Asymptotic Behaviour of Semigroups before 2000

Part 2: Positive Semigroups

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$(T(t))_{t\geq 0}$ C_0-semigroup on X.

$$T(t) \geq 0 :\Longleftrightarrow [f \geq 0 \Rightarrow T(t)f \geq 0] \ \& \\ X = L^p,\ 1 \leq p < \infty.$$

R. Nagel (ed.): *One-parameter Semigroups of Positive Operators*, Springer LN 1184 (1986).

Spectral bound = Growth bound

Theorem (L. Weis 1995)

$X = L^p$, $T(t) \geq 0$, $\lambda < 0$ for all $\lambda \in \sigma(A)$.
$\Longrightarrow \exists \varepsilon > 0$, $M \geq 0$ such that

$$\|T(t)\| \leq M\mathrm{e}^{-\varepsilon t} \quad (t \geq 0).$$

Wrong on $X = L^p(\Omega) \cap L^q(\Omega)$, $p \neq q$.
Wrong on $X = H^1(0,1)$ (Sobolev space).

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- X reflexive, $\|T(t)\| \leq M \Rightarrow T$ is completely ergodic.
- The Gaussian semigroup of $L^1(\mathbb{R})$ is positive, bounded, not ergodic.

Asymptotic Almost Periodicity

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General ABLV-Theorem

(a) $\|T(t)\| \leq M$;

(b) $\sigma(A) \cap i\mathbb{R}$ countable;

(c) T is completely ergodic.

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Remark

If $(\lambda - A)^{-1}$ is compact, then (a) $\Rightarrow$ (b) and (c).

Perron-Frobenius Theory

1. An Every-Day Perron-Frobenius Theorem
2. A Dynamical System Theorem
3. Kernel Operators

The Every-Day Perron-Frobenius Theorem

Theorem (Greiner 1981)

$T(t) \geq 0$, $\|T(t)\| \leq M$, $i\eta \in \sigma(A)$.
$\Longrightarrow i\eta m \in \sigma(A)\ \forall m \in \mathbb{Z}$ *(cyclicity of the boundary spectrum).*

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Theorem (Every-Day Perron-Frobenius Theorem)

(a) $T(t) \geq 0$;

(b) $T\colon (0,\infty) \to \mathscr{L}(L^p)$ *norm-continuous;*

(c) $\|T(t)\| \leq M$;

(d) T *ergodic.*

$\Longrightarrow \lim_{t\to\infty} T(t)f$ *exists for all* $f \in L^p$.

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Proof.

T norm-continuous $\Rightarrow \sigma(A) \cap i\mathbb{R}$ bounded. □

Norm-continuity

Theorem (Goersmeyer-Weis 1999)

$X = L^p(\Omega)$, $1 \le p < \infty$, $T(t) \ge 0$, $\lim\limits_{|\eta| \to \infty} \|(i\eta - A)^{-1}\| = 0$.

$\Longrightarrow$ *T norm-continuous.*

false on Banach spaces: Tamás Matrai 2008, Chill-Tomilov 2009

true on Hilbert spaces: Puhong-You 1992, El-Mennaoui-Engel 1996

Complete Ergodicity

$X = L^p$, $1 \leq p < \infty$.

Theorem (Batty, W.A. 1992)

$T(t) \geq 0$, $\|T(t)\| \leq M$, ergodic
$\Longrightarrow$ completely ergodic

false on $\mathrm{C}(K)$.

Irreducibility

A closed ideal of $L^p(\Omega)$ is a set of the form
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Definition

T is *irreducible* if

$$T(t)L^p(S) \subset L^p(S) \Longrightarrow \mu(S) = 0 \text{ or } \mu(\Omega \setminus S) = 0.$$

The Dynamical System Result

Theorem

$T(t) \geq 0$, $\|T(t)\| \leq M$, T irreducible, ergodic, $\sigma(A) \cap i\mathbb{R}$ countable.
$\Longrightarrow X = X_0 \oplus X_{\mathrm{ap}}$ and $T(t)|_{X_{\mathrm{ap}}}$ is a periodic group.

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Theorem (Greiner 1981)

$T(t) \geq 0$, $\|T(t)\| \leq M$, T irreducible, $\sigma_p(A) \cap i\mathbb{R} \not\subset \{0\}$.
$\Longrightarrow \exists \tau > 0$: $\sigma_p(A) \cap i\mathbb{R} \subset i\frac{2\pi}{\tau}\mathbb{Z}$.

Now:

$$T(t)x = \mathrm{e}^{i\eta\frac{2\pi}{\tau}t}x \Longrightarrow T(t+\tau)x = T(t)x$$

Trivial Point Spectrum

Theorem (E.B. Davies JEE 2005)

$T(t) \geq 0$, $\|T(t)\| \leq 1$.

(a) $X = \ell^p$

(b) *or* $X = L^p(\Omega)$, $\Omega \subset \mathbb{R}^d$ *open*, $T(t)f \in \mathrm{C}(\Omega)$ $\forall t > 0$, $f \in L^p(\Omega)$.

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Theorem

$T(t) \geq 0$, $\|\lambda(\lambda - A)^{-1}\| \leq M$ $(\lambda > 0)$,
$T(t_0)$ kernel operator for some $t_0 > 0$.
$\Longrightarrow \sigma(A) \cap i\mathbb{R} \subset \{0\}$.

Remark

(a) or (b) $\Rightarrow$ $T(t)$ is a kernel operator.

Convergence Result for Kernel Operators

Theorem

$X = L^p$, $T(t) \geq 0$, $\|T(t)\| \leq M$, $T(t)$ kernel operator, T irreducible, $\operatorname{Ker} A \neq \{0\}$. Then $\lim_{t\to\infty} T(t)f = Pf$, $f \in L^p$, P a rank-1-projection.

G. Greiner: Spektrum and Asymptotik stark stetiger Halbgruppen positiver Operatoren. Sitzungsberichte der Heidelberger Akademie der Wissenschaften. Mathematisch-naturwissenschaftliche Klasse (1982), 3. Abhandlung.

W.A.: Positivity 2008

The End

Thank you for your attention!