>
</tr>
<tr>
<td>$\Delta X_{ij}$</td>
<td>29.99</td>
<td>1.97</td>
<td>0.35</td>
<td>0.09</td>
<td>0.06</td>
<td>0.03</td>
</tr>
</tbody>
</table>

Obviously the correspondence between the two methods is virtually perfect in our example. This leads us to the following suggestion:

If a swing option has to be valued for many different spot prices, it is sufficient to carry out the Longstaff Schwartz routine for one single (but suitable) spot price. For all other spot prices the swing option can be valued by simulating the optimal strategy. The computing time could be significantly decreased by this.

The Empirical Error of the Optimal Thresholds for Different Cutting Angles

As stated above one would expect an increasing empirical error for the thresholds in the strategy matrix with decreasing cutting angle (see §4.1.4). In order to confirm this hypothesis I have calculated optimal strategy matrices for a variety of cutting angles. Since showing all thresholds would only confuse the reader I focus on the arithmetic average of the threshold values and their empirical errors.

Table 4.4 shows the results for various simulations which have been carried out with start value 20.

It is clearly discernible that the empirical error decreases with increasing cutting angle. I have, however, performed many simulations and found out that this relationship is not strict. As mentioned above, the empirical error also depends on the difference between start value and threshold. For example, for start values far above $X$ the errors of the thresholds are relatively small, even for small cutting angles.\(^8\)

\(^8\)Note that for small cutting angles the thresholds are very high and therefore start values far above the threshold are not very interesting.
Anyway, $\phi$ is a good indicator for the “sharpness” of the optimal exercise strategy.

While the empirical error of the threshold is very high for small cutting angles, the error of the Swing option value is still small (although slightly larger than for large angles). That means that convergence of the Swing value does not necessarily require convergence of the strategy matrix.

Although there is no obvious meaning of the arithmetic average of the thresholds in the strategy matrix it is noticeable that this value lies very close to the threshold $X$ for the simple Bermudan case as described in §4.1. Actually, not only the average but also each single threshold exhibits qualitatively the same dependence on $\alpha$ and $\sigma$ as $X$ (see §§4.1.2 and §4.1.3).

### 4.2.2 Comparison of Different Exercise Strategies

In this section we will now investigate how the expected payoff depends on the actual exercise strategy. In particular, we compare the following simple strategies with the optimal strategy:

- coincidental (random) exercise ($C$)
- exercise immediately ($I$)
- high threshold early exercise ($H$)

Coincidental exercise is not a proper strategy and cannot be described by a strategy matrix. This is where the holder’s exercise decision is driven by external constraints such as need for electricity. In this case he cannot optimize the option payoff by applying an appropriate strategy.

The simulation is performed as follows: first the exercise times are drawn for each simulated path, i.e. one of the 20 possible combinations ($j_1, j_2, j_3$) where $1 \leq j_1 < j_2 < j_3 \leq 6$ is chosen randomly. Thereby it is assumed that all combinations occur with the same probability. Then for each path the cashflows are evaluated by applying the function $\max(S(t_j) - K, 0)$ to the spot price at the $j$-th timestep, where $j \in \{j_1, j_2, j_3\}$. Finally the cashflows are added for each path and the average over the sums is the expected payoff.

Immediate exercise means that the holder exercises immediately when the option is in the money. The corresponding strategy matrix reads

$$
\Sigma_I = \begin{pmatrix}
  K & K & K & K & K & K \\
  \% & K & K & K & K & K \\
  \% & \% & K & K & K & K 
\end{pmatrix}
$$

(4.39)
High threshold early exercise corresponds to the exercise matrix

$$\Sigma_H = \begin{pmatrix} 4K & 3K & 2K & K & K & K \\ \% & 4K & 3K & 2K & K & K \\ \% & \% & 4K & 3K & 2K & K \end{pmatrix}$$ (4.40)

We define the success $s$ of a particular strategy by

$$s_{\text{Strategy}} = \frac{\text{Exp Payoff using Strategy}}{\text{Exp Payoff using Optimal Strategy}}$$ (4.41)

It turns out that the different strategies exhibit characteristic relationships between $s$ and the actual spot price $S(t_0)$. In particular, for the $C$ strategy $s$ tends to increase with increasing spot price. Above the strike this increase is not always observed. Depending on $\alpha$ and $\sigma$ the success can remain more or less constant or even pass through a maximum.

In the case of the $I$ strategy $s$ always passes through a minimum which is located at the strike price. This is not surprising, since early exercise with very small payoff becomes probable if the option is at the money. If the option is far in or far out of the money this 'wrong' (with respect to the optimal strategy) exercise decision becomes less likely, especially at the first opportunity. It is noticeable that the call strip exhibits the same behaviour as the $I$ strategy, i.e. the ratio between the call strip value (see §3.3.4) and the value of the Swing is minimal when the option is at the money.

For the $H$ strategy $s$ decreases monotonously with increasing spot price – at least in the range between 5 and 50. This can be understood by realizing that the expected missed benefit from early exercise increases with increasing spot price. However, when the spot price is far enough above the first threshold of $4K$ the $H$ strategy is getting better. For $S \to \infty$ the $H$ strategy must approach the optimal strategy since early exercise (which is then optimal) is applied for all paths.\footnote{Actually this convergence is quite slow. For $\alpha = 1$, $\sigma = 0.2$ we arrive at $s = 90.5\%$ for $S = 100000$.}

Figure 4.4 shows typical plots of $s$ as a function of the spot price for the different strategies.

In order to judge the suitability of the $I$ and $H$ strategies it is sensible to take the minimum of $s$ in the observed range since this minimum represents the inadequacy of the strategy, i.e. the holder’s error when applying his (suboptimal) strategy. If this error is too large, the holder has to refine his strategy.

On the other hand, for the $C$ strategy it is interesting to examine the value of $s$ for different spot prices. In this case the problem is different. The potential buyer of the
option has no influence on the strategy, but he might make his decision of purchase dependent on the expected payoff compared to the optimal value.

In order to investigate the success of the different strategies I have performed simulations with various parameter sets. The results are summarized in Table 4.5 which shows the minimum success of the $I$ and $H$ strategies and the minimum and maximum success of the $C$ strategy together with the process parameters, the threshold $X$ of the simple Bermudan problem and the cutting angle as defined by (4.33).

**Success of Coincidental Exercise**

First, we take a look on the $C$ strategy. For small $\alpha$ and thus small cutting angles $\phi$ the success $s$ rises up to more than 90% of the Swing value when the option is in the money. With increasing mean reversion speed, this difference decreases since the minimum success rises and the maximum success falls (see Figure 4.5). The decrease of the maximum success is significantly larger than the increase of the minimum success. This can be explained as follows.

For large spot prices, mean reversion impacts the expected payoff in a twofold manner. With increasing $\alpha$

- the spot price is pulled down more strongly towards the mean reversion level

and

- the sensitivity of the payoff with respect to suboptimal (coincidental) early exercises increases

The second effect is clearly discernible for $(\alpha, \sigma) = (0.05, 0.4)$. Since the threshold is located at 98.79 the vast majority of coincidental exercises must be suboptimal for
Table 4.5: Results of the strategy simulations. For all parameter sets \((\alpha, \sigma)\) the simple threshold \(X\) and the cutting angle \(\phi\) are shown in addition. The subscripts \(\min\) and \(\max\) refer to the observed spot price range of \(5 \leq S \leq 50\). However, for immediate exercise \(s'_{\min}\) this is also the global minimum.
Figure 4.5: Success of coincidental exercise as a function of spot price for various mean reversion speeds. With increasing $\alpha$, the difference between $s_{\text{min}}$ and $s_{\text{max}}$ decreases. Note that the lines only serve as guide to the eye.

$S(t_0)=50$. Nevertheless, the success is 93%. On the other hand, for $(\alpha,\sigma)=(1.0,0.2)$ the threshold is 21.88 and thus the proportion of optimal exercises is expected to be quite high. The success for $S(t_0) = 50$, however, is only 63.9%.

For low spot prices (i.e. when the option is out of the money) coincidental exercises with zero payoff lead to a significant decrease of the expected payoff even if $\alpha$ is small. With increasing mean reversion speed, the spot price is pulled up more strongly towards the mean reversion level and these zero payoff exercises become less likely. Altogether, the effects of increasing sensitivity to suboptimal exercise and amplified upward movement eliminate each other more or less on the low spot price side.

**Success of Immediate Exercise**

With the $I$ strategy there are two main observations:

- Keeping $\alpha$ constant the success decreases with increasing $\sigma$

- Keeping $\sigma$ constant the success increases with increasing $\alpha$
The first result is easy to understand. Since the threshold increases with increasing $\sigma$ immediate exercise becomes less optimal and therefore the expected payoff decreases.

However, the second result is surprising at first glance. Since the sensitivity of the expected payoff on the deviation from the optimal strategy is increasing with increasing mean reversion speed one could expect a decrease of the success. On the other hand, even if a holder applies the optimal strategy he may miss one or another possible exercise. This is the case when the spot price was above the strike but below the (optimal) threshold at a certain opportunity but subsequently decreased and remained below all thresholds. With increasing $\alpha$ the number of these missed exercises is expected to increase. A holder who applies the $I$-strategy, however, will never miss any exercise.

**Success of High Threshold Early Exercise**

Compared to the $I$ strategy the $H$ strategy exhibits inverse behaviour, i.e.:

- With increasing $\sigma$ (and constant $\alpha$) the success decreases
- With increasing $\alpha$ (and constant $\sigma$) the success decreases

The only exception is to be seen for $(\alpha,\sigma) = (0.05,0.6)$, but note that in this case the threshold is at 695.21 and thus far away from probable spot prices in the simulation.\(^\text{10}\)

Since the threshold increases with increasing $\sigma$ the $H$ strategy gets closer to the optimal strategy and therefore the success increases - as long as $\alpha$ remains constant. With increasing $\alpha$ there are two effects. First the threshold decreases and thus the $H$ strategy becomes less optimal and, second, suboptimal exercise decisions gain in importance since the sensitivity increases. Note that for $(\alpha,\sigma) = (0.15,0.2)$ the threshold is 25.16 and the success 69.6% while for $(\alpha,\sigma) = (0.5,0.4)$ the threshold of 26.04 is very close but the success of 52.2% remarkably worse.

**Concluding Remarks**

A close look on Table 4.5 yields that the $I$ strategy is less dangerous than the $H$ strategy. Even under unfavourable conditions concerning the process parameters the success of the $I$ strategy lies around 80% and is always significantly greater than the success of the $C$ strategy. Remarkably, the latter is not the case with the $H$ strategy.

\(^\text{10}\)It should be emphasized that the minimum success of the $H$ strategy is the minimum taken over the observed range $5 \leq S(t_0) \leq 50$ and not the total minimum. This is in contrast to the $I$ strategy where the minimum at the strike price is the total minimum.
strategy. For large mean reversion speed, suboptimal exercise leads to a success which is significantly worse than the success of coincidental exercise.

On the other hand, for certain sets of process parameters (implying a high but not too high optimal threshold) the $H$ strategy is very successfull and reaches up to $\sim 99\%$ of the optimal value. This is not the case with immediate exercise. Even if the conditions are good the maximum success remains around 90%.

From my personal point of view I am quite surprised that the success of coincidental exercises is that high. From intuition I would not have expected values as high as 60-85% since coincidental exercise is completely irrational and leads occasionally to early exercise even when the option is out of the money.
Chapter 5

Summary and Outlook

5.1 Summary

In the present thesis, Swing options have been investigated using Monte Carlo simulation techniques. For that purpose two different approaches have been pursued:

- least squares Monte Carlo in order to find the option value with respect to the optimal exercise strategy;
- strictly forward evolving strategy simulation in order to find the expected payoff for a particular (suboptimal) exercise strategy.

The first approach was realized by extending the Longstaff Schwartz algorithm for Bermudan or American options to the Swing option case where the holder has \( m \) exercise rights at \( N \) opportunities. This was done using two different stochastic processes for the underlying price, namely a one factor and a two factor mean–reverting process.

Both processes were discussed with respect to the solutions of their stochastic differential equations, the expected values of the solutions (forward prices) and the values of the corresponding vanilla call options.

The extended Longstaff Schwartz algorithm was applied to Swing options with upswings (call option features), downswings (put option features), and penalty functions.

For the one factor process the results have been compared to those obtained by finite–differences in a former thesis, and no significant deviations have been found.

In a computer experiment an upper boundary for Swing options with \( m \) upswings at \( N \) opportunities has been found. This boundary is smaller than \( m \) times the value of a Bermudan call option which is frequently discussed in the literature. Since
the boundary is valid for both processes under examination and contains no process
dependent parameters, it is concluded that it results solely from the exercise structure
and is thus universally valid.

In the second approach a particular exercise strategy (which is defined by a strat-
egy matrix) is used as input parameter and the expected payoff of the Swing option
is determined by simulating the exercise strategy for a large number of paths followed
by the spot price. All strategy simulations have been performed using the one factor
process for the spot price.

The success of different simple strategies, i.e. the expected payoff relative to the
expected payoff using the optimal strategy, has been investigated with respect to its
dependence on process parameters like mean–reversion speed and volatility.

As a first step in this context, the optimal exercise threshold was investigated for
a simple Bermudan option with two exercise opportunities. This can partly be done
analytically, and the main results concerning the dependence of the optimal threshold
on mean–reversion speed and volatility can be applied to Swing options as well. This
is also true for the sensitivity of the expected payoff with respect to the strategy
which depends strongly on the process parameters.

Three sample strategies have been compared with each other. This was done for
various sets of process parameters and start values for the spot price. The results
for the success of the strategies were explained in the context of process properties,
optimal exercise thresholds and sensitivity of the expected payoff with respect to the
strategy.

5.2 Outlook

Based on the present work, two main issues could be addressed:

- Implementation of a valuation tool for Swing options which is designated for
  practical purposes;

- Finding a (semi–analytic) approximation formula for Swing options.

The first issue requires the implementation of an underlying stochastic process which
can be fitted to real market data. Moreover, since Swing options in practice have
typical lifetimes of several months and thus a large number of exercise opportunities,
it seems to be necessary to care for convergence acceleration in order to keep the
number of paths small and thus the computing time acceptable. For example, this
could be done with the help of variance reduction by using control–variates. A further
important aspect is that, in practice, exercise decisions have to be made one day in advance. It seems therefore appropriate to consider the forward rather than the spot price process.

The second issue has already been sketched in Section 3.3.4. It seems to be possible to find an approximation which expresses the Swing option value in terms of the corresponding Bermudan option and the corresponding vanilla call options. This approximation is not fully analytic, since the value of the Bermudan option cannot be calculated analytically.
Appendix A

Proof of the Statement: Threshold for Early Exercise > Mean Reversion Level

The following proof is done according to [3].

With the notation

\[ A = S^{e^{-\alpha t}} F^{1-e^{-\alpha t}} \]  
\[ v = \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha t}) \]

where \( F \) is the mean reversion level, we can write the value of a call option as

\[ c(A, \sqrt{v}) = Ae^{\frac{1}{\sqrt{v}} (\ln \frac{A}{K} + v)} \int_{-\infty}^{\frac{1}{\sqrt{2\pi}} \ln \frac{A}{K}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - K \int_{-\infty}^{\frac{1}{\sqrt{2\pi}} \ln \frac{A}{K}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \]

where \( K \) is the strike price, \( \alpha \) the mean reversion speed and \( \sigma \) the volatility.\(^1\)

We will now show that

\[ S < F \Rightarrow c > S - K \]  
\[ (\sqrt{v}, A = Ke^{z\sqrt{v}}) \rightarrow (\sqrt{v}, z) \]

\(^1\)It is evident that all these quantities are positive.
Hence we can rewrite Eq. (A.4) as

\[ c = K \left( e^{z + \frac{1}{2} v} \int_{-\infty}^{z + \sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx - \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right) \]  

(A.8)

i.e. as a function of the two independent variables \( z \) and \( \sqrt{v} \).

We now want to maximize \( c \) with respect to \( \sqrt{v} \), i.e. we search for a zero of the partial derivative

\[ \frac{1}{K} \frac{\partial c}{\partial \sqrt{v}} = (z + \sqrt{v}) e^{z + \frac{1}{2} v} \int_{-\infty}^{z + \sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx + e^{z + \frac{1}{2} v} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z + \sqrt{v})^2} \]  

(A.9)

Setting (A.9) to zero yields

\[ 0 = \int_{-\infty}^{z + \sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx + \frac{1}{z + \sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z + \sqrt{v})^2} \]  

(A.10)

The right hand side of Eq. (A.10) can be written as a function of \( y := z + \sqrt{v} \):

\[ f(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx + \frac{1}{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \]  

(A.11)

In the limit \( y \to \infty \) we obtain

\[ \lim_{y \to \infty} f(y) = 1 \]  

(A.12)

On the other hand,

\[ \frac{df}{dy} = -\frac{1}{\sqrt{2\pi} y^2} e^{-\frac{y^2}{2}} < 0 \]  

(A.13)

meaning \( f \) is a strictly monotonically decaying function of \( y \).

Together with (A.12) this yields that \( f \) has no zeroes and thus \( c(\sqrt{v}, z) \) has no maxima with respect to \( \sqrt{v} \). Therefore the same is true for \( c(A, \sqrt{v}) \) and since

\[ \lim_{\sqrt{v} \to \infty} c(A, \sqrt{v}) = \infty \]  

(A.14)

we obtain

\[ c(A, \sqrt{v}) \geq \lim_{\sqrt{v} \to 0} c(A, \sqrt{v}) = A - K \]  

(A.15)

On the other hand

\[ A = S e^{-\alpha t} F^{1-e^{-\alpha t}} = S \left( \frac{S}{F} \right)^{1-e^{-\alpha t}} > S \]  

(A.16)

since \( S < F \).

Thus we have shown that

\[ c > S - K \]  

(A.17)

i.e. \( c = S - K \) cannot be true if \( S < F \). Therefore we obtain

\[ S > F \]  

(A.18)
Appendix B

The MATLAB Routines

In the following the most important MATLAB routines used in this thesis are sketched. However, this does not include all programs used in this thesis. For example, simple functions which calculate analytic values of single call options or the value of a callstrip are omitted.

ls_upswing.m

This function calculates the value of a Swing option by using the extended Longstaff Schwartz algorithm as described in Sec. 3.2.1. The Swing option consists only of upswings and there is no penalty function, and the underlying stochastic process is the one-factor mean-reverting process discussed in Sec. 2.1.

Input Parameters

<table>
<thead>
<tr>
<th>paths</th>
<th>number of paths included in each simulation loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>timesteps</td>
<td>number of exercise opportunities</td>
</tr>
<tr>
<td>delta_t</td>
<td>vector of time periods between exercise opportunities</td>
</tr>
<tr>
<td>start_value</td>
<td>spot price</td>
</tr>
<tr>
<td>f</td>
<td>mean reversion level</td>
</tr>
<tr>
<td>alpha</td>
<td>mean reversion speed</td>
</tr>
<tr>
<td>sigma</td>
<td>volatility</td>
</tr>
<tr>
<td>exercises</td>
<td>number of upswings</td>
</tr>
<tr>
<td>strikes</td>
<td>vector of strike prices</td>
</tr>
</tbody>
</table>

Functionality and Subroutines

In each of the 10 simulation loops the following is done:
• Creation of the spot price matrix by the subroutine `create_paths_anti.m`. This function uses antithetic sampling, i.e. the number of paths must be even. As can be seen from the input parameters, the periods between the opportunities can differ from each other.

• Performing the LS algorithm as described in 3.2.1. Note that the strike prices can vary from opportunity to opportunity. The least squares regression is carried out by the subroutine `cont_values.m` which uses special built-in functions provided by MATLAB. After exercising the reset of the cashflow matrices is done by the subroutine `reset_cf_matrix`.

**Output**

The output is a $u \times 2$–matrix where $u$ is the number of upswings. In the first (second) column the average (standard deviation) of the 10 simulation results are shown. The numbers in the $i$–th row correspond to a Swing option with $i$ upswings. As explained in Sec. 3.2.1, we calculate simultaneously the value of $U$ different Swing option with one to $U$ upswings.

**ls_upswing_2f.m**

This function is very similar to `ls_upswing.m`. The only difference is the paths sampling which is now done for the two factor mean-reverting process as discussed in Sec. 2.2.

**Input Parameters**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>paths</td>
<td>number of paths included in each simulation loop</td>
</tr>
<tr>
<td>timesteps</td>
<td>number of exercise opportunities</td>
</tr>
<tr>
<td>tau</td>
<td>vector of time periods between exercise opportunities</td>
</tr>
<tr>
<td>start_value</td>
<td>spot price</td>
</tr>
<tr>
<td>start_delta</td>
<td>initial convenience yield $\delta_0$</td>
</tr>
<tr>
<td>alpha</td>
<td>mean reversion speed</td>
</tr>
<tr>
<td>kappa</td>
<td>??</td>
</tr>
<tr>
<td>sigma_s</td>
<td>volatility of the spot price</td>
</tr>
<tr>
<td>sigma_d</td>
<td>volatility of the convenience yield</td>
</tr>
<tr>
<td>rho</td>
<td>correlation between spot price and convenience yield</td>
</tr>
<tr>
<td>r</td>
<td>interest rate</td>
</tr>
<tr>
<td>exercises</td>
<td>number of upswings</td>
</tr>
<tr>
<td>strikes</td>
<td>vector of strike prices</td>
</tr>
</tbody>
</table>
Functionality and Subroutines

The function works exactly in the same way as \texttt{ls\_upswing.m}, except that the path sampling is performed by the subroutine \texttt{create\_paths\_2f.m}. This subroutine does not use antithetic sampling.

Output

The output is equivalent to \texttt{ls\_upswing.m}.

\texttt{ls\_updownswing.m}

This function calculates the value of a Swing option with \( u \) upswings, \( d \) downswing and includes an (arbitrary) penalty function \( \phi \).

Input Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>paths</td>
<td>number of paths included in each simulation loop</td>
</tr>
<tr>
<td>timesteps</td>
<td>number of exercise opportunities</td>
</tr>
<tr>
<td>delta_t</td>
<td>vector of time periods between exercise opportunities</td>
</tr>
<tr>
<td>start_value</td>
<td>spot price</td>
</tr>
<tr>
<td>( f )</td>
<td>mean reversion level</td>
</tr>
<tr>
<td>alpha</td>
<td>mean reversion speed</td>
</tr>
<tr>
<td>sigma</td>
<td>volatility</td>
</tr>
<tr>
<td>ups</td>
<td>number of upswings</td>
</tr>
<tr>
<td>ustrikes</td>
<td>vector of strike prices for the upswings</td>
</tr>
<tr>
<td>downs</td>
<td>number of downswings</td>
</tr>
<tr>
<td>dstrikes</td>
<td>vector of strike prices for the downswings</td>
</tr>
</tbody>
</table>

Functionality and Subroutines

As in the case of \texttt{ls\_upswing.m}, the path sampling is done with the subroutine \texttt{create\_paths\_anti.m}.

However, the least squares algorithm is slightly different. As discussed in Sec. 3.2.2, the penalty function has to be evaluated in the initial iteration step. Calculating the penalty is carried out with the subroutine \texttt{penalty.m} which can easily be edited in order to implement an arbitrary form of \( \phi \).

The least squares regression is performed by the subroutine \texttt{cont\_values.m} (as for \texttt{ls\_upswing.m}), and the reset of the cashflow matrices by the function \texttt{reset\_cfmatrix\_ud.m}. The latter subroutine handles the four dimensional cashflow matrix as discussed in
Sec. 3.2.2. Note that the strike prices for the upswings need not be the same as for the downswings.

Output

The output is a $1 \times 2$-vector where the first (second) element is the average (standard deviation) of the ten simulation results. In contrast to \texttt{ls\_upswing.m}, only the Swing with $u$ upswings and $d$ downswings is returned by the routine (see also Sec. 3.2.2).

\texttt{ls\_updownswing\_2f.m}

This function is similar to \texttt{ls\_updownswing.m}. The only difference is that the sampling is done assuming the two factor mean-reverting process.

Input Parameters

<table>
<thead>
<tr>
<th>paths</th>
<th>number of paths included in each simulation loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>timesteps</td>
<td>number of exercise opportunities</td>
</tr>
<tr>
<td>tau</td>
<td>vector of time periods between exercise opportunities</td>
</tr>
<tr>
<td>start_value</td>
<td>spot price</td>
</tr>
<tr>
<td>start_delta</td>
<td>initial convenience yield $\delta_0$</td>
</tr>
<tr>
<td>alpha</td>
<td>mean reversion speed</td>
</tr>
<tr>
<td>kappa</td>
<td>$??$</td>
</tr>
<tr>
<td>sigma_s</td>
<td>volatility of the spot price</td>
</tr>
<tr>
<td>sigma_d</td>
<td>volatility of the convenience yield</td>
</tr>
<tr>
<td>rho</td>
<td>correlation between spot price and convenience yield</td>
</tr>
<tr>
<td>r</td>
<td>interest rate</td>
</tr>
<tr>
<td>exercises</td>
<td>number of upswings</td>
</tr>
<tr>
<td>ups</td>
<td>number of upswings</td>
</tr>
<tr>
<td>ustrikes</td>
<td>vector of strike prices for the upswings</td>
</tr>
<tr>
<td>downs</td>
<td>number of downswings</td>
</tr>
<tr>
<td>dstrikes</td>
<td>vector of strike prices for the downswings</td>
</tr>
</tbody>
</table>

Functionality and Subroutines

The path sampling is the same as for \texttt{ls\_upswing\_2f.m} and the LS algorithm is the same as for \texttt{ls\_updownswing.m}.

Output

The output is equivalent to the output of \texttt{ls\_updownswing.m}.
ls_upswing_strat.m

This is an extension of ls_upswing.m. The value of the Swing option (only upswings, no penalty) remains unchanged, but in addition to the option value the elements of the strategy matrix and their standard deviations are returned (see Sec. 4.2.1).

Input Parameters

The input parameters are the same as for ls_upswing.m.

Functionality and Subroutines

The valuation of the Swing option is exactly the same as in the case of ls_upswing.m where the cashflow matrix is determined by the LS algorithm. Additionally, the optimal strategy matrix is calculated from the cashflow matrix. This evaluation is performed by the subroutine find_strategy.m.

Output

If there are $u$ upswings and $m$ opportunities, the output is a matrix of dimension $u \times 2(m+2)$. As before, the first two columns contain the average and standard deviation of the simulations for Swing options from 1 to $u$ upswings. The next $m$ columns built the strategy matrix as defined in 4.2.1. As discussed in 4.2.1 not all elements of the strategy matrix are defined. For the undefined elements zeros are returned. The last $m$ columns represent the standard deviations of the elements of the strategy matrix.

strat_upswing.m

This function performs the valuation of a Swing option in terms of a certain exercise strategy as discussed in Sec. 4.2.1 and 4.2.2, i.e. the expected payoff with respect to a certain strategy.
### Input Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>paths</td>
<td>number of paths included in each simulation loop</td>
</tr>
<tr>
<td>timesteps</td>
<td>number of exercise opportunities</td>
</tr>
<tr>
<td>delta_t</td>
<td>vector of time periods between exercise opportunities</td>
</tr>
<tr>
<td>start_value</td>
<td>spot price</td>
</tr>
<tr>
<td>f</td>
<td>mean reversion level</td>
</tr>
<tr>
<td>alpha</td>
<td>mean reversion speed</td>
</tr>
<tr>
<td>sigma</td>
<td>volatility</td>
</tr>
<tr>
<td>exercises</td>
<td>number of upswings</td>
</tr>
<tr>
<td>strikes</td>
<td>vector of strike prices</td>
</tr>
<tr>
<td>strat_matrix</td>
<td>$u \times m$ strategy matrix</td>
</tr>
<tr>
<td>loops</td>
<td>number of simulation loops</td>
</tr>
</tbody>
</table>

### Functionality and Subroutines

The path sampling is carried out using the subroutine `create_paths_anti`. With the spot price matrix, early exercise is performed according to the strategy matrix. Unlike the LS algorithm this routine steps forward in time. Besides the path sampling there is no further subroutine.

### Output

The output is a $(1 \times 2)$ vector where the first (second) element is the average (standard deviation) of the simulation. Note that the number of loops is here an input parameter.

**coinc_strat_upswing.m**

This function calculates the expected payoff of a Swing option assuming random (coincidental) exercise. As explained in Sec. 4.2.2 random exercise cannot be described by a strategy matrix. If there are $u$ upswings a $u$-tupel of exercise times is chosen randomly. All possible sets of exercise times are assumed to have equal probabilities.

### Input Parameters

The input parameters are the same as for `ls_upswing.m`. 
Functionality and Subroutines

Again, the path sampling is performed using `create_paths_anti`. The random choice of the exercise time set is realized with the help of special built-in functions provided by MATLAB. For the simulation of early exercise, no further subroutine is used.

Output

The output is the same as for `strat_upswing.m`. 
References


