A minimisation method of W. M. Schmidt

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Let K denote a $\mathfrak{p}$-adic field with ring of integers $\mathcal{O}_K$, and residue class field $\mathbb{F}_q$. Let π denote a uniformizer for $\mathcal{O}_K$. If $\alpha \in K - \{0\}$, we may write $\alpha = \pi^s u$, where u is a unit in K. We define the π-adic order $v(\cdot)$ by setting $v(\alpha) = s$.

We let

$$F_1(\mathbf{x}), \ldots, F_r(\mathbf{x}) \in K[\mathbf{x}]$$

denote a system of homogenous polynomials (forms) of degrees $d_1, \ldots, d_r$ respectively in the variables $\mathbf{x} = (x_1, \ldots, x_n)$. From now on we shall assume that $d_1 \geq \ldots \geq d_r$ and for brevity write the above system of forms as $\mathbf{F}$.

We are interested in determining the existence of a point $\mathbf{x} \in K^n - \{\mathbf{0}\}$ such that $\mathbf{F}(\mathbf{x}) = \mathbf{0}$. Clearly we may assume that the coefficients of the forms $\mathbf{F}$ and the variables $\mathbf{x}$ are in $\mathcal{O}_K$, since this does not effect existence of a zero.

Let $\tau \in GL(n, \mathcal{O}_K)$ and we write $\mathbf{F}_\tau$ to denote $\mathbf{F}(\tau\mathbf{x}) = (F_1(\tau\mathbf{x}), \ldots, F_r(\tau\mathbf{x}))^t$. Also we write $T = (t_{ij})$ to denote the $(r \times r)$ upper triangular matrix with entries $t_{ij}(\mathbf{x}) = \pi^{-c_i} G_{ij}(\mathbf{x})$, where $G_{ij} \in \mathcal{O}_K[\mathbf{x}]$ denotes an arbitrary form of degree $\deg F_i - \deg F_j \geq 0$, for $i \leq j$ and $c_i \geq 0$.

For $\omega_1, \ldots, \omega_r > 0$ we write

$$\mathbf{F} \underset{\omega_i}{\succ} \mathbf{F}',$$

if $\mathbf{F}$ and $\mathbf{F}'$ are both defined over $\mathcal{O}_K$ and

$$\mathbf{F}' = T\mathbf{F}_\tau$$

with

$$\sum c_i \omega_i - s > 0$$

where $s = v(\det \tau)$.

We say that $\mathbf{F}$ is (ω_i)-bottomless if there is an infinite chain

$$\mathbf{F} \underset{\omega_i}{\succ} \mathbf{F}^{(\mathbf{1})} \underset{\omega_i}{\succ} \mathbf{F}^{(\mathbf{2})} \underset{\omega_i}{\succ} \cdots$$

otherwise, $\mathbf{F}$ will be called (ω_i)-bottomed. We also say that $\mathbf{F}$ is (ω_i)-reduced if there does not exist any $\mathbf{F}'$ such that

$$\mathbf{F} \underset{\omega_i}{\succ} \mathbf{F}'.$$

We say that two systems $\mathbf{F}$ and $\mathbf{F}'$ are equivalent, if both systems are defined over $\mathcal{O}_K$ and

$$\mathbf{F}' = T\mathbf{F}_\tau$$

where $c_i = 0$ for all $1 \leq i \leq r$ and $v(\det \tau) = 0$ in T and τ as above. We define the "h-invariant" for a system $\mathbf{F}$, denoted $h(\mathbf{F})$ as the least integer h such that we can write

$$F_i(\mathbf{x}) = x_1 H_{i1} + \ldots + x_h H_{ih} \pmod{\pi}, \qquad 1 \leq i \leq r$$

for all systems equivalent to $\mathbf{F}$.

In this talk we shall deal with the simple case of when $\mathbf{F}$ is a system of a cubic and quadratic form and discuss how this gives us "useful" information about the h-invariant over the residue class field. Moreover we use this to verify Artin's conjecture for this case provided $q > 293$ viz. every cubic and quadratic form has a simultaneous zero in K for $n > 2^2 + 3^2$ and $q > 293$.

Problem 1. *Let* $\mathbf{F} = (F, G)^t$ *denote a system of a cubic and quadratic form over* $\mathcal{O}_K$*. Suppose* $\mathbf{F}$ *is* (ω_1, ω_2)*-reduced where* $\omega_1 > 3$ *and* $\omega_2 > 2$*. Show that*

$$h(\mathbf{F}) > 5, \qquad h(F - LG) > 3, \qquad h(G) > 2$$

for all linear forms $L(\mathbf{x}) \in \mathcal{O}_K[\mathbf{x}]$*.*