Pseudo-uniform stability of bounded continuous semigroups

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Plan

1. Background

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2. Pseudo-uniform stability
3. Proof

Background

Let $S(t)$ be a strongly continuous semigroup on a Banach space X with generator A. Assume that $S(t)$ is bounded:

$$\sup_{t \geq 0} \|S(t)\|_{X \to X} = \widetilde{C} < \infty.$$

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Stability of $S(t)$ i.e. $\lim_{t\to+\infty} S(t) = 0$ in some sense

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$$\updownarrow$$

$$\text{Spectrum and Resolvent of } A \text{ on } i\mathbb{R}?$$

Strong stability

The following theorem [Arendt-Batty, Lyubich-Phóng, 1988] is a sufficient condition for strong (pointwise) stability:

Theorem

If $\sigma(A) \cap i\mathbb{R}$ is at most countable and A^* does not have any eigenvalue on $i\mathbb{R}$ then

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- Not a necessary condition (example: left shift on $L^2(0, +\infty)$).
- There is no purely spectral necessary and sufficient condition for strong stability.
- Does not give the rate of decay.

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Assume that X is an Hilbert space and S bounded. Then:
$S(t)$ is uniformly stable

$$\iff \sigma(A) \cap i\mathbb{R} = \emptyset \text{ and } \sup_{\tau\in\mathbb{R}} \|(A - i\tau)^{-1}\| < \infty.$$

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$\Longleftarrow$ does not hold on general Banach space.

Damped waves

Let us give a typical example. Consider the equation:

$$\partial_t^2 u - \Delta u + a\partial_t u = 0, \quad u_{\restriction \partial\Omega} = 0$$

$$u_{\restriction t=0} = u_0 \in H_0^1(\Omega), \quad \partial_t u_{\restriction t=0} = u_1 \in L^2(\Omega).$$

where Ω is a bounded smooth domain and $a(x) \geq 0$. Here $X = H_0^1 \times L^2$ and the corresponding semigroup is dissipative.

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- In some cases there exists $\alpha > 0$ s.t.

$$\|(u(t), \partial_t u(t))\|_X \leq \frac{C}{t^\alpha} \|(u_0, u_1)\|_{D(A)}.$$

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- Uniform stability $\Longrightarrow$ pseudo-uniform stability $\Longrightarrow$ strong stability.
- Pseudo-uniform stability $\iff$ $\lim_{t \to +\infty} \|S(t)\|_{D(A^k) \to X} = 0$, $k > 0$.

Spectral condition

Let $S(t)$ be a bounded semigroup with generator A. Then

Theorem (C. Batty, TD)

The following conditions are equivalent.

1. $S(t)$ *is pseudo-uniformly stable.*
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- (1)$\Longrightarrow$(2) seemed unknown.
- Purely spectral criterion (not the case for pointwise stability and uniform stability).

Rate of decay

If $S(t)$ is pseudo-uniformly stable, we have

$$\|S(t)x\| \leq m(t)\|x\|_{D(A)}, \text{ where}$$

$$\lim_{t \to +\infty} m(t) = 0, \quad m(t) := \|S(t)\|_{D(A) \to X} .$$

Bound on $m(t)$?

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$$M(\lambda) := \sup_{\tau \in [-\lambda, \lambda]} \left\| (A - i\tau)^{-1} \right\|_{X \to X} < \infty.$$

The proof of (1)$\Longrightarrow$(2) implies that for some $C > 0$

$$M(\lambda) \leq C m^{-1}\left(\frac{1}{C\lambda}\right), \quad \lambda \gg 1.$$

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$$M(\lambda) \leq Cm^{-1}\left(\frac{1}{C\lambda}\right), \quad \lambda \gg 1.$$

We want a "converse" bound, maybe $m(t) \leq C/M^{-1}(t/C)$.

Rate of decay

Recall $M(\lambda) := \sup_{\tau\in[-\lambda,\lambda]} \left\|(A - i\tau)^{-1}\right\|_{X\to X} < \infty$ (nondecreasing). Introduce

$$M_{\log}(\lambda) := M(\lambda)\left[\log(1 + M(\lambda)) + \log(1+\lambda)\right]$$

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This is not the optimal excepted decay (M^{-1} replaced by $M_{\log}^{-1}$). The “smoother” the initial condition, the faster the decay:

$$\|S(t)\|_{D(A^k)\to X} \le C_k \left(M_{\log}^{-1}\left(\frac{t}{C_k}\right)\right)^{-k}, \quad t \gg 1.$$

Examples and previous results

- Logarithmic decay: if $M(\lambda) \leq C \exp(C\lambda)$, we obtain

$$\|S(t)x\| \leq \frac{C}{\log(2+t)} \|x\|_{D(A)}.$$

The log loss is invisible. Already known in Hilbert spaces [Lebeau 96, Burq 98]. Example: wave equation on a bounded connex domain with a localized damping term.

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- Polynomial decay: if $M(\lambda) = \lambda^s$, $s > 0$, we obtain

$$\|S(t)x\| \leq C\left(\frac{\log t}{t}\right)^{1/s}\|x\|_{D(A)}, \quad t \gg 1.$$

Generalizes previous results of [Batkai-Engel-Prüss-Schnaubelt 06], [Liu-Rao 07]. Example: wave equation with a localized damping term in a rectangle domain.

Optimality

Assume that $\sigma(A) \cap i\mathbb{R} = \emptyset$, that X is a Hilbert space and

$$\sup_{\tau\in\mathbb{R}} \left\|(A - i\tau)^{-1}\right\|_{X\to X} < \infty.$$

Then by the classical theorem on uniform stability

$$\|S(t)x\| \leq Ce^{-ct}\|x\|. \tag{*}$$

Counter-examples exist if X is not an Hilbert space.

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Counter-examples exist if X is not an Hilbert space.
On the other hand $M_{\log}(\tau) \approx \log(\tau)$, and thus by our previous theorem,

$$\|S(t)x\| \leq Ce^{-ct}\|x\|_{D(A)}.$$

By (*), this is not optimal in the Hilbert space case.
Conjecture: one can get better result in the Hilbert space case. (Maybe the optimal decay without the log loss?). Already known for normal operators in the polynomial case [Batkai-Engel-Prüss-Schnaubelt 06].

Optimal polynomial decay

The conjecture is true in the case of polynomial decay [Borichev, Tomilov, preprint 2009].

Theorem

If X is an Hilbert space, $\sigma(A) \cap i\mathbb{R} = \emptyset$ *and* $\exists s > 0,\ \left\|(A - i\tau)^{-1}\right\|_{X\to X} \leq C\tau^s$. *Then*

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- See next talk!

Reminder of the main result

Recall

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The proof is essentially the initial step of the induction of [Arendt-Batty 1988] (inspired by a Tauberian theorem of Ingham).

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- to bound other terms we use a trick due to Newman and Korevaar.

Proof of main result

Assuming $i\mathbb{R} \cap \sigma(A) = \emptyset$, we will bound $\|A^{-1}S(t)\|_{X \to X}$.
Fix $t \gg 1$, let $R \gg 1$ (depending on t). Recall

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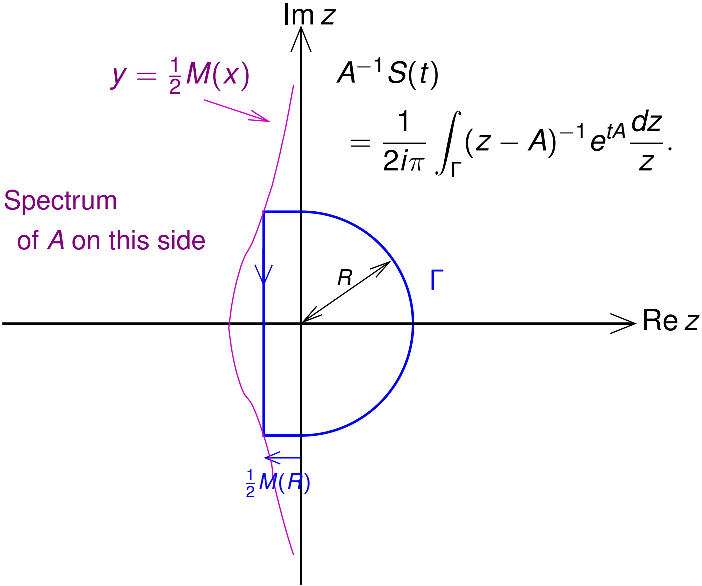
Let $\Gamma \subset F$ be a contour (0 inside Γ). By Cauchy formula

$$S(t)A^{-1} = e^{tA}A^{-1} = \frac{1}{2i\pi}\int_\Gamma (z-A)^{-1}e^{tA}\frac{dz}{z}.$$

The trick of Newman and Korevaar



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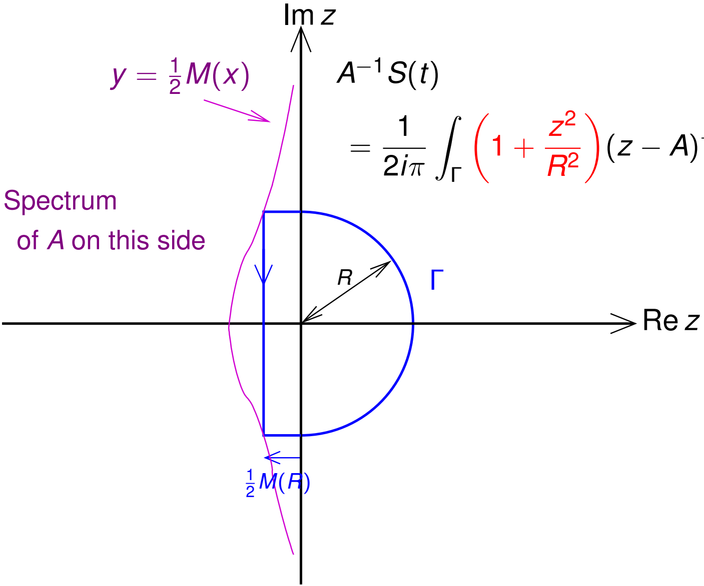
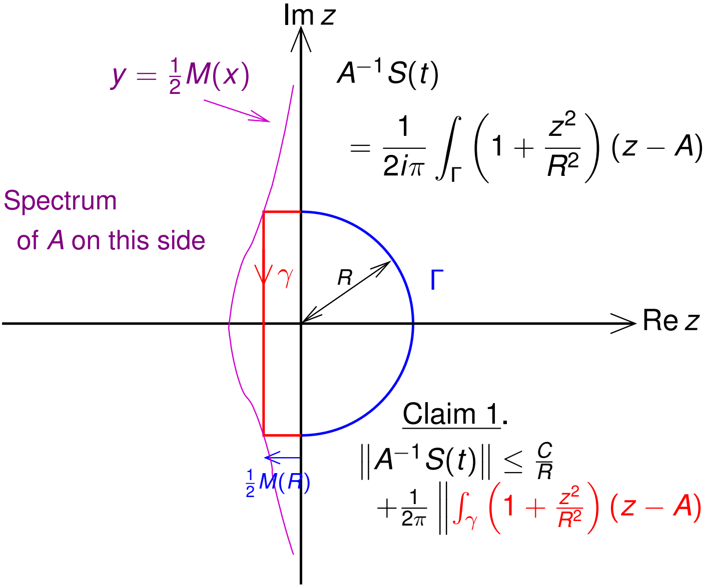


$$A^{-1}S(t) = \frac{1}{2i\pi}\int_\Gamma \left(1 + \frac{z^2}{R^2}\right)(z - A)^{-1}e^{tA}\frac{dz}{z}.$$

The trick of Newman and Korevaar



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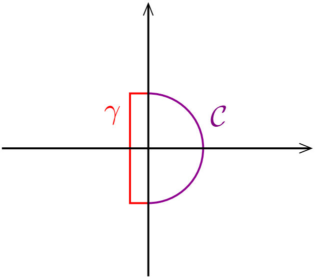
$$= \frac{1}{2i\pi}\int_\Gamma \left(1 + \frac{z^2}{R^2}\right)(z - A)^{-1} e^{tA}\frac{dz}{z}.$$

<u>Claim 1</u>.

$$\left\|A^{-1}S(t)\right\| \leq \frac{C}{R} + \frac{1}{2\pi}\left\|\int_\gamma \left(1 + \frac{z^2}{R^2}\right)(z - A)^{-1} e^{tz}\frac{dz}{z}\right\|.$$

Proof of Claim 1 (A)



$$S(t)A^{-1} - \frac{1}{2\pi i}\int_\gamma \left(1+\frac{z^2}{R^2}\right)(z-A)^{-1}e^{tz}\frac{dz}{z}$$

$$= \frac{1}{2\pi i}\int_{\mathcal{C}\cup\gamma} \left(1+\frac{z^2}{R^2}\right)(z-A)^{-1}e^{tA}\frac{dz}{z}$$

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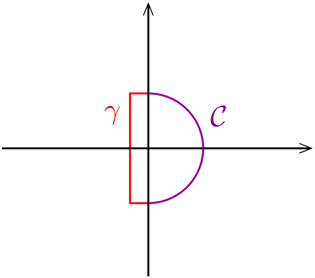

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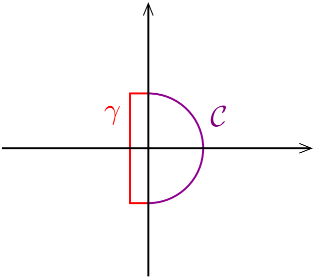

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Goal: bound this two terms by $\frac{C}{R}$.

We will only bound the first term.

Proof of Claim 1 (B)

$$\left\|(z-A)^{-1}e^{tA}\right\| = \left\|e^{tz}\int_t^{+\infty} e^{-(z-A)s}\,ds\right\|$$

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On $\mathcal{C}$, $z = Re^{i\theta}$, $\theta \in [-\pi/2, \pi/2]$. Thus:

$$\begin{aligned}\left\|e^{tz}\int_t^{+\infty} e^{-(z-A)s}\,ds\right\| \\ \leq \left|e^{tz}\int_t^{+\infty}\widetilde{C}\left|e^{-zs}\right|\,ds\right| = \left|e^{tz}\int_t^{+\infty}\widetilde{C}e^{-R(\cos\theta)s}\,ds\right| \\ \leq \frac{\widetilde{C}}{R\cos\theta}.\end{aligned}$$

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$$\leq \frac{\widetilde{C}}{R\cos\theta}.$$

Using that $\left|1+\frac{z^2}{R^2}\right| = 2|\cos\theta|$, we get

$$\left|\int_{\mathcal{C}}\left(1+\frac{z^2}{R^2}\right)(z-A)^{-1}e^{tA}\frac{dz}{z}\right| \leq \frac{2\widetilde{C}}{R}.$$

Proof of Claim 1 (C)

Bounding the other term we get Claim 1:

$$\left\| S(t)A^{-1} \right\| \leq \frac{C}{R} + \frac{1}{2\pi} \left\| \int_\gamma \left(1 + \frac{z^2}{R^2} \right) (z - A)^{-1} e^{tz} \frac{dz}{z} \right\|.$$

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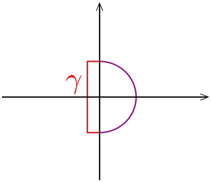
Claim 2: $\left\| \int_\gamma \left(1 + \frac{z^2}{R^2}\right)(z-A)^{-1} e^{tz} \frac{dz}{z} \right\| \leq 16 e^{-t/M(R)} RM(R)^2 + \frac{C}{R}$.

Indeed



$$\int_\gamma \cdots = \int_{\substack{\operatorname{Re} z = -\frac{1}{2M(R)} \\ -R \leq \operatorname{Im} z \leq R}} \cdots + \int_{\substack{-\frac{1}{2M(R)} \leq \operatorname{Re} z \leq 0 \\ |\operatorname{Im} z| = R}} \cdots$$

Proof of Claim 2

$$\left\| \int_{\substack{\operatorname{Re} z = -\frac{1}{2M(R)} \\ -R \leq \operatorname{Im} z \leq R}} \cdots \right\| \leq \int_{\substack{\operatorname{Re} z = -\frac{1}{2M(R)} \\ -R \leq \operatorname{Im} z \leq R}} \underbrace{\left| 1 + \frac{z^2}{R^2} \right|}_{\leq 4} \underbrace{\left\| (z - A)^{-1} \right\|}_{\leq 2M(R)} \underbrace{\left| e^{tz} \right|}_{= e^{-\frac{t}{2M(R)}}} \frac{dz}{z}$$

$$\leq 16 R M(R)^2 e^{-\frac{t}{2M(R)}}.$$

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Combining with the straightforward bound

$$\left\| \int_{\substack{-\frac{1}{2M(R)} \le \operatorname{Re} z \le 0 \\ |\operatorname{Im} z| = R}} \left(1 + \frac{z^2}{R^2} \right) (z-A)^{-1} e^{tz} \frac{dz}{z} \right\| \le \frac{C}{R},$$

we get Claim 2.

Proof of Claim 2

$$\left\| \int_{\substack{\mathrm{Re}\, z=-\frac{1}{2M(R)} \\ -R\leq \mathrm{Im}\, z\leq R}} \cdots \right\| \leq \int_{\substack{\mathrm{Re}\, z=-\frac{1}{2M(R)} \\ -R\leq \mathrm{Im}\, z\leq R}} \underbrace{\left|1+\frac{z^2}{R^2}\right|}_{\leq 4} \underbrace{\left\|(z-A)^{-1}\right\|}_{\leq 2M(R)} \underbrace{\left|e^{tz}\right|}_{=e^{-\frac{t}{2M(R)}}} \frac{dz}{z}$$

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$$\left\| \int_{\substack{-\frac{1}{2M(R)}\leq \mathrm{Re}\, z\leq 0 \\ |\,\mathrm{Im}\, z|=R}} \left(1+\frac{z^2}{R^2}\right)(z-A)^{-1} e^{tz} \frac{dz}{z} \right\| \leq \frac{C}{R},$$

we get Claim 2. Claim 1 and Claim 2 imply:

$$\left\|S(t)A^{-1}\right\| \leq \frac{C}{R} + CRM(R)^2 e^{-\frac{t}{2M(R)}}.$$

End of the proof

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Implies immediately $\lim_{t\to+\infty} \left\|S(t)A^{-1}\right\| = 0$... exactly the first step of the proof of [Arendt-Batty 88].

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Quantitative version: chose R such that

$$t = 4M(R)\Big[\log(1+M(R)) + \log(1+R)\Big] = 4M_{\log}(R).$$

Then

$$\left\|S(t)A^{-1}\right\| \leq \frac{C}{R} \leq \frac{C}{M_{\log}^{-1}\left(\frac{t}{4}\right)}.$$

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The proof is complete.