Strong stability of semigroups: a personal (over-)view

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Throughout, $(T(t))_{t\geq 0}$ is a **bounded** C_0-semigroup on a Banach space X with generator A

The C_0-semigroup $(T(t))_{t\geq 0}$ is **stable** if

$$\lim_{t\to\infty} \|T(t)x\| = 0 \quad \text{for every } x \in X.$$

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Three reasons to study stability:

- important subject of operator theory;
- rich in methods and ideas;
- interesting for its own sake.

Spectral conditions for stability

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Basic answer: If A generates a bounded C_0-semigroup and if

- the boundary spectrum $\sigma(A) \cap i\mathbb{R}$ is empty, or
- the boundary spectrum is countable and contains no residual spectrum (Arendt-Batty-Lyubich-Vu Theorem)

then the semigroup is stable.

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Possible approaches:

- contour integral method;
- limit isometric semigroup method;
- functional calculi method (Katznelson-Tzafriri).

Spectral Principle:

small boundary spectrum $\Rightarrow$ *better stability properties*

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Boundary Value Principle:
"good" boundary value of resolvent $\Rightarrow$ *better stability properties*

Laplace and Fourier transforms: a formal set-up

Definition
For every bounded and measurable $f : \mathbb{R}_+ \to X$ define the **Laplace transform** $\hat{f}$ by

$$\hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\, dt, \quad \lambda \in \mathbb{C}_+,$$

where $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > 0\}$ is the open right-half plane. The Laplace transform $\hat{f}$ is analytic in $\mathbb{C}_+$.

If A generates a bounded C_0-semigroup $(T(t))_{t\geq 0}$, then $\hat{T}(\lambda)x = R(\lambda, A)x$, that is, **the resolvent is the Laplace transform of the semigroup.**

Define the Fourier transform $\mathcal{F}\varphi$ of an **integrable** $\varphi : \mathbb{R} \mapsto X$ by

$$\mathcal{F}\varphi(\beta) := \int_{\mathbb{R}} e^{-i\beta t}\varphi(t)\, dt, \quad \beta \in \mathbb{R}.$$

The Fourier transform $\mathcal{F}f$ of a **bounded** measurable function $f : \mathbb{R} \mapsto X$ is defined in the distributional sense. It is always true that

$$\mathcal{F}f = \lim_{\alpha\to 0+} \mathcal{F}(e^{-\alpha\cdot} f) = \lim_{\alpha\to 0+} \hat{f}(\alpha + i\cdot)$$

in the distributional sense.

Stronger convergence implies stability

Theorem [Ingham]
Let $f \in BUC(\mathbb{R}_+; X)$. Assume that $\hat{f}$ has a locally integrable extension on $i\mathbb{R}$ in the sense that

$$\lim_{\alpha \to 0+} \hat{f}(\alpha + i\cdot) = \hat{f}(i\cdot) \qquad \text{in } L^1_{loc}(\mathbb{R}; X).$$

Then $f \in C_0(\mathbb{R}_+; X)$.

Proof:

Let $\varphi \in \mathcal{S}(\mathbb{R})$ be such that $\mathcal{F}\varphi \in \mathcal{D}(\mathbb{R})$. Then

$$
\begin{aligned}
f * \varphi(t) &= \int_0^\infty f(s)\varphi(t-s)\, ds \\
&= \lim_{\alpha\to 0+} \int_0^\infty e^{-\alpha s} f(s)\varphi(t-s)\, ds \\
\text{(by Parseval)} &= \lim_{\alpha\to 0+} \int_{\mathbb{R}} \hat{f}(\alpha + i\beta) e^{i\beta t} \mathcal{F}^{-1}\varphi(\beta)\, d\beta \\
\text{(by assumption)} &= \int_{\mathbb{R}} \hat{f}(i\beta) e^{i\beta t} \mathcal{F}^{-1}\varphi(\beta)\, d\beta .
\end{aligned}
$$

By the Lemma of Riemann-Lebesgue,

$$
\lim_{|t|\to\infty} f * \varphi(t) = 0.
$$

Choose an approximate unit of appropriate test functions and use that f is bounded and uniformly continuous.

Corollary

If the boundary spectrum $\sigma(A) \cap i\mathbb{R}$ is empty, then the semigroup is stable.

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Proof of Ingham can be adapted in order to prove also the ABLV theorem or the Katznelson-Tzafriri theorem.

Theorem [ABLV]
If the boundary spectrum $\sigma(A) \cap i\mathbb{R}$ is **countable**, and if $\mathrm{Rg}\,(i\beta - A)$ **is dense in** X for every $\beta \in \mathbb{R}$, then the semigroup is stable.

Theorem [Katznelson-Tzafriri]
If $f \in L^1(\mathbb{R}_+)$ is of **spectral synthesis** with $\sigma(A) \cap i\mathbb{R}$, then

$$\lim_{t\to\infty} \|T(t)\hat{f}(T)\| = 0.$$

Large boundary spectrum: a motivation for further study

Let $\omega : \mathbb{R}_+ \to (0, \infty)$ a continuous and nonincreasing function such that
(i) $\lim_{t\to+\infty} \omega(t) = 0$, and
(ii) the function $1/\omega$ is of subexponential growth on $\mathbb{R}_+$.
$X_p := L^p(\mathbb{R}_+; \omega(t)dt)$ $(1 \le p < \infty)$, $(S(t))_{t\ge 0}$ is the **stable** right-shift C_0-semigroup defined by

$$(S(t)f)(s) := \begin{cases} f(s-t), & s \ge t \ge 0, \\ 0, & 0 \le s < t, \end{cases} \qquad f \in X_p. \tag{1}$$

If $w(t) = (\log(2+t))^{-1}$, then

- for every nonzero $f \in X_p$ and every $\beta \in \mathbb{R}$, the local resolvent $R(\cdot, D)f$ does not extend continuously near $i\beta$.
- the boundary spectrum is the whole imaginary axis
- every nonzero orbit $S(\cdot)f$ does not satisfy the conditions neither of Ingham's tauberian theorem nor of its generalization discussed before.

Such examples show that **FINER** *stability conditions are needed.*

Stability and complete trajectories

A function $F : \mathbb{R} \to X$ is a *complete trajectory* for a C_0-semigroup $(T(t))_{t\geq 0}$ if for all $t \geq 0$ and all $s \in \mathbb{R}$: $F(t+s) = T(t)F(s)$.

Theorem For a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X the following statements are equivalent:

(i) The semigroup $(T(t))_{t\geq 0}$ is stable,

(ii) There is only one bounded, complete trajectory for the adjoint semigroup $(T(t)^*)_{t\geq 0}$, namely $F = 0$.

Carleman transform as a tool

For every bounded measurable $f : \mathbb{R} \to X$ define the *Carleman transform* $\widehat{f}$ by

$$\widehat{f}(\lambda) := \begin{cases} \int_0^\infty e^{-\lambda t} f(t)\, dt, & \mathrm{Re}\,\lambda > 0, \\ -\int_{-\infty}^0 e^{-\lambda t} f(t)\, dt, & \mathrm{Re}\,\lambda < 0. \end{cases}$$

The Carleman transform $\hat{f}$ is analytic in $\mathbb{C} \backslash i\mathbb{R}$. The Carleman transform $\hat{f}$ is entire if and only if $f = 0$!

Stability, complete trajectories and Carleman transform

Theorem For a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X the following statements are equivalent:

(i) The semigroup $(T(t))_{t\geq 0}$ is stable,

(ii) The Carleman transform of every bounded, complete trajectory of the adjoint semigroup $(T(t)^*)_{t\geq 0}$ is entire.

Remark Let F be a bounded complete trajectory for $(T(t)^*)_{t\geq 0}$. Then for every $\lambda \in \mathbb{C}_+$ and every $\mu \in \mathbb{C}\backslash i\mathbb{R}$:

$$\hat{F}(\mu) = R(\lambda, A^*)F(0) + (\lambda - \mu)R(\lambda, A^*)\hat{F}(\mu).$$

Edge-of-the wedge theorems, or: how to kill a complete trajectory

Edge-of-the wedge theorems, or: how to kill a complete trajectory

Theorem Let $f : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ be analytic, and define $F : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ by $F(z) = f(z) - f(\bar{z})$. Assume that
(1) there exists a constant $m \geq 0$ such that

$$\sup_{\alpha\in\mathbb{R}} |f(\alpha + i\beta)| = O(|\beta|^{-m}), \quad \beta \to 0,$$

(2) there exist a measurable function $G : \mathbb{C}\backslash\mathbb{R} \to \mathbb{R}_+$ and a continuous function $H : \mathbb{C}\backslash\mathbb{R} \to \mathbb{R}_+$, such that $|F| \leq G \cdot H$,

$$\sup_{\beta\in(0,1)} \|G(\cdot + i\beta)\|_{L^1(-R,R)} < \infty \quad \text{for every } R > 0,$$

and there exists $\theta_0 \in (0, \frac{\pi}{2})$ such that

$$\lim_{\substack{z\to\alpha \\ z\in\alpha+\Sigma_{\theta_0}}} H(z) = 0 \quad \text{for every } \alpha \in \mathbb{R}.$$

Then the function f is entire.

Katarina
August 28, 2009

MAIN IDEA: The boundary behaviour of (local) resolvents determines the stability of the semigroup

$\swarrow$ $\searrow$

pointwise resolvent conditions reflect the boundary behaviour of the resolvent horizontally near every point of the imaginary axis

integral resolvent conditions reflect the boundary behaviour of integrals of resolvents along vertical lines near the imaginary axis

$\swarrow$ $\downarrow$

global integral conditions behaviour of integrals of local resolvents along **whole** vertical lines near imaginary axis

local integral conditions behaviour of integrals of local resolvents along bounded intervals of vertical lines near imaginary axis

Pointwise resolvent conditions in Banach space

Theorem [Pointwise resolvent condition in Banach space]
If there exists a dense set $M \subset X$ such that for every $x \in M$ and every $\beta \in \mathbb{R}$

$$\lim_{\alpha \to 0+} \alpha R(\alpha + i\beta, A)^2 x = 0,$$

then the semigroup is stable.

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then the semigroup is stable.

Corollary [Range condition in Banach space]
If

$$\bigcap_{\beta \in \mathbb{R}} \mathrm{Rg}\,(i\beta - A) \text{ is dense in } X,$$

then the semigroup is stable.

Illustration: positive semigroups on Banach lattices

Proposition Let $(T(t))_{t\geq 0}$ be a bounded positive C_0-semigroup on a Banach lattice X.

(i) If $x \in X_+$ and $\lim_{\alpha\to 0+} \alpha R^2(\alpha, A)x = 0$, then $T(t)x \to 0$, $t \to \infty$.

(ii) If $x \in X_+$ and $w - \lim_{\alpha\to 0+} R(\alpha, A)x$ exists, then $T(t)x \to 0, t \to \infty$.

(iii) If $x \in X_+$, $\sup_{\alpha>0} \|R(\alpha, A)x\| < \infty$, and if X is a KB space, then $T(\cdot)x \to 0$, $t \to \infty$.

(iv) If $x \in X$ and $\lim_{\alpha\to 0+} R(\alpha, A)x_\pm$ exist, then $T(t)x \to 0$, $t \to \infty$.

Remark By positivity, under (ii), (iii) or (iv): $\sup_{\lambda\in\mathbb{C}_+} \|R(\lambda, A)x\| < \infty$.

Integral resolvent conditions in Banach space

Theorem [Global integrability criterion in Banach space]
If for some $\gamma > 1$ and for every x from a dense subset of X

$$\lim_{\alpha \to 0+} \int_{\mathbb{R}} \|\alpha^{\gamma-1} R(\alpha + i\beta, A)^{\gamma} x\| \, d\beta = 0,$$

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then the semigroup is stable.

Theorem [Local integrability criterion in Banach space]
If for every $\beta \in \mathbb{R}$ there exists an open neighbourhood $U \subset \mathbb{R}$ of β and a dense set $M \subset X$ such that

$$\lim_{\alpha\to 0+} \int_{U} \|\alpha R(\alpha + i\beta', A)^{2} x\| \, d\beta' = 0 \quad \text{for every } x \in M,$$

then the semigroup is stable.

What about stability of Hilbert space semigroups ?

Specifics: *special geometric properties* of Hilbert spaces, for example the validity of Plancherel's theorem, unitary dilations, functional calculi.

Stability of semigroups on Hilbert spaces is *of independent interest* in operator theory (for example for the study of invariant subspaces).

Pointwise resolvent conditions in Hilbert space

Theorem [Pointwise resolvent conditions in Hilbert space]

(i) If there exists a dense set $M \subset X$ such that

$$\lim_{\alpha \to 0+} \sqrt{\alpha} R(\alpha + i\beta, A)x = 0 \ \ \forall x \in M, \forall \beta \in \mathbb{R},$$

then the semigroup is stable.

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then the semigroup is stable.

(ii) If

$$\bigcap_{\beta \in \mathbb{R}} \mathrm{Rg}\,(i\beta - A)^{\frac{1}{2}} \text{ is dense in } X,$$

then the semigroup is stable.

Theorem [Pointwise criterion, Hilbert space contractions]
$(T(t))_{t\geq 0}$ is a C_0-semigroup of completely nonunitary contractions.

(i) The semigroup is stable if and only if there exists a dense $M \subset X$ such that

$$\lim_{\alpha\to 0+} \sqrt{\alpha} R(\alpha + i\beta, A)x = 0 \ \ \forall x \in M \text{ and a.e. } \beta \in \mathbb{R}.$$

(ii) If there exists $E \subset \mathbb{R}$ of measure 0 such that

$$\bigcap_{\beta\in\mathbb{R}\setminus E} \mathrm{Rg}\,(i\beta - A)^{\frac{1}{2}} \text{ is dense in } X,$$

then the semigroup is stable.

Integral resolvent conditions in Hilbert space

Theorem [Global integrability criterion in Hilbert space]
The semigroup is stable *if and only if* for some $\gamma > \frac{1}{2}$ and every x from a dense subset of X,

$$\lim_{\alpha \to 0+} \int_{\mathbb{R}} \|\alpha^{\gamma - \frac{1}{2}} R(\alpha + i\beta, A)^{\gamma} x\|^2 \, d\beta = 0.$$

Integral resolvent conditions in Hilbert space

Theorem [Global integrability criterion in Hilbert space]
The semigroup is stable *if and only if* for some $\gamma > \frac{1}{2}$ and every x from a dense subset of X,

$$\lim_{\alpha\to 0+} \int_{\mathbb{R}} \|\alpha^{\gamma-\frac{1}{2}} R(\alpha+i\beta, A)^{\gamma} x\|^2 \, d\beta = 0.$$

Theorem [Local integrability criterion in Hilbert space]
The semigroup is stable *if and only if* for every $\beta \in \mathbb{R}$ there exists an open neighbourhood $U \subset \mathbb{R}$ of β and a dense set $M \subset X$:

$$\lim_{\alpha\to 0+} \int_{U} \|\alpha^{\frac{1}{2}} R(\alpha+i\beta', A) x\|^2 \, d\beta' = 0 \quad \text{for every } x \in M.$$

Pointwise resolvent conditions in Banach spaces with nontrivial Fourier type

Theorem [Pointwise resolvent condition]
X has Fourier type $p \in (1,2]$, $q := \frac{p}{p-1}$. If there exists a dense $M \subset X$ such that

$$\lim_{\alpha \to 0+} \|\alpha^{\frac{1}{q}} R(\alpha + i\beta, A)x\| = 0, \ \forall \beta \in \mathbb{R}, \forall x \in M,$$

then the semigroup is stable.

Proof
$F : \mathbb{R} \to X^*$ is a bounded complete trajectory for $(T(t)^*)_{t \geq 0}$,
$x \in M$, $f := \langle F, x \rangle$,
$\hat{F}$ and $\hat{f}$ are the Carleman transforms of F and f.

Local resolvent identity implies

$$\begin{aligned} & |\hat{f}(\alpha + i\beta) - \hat{f}(-\alpha + i\beta)| \\ = \; & |2\langle \alpha^{\frac{1}{p}} \hat{F}(-\alpha + i\beta), \alpha^{\frac{1}{q}} R(\alpha + i\beta, A)x\rangle| \\ \leq \; & G(\alpha + i\beta)\, H(\alpha + i\beta), \end{aligned}$$

with

$$G(\alpha + i\beta) := \|2\alpha^{\frac{1}{p}} \hat{F}(-\alpha + i\beta)\|$$

and

$$H(\alpha + i\beta) := \|\alpha^{\frac{1}{q}} R(\alpha + i\beta, A)x\|.$$

Boundedness of F + Hausdorff-Young inequality imply

$$\sup_{\alpha > 0} \|G(\alpha + i\cdot)\|_{L^q(\mathbb{R})} < \infty.$$

Now for every $\theta_0 \in (0, \frac{\pi}{2})$ and every $\beta \in \mathbb{R}$:

$$
\begin{aligned}
& \limsup_{\substack{\alpha\to 0+ \\ |\beta'-\beta|\le\alpha \tan\theta_0}} H(\alpha + i\beta') \\
\le\ & \limsup_{\substack{\alpha\to 0+ \\ |\beta'-\beta|\le\alpha \tan\theta_0}} \|\alpha^{\frac{1}{q}}\big(R(\alpha+i\beta',A) - R(\alpha+i\beta,A)\big)x\| \\
\le\ & \limsup_{\substack{\alpha\to 0+ \\ |\beta'-\beta|\le\alpha \tan\theta_0}} \|\alpha \tan\theta_0\, R(\alpha+i\beta',A)\,\alpha^{\frac{1}{q}} R(\alpha+i\beta,A)x\| \\
\le\ & \tan\theta_0 \sup_{t\ge 0}\|T(t)\| \limsup_{\alpha\to 0+} \|\alpha^{\frac{1}{q}} R(\alpha+i\beta,A)x\| \\
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\le\ & \limsup_{\substack{\alpha \to 0+ \\ |\beta' - \beta| \le \alpha \tan \theta_0}} \|\alpha^{\frac{1}{q}} \big(R(\alpha + i\beta', A) - R(\alpha + i\beta, A) \big) x\| \\
\le\ & \limsup_{\substack{\alpha \to 0+ \\ |\beta' - \beta| \le \alpha \tan \theta_0}} \|\alpha \tan \theta_0\, R(\alpha + i\beta', A)\, \alpha^{\frac{1}{q}} R(\alpha + i\beta, A) x\| \\
\le\ & \tan \theta_0 \sup_{t \ge 0} \|T(t)\| \limsup_{\alpha \to 0+} \|\alpha^{\frac{1}{q}} R(\alpha + i\beta, A) x\| \\
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Edge-of-the-wedge theorem $\Longrightarrow$ $\hat{f}$ is an entire function

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$\Longrightarrow f = \langle F, x \rangle = 0$ for every $x \in M$

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Edge-of-the-wedge theorem $\Longrightarrow$ $\hat{f}$ is an entire function
$\Longrightarrow$ $f = \langle F, x\rangle = 0$ for every $x \in M$
$\Longrightarrow$ $F = 0$ since M is dense, and there is no nontrivial bounded complete trajectory for $(T(t))^*)_{t \ge 0}$

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Edge-of-the-wedge theorem $\Longrightarrow$ $\hat{f}$ is an entire function
$\Longrightarrow$ $f = \langle F, x \rangle = 0$ for every $x \in M$
$\Longrightarrow$ $F = 0$ since M is dense, and there is no nontrivial bounded complete trajectory for $(T(t))^*)_{t \ge 0}$
$\Longrightarrow$ the semigroup $(T(t))_{t \ge 0}$ is stable.

Range condition in Banach spaces with nontrivial Fourier type

Corollary

X has Fourier type $p \in (1, 2]$. If

$$\bigcap_{\beta \in \mathbb{R}} \mathrm{Rg}\,(i\beta - A)^{\frac{1}{p}} \text{ is dense in } X,$$

then the semigroup is stable.

Integral resolvent conditions in Banach spaces with nontrivial Fourier type

Theorem [Global integral condition]
X has Fourier type $p \in [1, 2]$. If for some $\gamma > \frac{1}{p}$ and for every x from a dense subset of X,

$$\lim_{\alpha \to 0+} \int_{\mathbb{R}} \|\alpha^{\gamma - \frac{1}{p}} R(\alpha + i\beta, A)^{\gamma} x\|^p \, d\beta = 0,$$

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Theorem [Local integral condition]
X has Fourier type $p \in (1, 2]$, $q = \frac{p}{p-1}$. If for every $\beta \in \mathbb{R}$ there exists an open neighbourhood $U \subset \mathbb{R}$ of β and a dense set $M \subset X$ such that

$$\lim_{\alpha \to 0+} \int_{U} \|\alpha^{\frac{1}{q}} R(\alpha + i\beta', A) x\|^p \, d\beta' = 0 \quad \text{for every } x \in M,$$

then the semigroup is stable.

OPEN PROBLEM

Let $(T(t))_{t\geq 0}$ be a stable semigroup on a Hilbert space H. Does there exist a dense set $M \subset X$ such that

$$\lim_{\alpha\to 0+} \sqrt{\alpha} R(\alpha + i\beta, A)x = 0 \;\; \forall x \in M, \forall \beta \in \mathbb{R}?$$

Remark The converse statement holds
('pointwise resolvent criterion')

Remark It seems that there are stable semigroups on H such that

$$\bigcap_{\beta\in\mathbb{R}} \mathrm{Rg}\,(i\beta - A)^{\frac{1}{2}} = \{0\}$$

that is, the 'range condition' fails to be a criterion.