Ostrowski's Theorem and other diversions

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A **metric** on a set X is a map $d : X \times X \to \mathbb{R}_+$ such that

1. $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$, for all $z \in X$

A **norm** on a field F is a map $||.|| : F \to \mathbb{R}_+$ such that

1. $||x|| = 0$ iff $x = 0$
2. $||xy|| = ||x||.||y||$
3. $||x + y|| \leq ||x|| + ||y||$

It is straight forward to show that $d(x, y) = ||x - y||$ defines a metric on F. Suppose $x \in \mathbb{Q}$ then we may write $x = p^{v_p(x)} \frac{a}{b}$, where $p \nmid a, b$. We define $|.|_p$ by

$$|x|_p = p^{-v_p(x)}$$

for $x \neq 0$ and $|0|_p = 0$. Recall that we say the sequence $\{a_n\}$ is Cauchy in the metric space (X, d) if for all $\epsilon > 0$ there exists an N_ϵ such that $d(a_n, a_m) < \epsilon$ for all $m, n > N_\epsilon$.

Two metrics d_1, d_2 on X are **equivalent** if every sequence that is Cauchy with respect to d_1 is also Cauchy with respect to d_2. We also say two norms on a field are **equivalent** if they induce equivalent metrics.

Problem 1. *Let* $0 < c < 1$ *and* p *prime. Define* $||x|| = c^{v_p(x)}$ *for* $x \neq 0$ *and* $||0|| = 0$*, for all* $x \in \mathbb{Q}$*. Show that* $||.||$ *is equivalent to* $|.|_p$ *on* $\mathbb{Q}$*.*

Let $|.|_\infty$ denote the usual absolute value on $\mathbb{Q}$, which is clearly a norm. We say that a norm is trivial if $||0|| = 0$ and $||x|| = 1$ for $x \neq 0$.

Theorem 1 (Ostroski)**.** *Every non-trivial norm* $||.||$ *on* $\mathbb{Q}$ *is equivalent to* $|.|_p$ *for some prime* p *or* $|.|_\infty$*.*

Question: is there an equivalent to Ostroski's Theorem for any number field?

Problem 2. *Given an arithmetic progression of integers*

$$h, h+k, h+2k, \ldots, h+nk, \ldots$$

where $0 < k < 2009$*. If* $h+nk$ *is prime for* $n = t, t+1, \ldots, t+r$ *prove that* $r \leq 9$ *i.e. at most* 10 *consecutive terms of this progression can be primes. Can you generalise this by replacing* 2009 *by* N *and finding an upper bound on* r*?*

Problem 3. *Prove the following generalisation of Wilson's theorem*

$$(p-k)!(k-1)! \equiv (-1)^k \pmod{p}$$

for $1 \leq k \leq p-1$*.*

Problem 4. *Prove that for an odd prime* p*,*

$$\frac{2^{p-1}-1}{p} \equiv \sum_{j=1}^{p-1} \frac{(-1)^{j+1}}{2j} \pmod{p}$$

Deduce that $2^{p-1} \equiv 1 \pmod{p^2}$ *iff the numerator of*

$$1 - \frac{1}{2} + \frac{1}{3} - \ldots - \frac{1}{p-1}$$

is divisible by p*.*