Asymptotic properties of semigroup orbits

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Theorem

Let $T \in B(X)$, $a_n \searrow 0$. *Then there exists* $x \in X$ *such that*

$$\|T^n x\| \geq a_n r(T)^n \qquad (n \in \mathbb{N})$$

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Theorem

Let $T \in B(X)$, $a_n \geq 0$, $\sum a_n < \infty$. Then there exists $x \in X$ such that

$$\|T^n x\| \geq a_n \|T^n\| \qquad (n \in \mathbb{N})$$

(if X is a Hilbert space then it is sufficient to require $\sum a_n^2 < \infty$)

Theorem

Let $\mathcal{T} = (T(t))_{t\geq 0}$ *be a strongly continuous semigroup on a Banach space* X*, let* $f \in L^\infty$, $f \geq 0$, $\lim_{t\to\infty} f(t) = 0$. *Then there exists* $x \in X$ *such that*

$$\|T(t)x\| \geq f(t)e^{\omega_0 t} \qquad (t \geq 0)$$

where ω_0 *is the growth bound of the semigroup* $\mathcal{T}$.

Theorem

Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X, let $f \in L^1$, $f(t) \searrow 0 \quad (t \to \infty)$. Then there exists $x \in X$ such that

$$\|T(t)x\| \geq f(t)e^{\omega_0 t} \qquad (t \geq 0)$$

If X is a Hilbert space then it is possible to take $f \in L^2$.

Theorem

Let H be a Hilbert space, $T \in B(X)$, $T^n \to 0 \quad (WOT)$, $r(T) = 1$. Then there exist $x, y \in H$ such that

$$|\langle T^n x, y\rangle| \geq a_n \qquad (n \in \mathbb{N})$$

Theorem

Let H be a Hilbert space, $\mathcal{T} = (T(t))_{t\geq 0}$ a strongly continuous semigroup on H with generator A, $\mathcal{T}$ weakly stable, $s(A) = 0$. Let $f \in L^\infty$, $f(t) \searrow 0 \quad (t \to \infty)$. Then there exist $x, y \in H$ such that

$$|\langle T(t)x, y\rangle| \geq a_n \qquad (n \in \mathbb{N})$$

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Let H be a Hilbert space, $\mathcal{T} = (T(t))_{t\geq 0}$ a strongly continuous semigroup on H with generator A, $\mathcal{T}$ weakly stable, $s(A) = 0$. Let $f \in L^\infty$, $f(t) \searrow 0 \quad (t \to \infty)$. Then there exist $x, y \in H$ such that

$$|\langle T(t)x, y\rangle| \geq a_n \qquad (n \in \mathbb{N})$$

Problem

Is it sufficient to assume that $\omega_0 = 0$, i.e., $r(T(t)) = 1$ for all $t \geq 0$?

Theorem

Let $T \in B(X)$, $a_n \geq 0$, $\sum \sqrt{a_n} < \infty$. Then there exist $x \in X$ and $x^* \in X^*$ such that

$$|\langle T^n x, x^* \rangle| \geq a_n \|T^n\| \qquad (n \in \mathbb{N})$$

(if X is a Hilbert space, then it is sufficient to require that $\sum a_n < \infty$)

Theorem

Let H be a Hilbert space, $\mathcal{T} = (T(t))_{t\geq 0}$ uniformly continuous semigroup and $\varepsilon > 0$ Then there exist $x, y \in H$ such that

$$|\langle T(t)x, y\rangle| \geq \frac{1}{(t+1)^{2+\varepsilon}} \|T(t)\|$$

for all $t \geq 0$.

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Problem

Is it sufficient to assume that $\mathcal{T}$ is strongly continuous?