Asymptotic Behaviour of Semigroups before 2000
Part 1 : General Results

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Background

Abstract Cauchy problem

$$\begin{aligned} u'(t) &= Au(t) \quad (t \in \mathbb{R}_+), \\ u(0) &= x. \end{aligned} \tag{ACP}$$

$u : \mathbb{R}_+ \to X$ (Banach space); A closed linear operator on X; $x \in X$.

Mild solution of (ACP):

$$u(t) = x + A\left(\int_0^t u(s)\,ds\right).$$

We assume that u is continuous. Then u is a classical solution of (ACP) if and only if u is continuously differentiable.

(ACP) is well-posed if and only if A generates a C_0-semigroup $\{T(t) : t \geq 0\}$. Then

- T strongly continuous,
- $T(s)T(t) = T(s+t) \quad (s, t \geq 0)$,
- $T(0) = I$,
- $Ax = \lim_{t \to 0+} t^{-1}(T(t)x - x)$ if this exists,
- $R(\lambda, A)x := (\lambda I - A)^{-1}x = \int_0^\infty e^{-\lambda t} T(t)x \, dt$ $(\operatorname{Re} \lambda > \omega)$,
- $u(t) := T(t)x$ is a mild solution of (ACP) for all $x \in X$; it is a classical solution if and only if $x \in D(A)$.

Growth bounds and spectral bounds

$$\omega_0(T) = \inf\Big\{\omega \in \mathbb{R} : \text{there exists } M_\omega \text{ such that } \|T(t)\| \le M_\omega e^{\omega t} \text{ for all } t \ge 0\Big\}$$

$$= \inf\Big\{\omega \in \mathbb{R} : \text{for all } x \in X \text{ there exists } M_{\omega,x} \text{ such that } \|T(t)x\| \le M_{\omega,x} e^{\omega t} \text{ for all } t \ge 0\Big\},$$

$$\omega_1(T) = \inf\Big\{\omega \in \mathbb{R} : \text{there exists } M_\omega \text{ such that } \|T(t)R(\mu, A)\| \le M_\omega e^{\omega t} \text{ for all } t \ge 0\Big\}$$

$$= \inf\Big\{\omega \in \mathbb{R} : \text{for all } x \in D(A) \text{ there exists } M_{\omega,x} \text{ such that } \|T(t)x\| \le M_{\omega,x} e^{\omega t} \text{ for all } t \ge 0\Big\}$$

$$s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\},$$

$$s_0(A) = \inf\left\{\omega \in \mathbb{R} : \{\operatorname{Re}\lambda > \omega\} \subseteq \rho(A), \sup_{\operatorname{Re}\lambda > \omega} \|R(\lambda, A)\| < \infty\right\}.$$

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Easy facts

$$s(A) \le s_0(A) \le \omega_0(T) < \infty; \qquad s(A) \le \omega_1(T) \le \omega_0(T).$$

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$s(A) = \omega_0(T)$ if T is a holomorphic semigroup (or eventually norm-continuous)

Theorem (Gearhart 1978, Prüss and others c.1984)

$s_0(A) = \omega_0(T)$ if X is a Hilbert space.

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For any C_0-semigroup, $\omega_1(T) \le s_0(A)$.

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This can be proved by the complex inversion formula for Laplace transforms (with a subtly chosen contour or function) (van Neerven). In fact this approach works for functions (Blake).

If $f : \mathbb{R}_+ \to X$ is exponentially bounded and its Laplace transform $\widehat{f}$ has a bounded holomorphic extension to $\{\operatorname{Re}\lambda > \omega\}$, then

$$\widehat{f}(\lambda) = \lim_{t\to\infty} \int_0^t e^{-\lambda s} f(s)\, ds \quad (\operatorname{Re}\lambda > \omega).$$

Moreover,

$$\left\| \int_0^t e^{-\omega s} f(s)\, ds \right\| \le c(1+t).$$

Note: The result is not true for arbitrary Laplace-transformable functions (Bloch, 1949).

Countable spectrum and stability

Theorem (Arendt-Batty, Lyubich-Vũ, 1988)

Let T be a bounded semigroup with generator A and suppose that $\sigma(A) \cap i\mathbb{R}$ is countable and $P\sigma(A^*) \cap i\mathbb{R}$ is empty. Then

$$\lim_{t\to\infty} \|T(t)x\| = 0$$

for all $x \in X$.

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The assumptions that T is bounded and $P\sigma(A^*) \cap i\mathbb{R}$ is empty are necessary for the conclusion.

Countability of $\sigma(A) \cap i\mathbb{R}$ is not necessary [Example: Left-translations on $C_0(\mathbb{R}_+)$] but it is best possible in a certain sense [Given an uncountable closed subset E of $i\mathbb{R}$, there is a unitary group with $\sigma(A) \subseteq E$.]

Earlier results

- A bounded (Sklyar-Shirman 1982)
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Proofs

LV Functional analytic construction - very neat

AB Laplace transforms (Ingham, Allan-O'Farrell-Ransford) and (horrible) transfinite induction

Others (Esterle-Strouse-Zouakia 1992, Batty-Chill-Tomilov 2002)

Generalisations, etc (Allan, Arendt, Batty, Chill, El Mennaoui, Greenfield, Huang, Hutter, Kérchy, Léka, Lyubich, Nagel, van Neerven, Prüss, Räbiger, Ransford, Vũ, Yeates .....)

- asymptotically almost periodic semigroups (relatively compact orbits)
- Volterra equations
- integrated semigroups
- periodic non-autonomous problems
- inhomogeneous Cauchy problems
- weighted versions
- representations of abelian semigroups
- individual bounded, uniformly continuous, mild solutions
- bounded functions and Laplace transforms

For a bounded semigroup T and $f \in L^1(\mathbb{R}_+)$, define $f(T) \in \mathcal{B}(X)$ by

$$f(T)x = \int_0^\infty f(t)T(t)x\,dt.$$

Theorem (Katznelson-Tzafriri Theorem for semigroups)

Let T be a bounded semigroup, and let $f \in L^1(\mathbb{R}_+)$ be of spectral synthesis with respect to $i\sigma(A) \cap \mathbb{R}$. Then

$$\|T(t)f(T)\| \to 0 \text{ as } t \to \infty.$$