Semigroup Growth Bounds

First Meeting on Asymptotics of Operator Semigroups

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There are several distinct issues for a one-parameter semigroup $T_t = e^{At}$ acting in a Banach space $\mathcal{B}$.

- The long time asymptotics of $\|T_t\|$;
- The short time asymptotics of $\|T_t\|$;
- The intermediate time behaviour of $\|T_t\|$;
- The spectrum of $A$;
- The behaviour of the norms of the resolvent operators $R_z = (zl - A)^{-1}$. 

E.B. Davies (KCL)
Every semigroup has a bound of the form

$$\| T_t \| \leq Me^{at} \quad \text{for all } t \geq 0.$$  

This implies that

$$\| R_z \| \leq M(\text{Re}(z) - a)^{-1}$$

for all $z$ satisfying $\text{Re}(z) > a$.

The precise form of the converse was proved by Feller, Miyadera and Phillips.
Dissipativity

If $B$ is a Hilbert space and
\[ \text{Re} \langle Af, f \rangle \leq a \]
for all $f \in \text{Dom}(A)$ then
\[ \| T_t \| \leq e^{at} \quad \text{for all } t \geq 0 \]
and conversely.
The Asymptotic Growth Rate

The infimum of all possible $a$ is given by

$$\omega_0 = \lim_{t \to +\infty} t^{-1} \log(\| T_t \|).$$

This implies that

$$\text{Spec}(A) \subseteq \{ z : \text{Re}(z) \leq \omega_0 \}$$

and

$$\| T_t \| \geq e^{\omega_0 t} \text{ for all } t > 0.$$
Zabczyk’s Example

\[ \text{Spec}(T_t) \supseteq \{ e^{zt} : z \in \text{Spec}(A) \} . \]

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There exists a one-parameter group \( T_t \) acting in a Hilbert space \( \mathcal{H} \) such that

\[ \text{Spec}(A) \subseteq i\mathbb{R} \]

but

\[ \| T_t \| = e^{|t|} \quad \text{for all } t \in \mathbb{R}. \]

\(^1\) Zabczyk 1975
The Schrödinger Group

The operators $T_t = e^{i\Delta t}$ are unbounded on $L^p(\mathbb{R}^n)$ for all $p \neq 2$ and $0 \neq t \in \mathbb{R}$ in spite of the fact that $\text{Spec}(\Delta) \subseteq \mathbb{R}$.

Hormander 1960
The operators $T_t = e^{i\Delta t}$ are unbounded on $L^p(\mathbb{R}^n)$ for all $p \neq 2$ and $0 \neq t \in \mathbb{R}$ in spite of the fact that

$$\text{Spec}(\Delta) \subseteq \mathbb{R}.$$ 

The resolvents of $\Delta$ satisfy

$$\|R_z\| \leq c_p|\text{Im}(z)|^{-1}$$

for all $z \notin \mathbb{R}$, where $c_p \to 1$ as $p \to 2$. 

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$^2$Hormander 1960
If $\mathcal{H} = L^2(-\pi, \pi)$ and $0 < \varepsilon < 2$ and

$$(Lf)(\theta) = \varepsilon \frac{d}{d\theta} \left\{ \sin(\theta) \frac{df}{d\theta} \right\} + \frac{df}{d\theta}$$

then

$$\frac{df}{dt} = Lf(t)$$

describes the evolution of a thin fluid layer inside a rotating cylinder.

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An Indefinite ODE$^4$

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If $\varepsilon > 2$ then the spectrum of $L$ includes the entire imaginary axis and probably the entire complex plane.

If $0 < \varepsilon < 2$ the resolvent operators are all compact but $e^{Lt}$ is unbounded for all $t \neq 0$. 

\footnote{Benilov, O’Brien, Sazonov, Weir et al. 2000-2008}
An Example with an Oscillating Norm

Let

\[(T_t f)(x) = \frac{a(x + t)}{a(x)} f(x + t)\]

for all \(f \in L^2(0, \infty)\) and all \(t \geq 0\).
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If \(c > 1\) then the choice
\[a(x) = 1 + (c - 1) \sin^2(\pi x/2)\]
leads to \(\|T_{2n}\| = 1\) and \(\|T_{(2n+1)}\| = c\) for all positive integers \(n\).
Definition of $N(t)^5$

We define $N(t)$ to be the upper log-concave envelope of $\|T_t\|$. In other words $\nu(t) = \log(N(t))$ is defined to be the smallest concave function satisfying $\nu(t) \geq \log(\|T_t\|)$ for all $t \geq 0$.

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It is immediate that $N(t)$ is continuous for $t > 0$, and that

$$1 = N(0) \leq \lim_{t \to 0^+} N(t).$$

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Normalization

We replace $T_t$ by $T_t e^{-\omega_0 t}$ or, equivalently, normalize our problem by assuming that $\omega_0 = 0$.

This implies that $\text{Spec}(A) \subseteq \{z : \text{Re}(z) \leq 0\}$.

It also implies that $\|T_t\| \geq 1$ for all $t \geq 0$. 
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$N(t)$ is increasing function of $t$ but it increases sub-exponentially as $t \to +\infty$. 
The Legendre Transform

We study the function $N(t)$ via a transform, defined for all $\omega > 0$ by

$$M(\omega) = \sup \{ \| T_t \| e^{-\omega t} : t \geq 0 \}.$$ 

$M(\omega)$ is a monotonic decreasing function of $\omega$ which satisfies

$$\lim_{\omega \to +\infty} M(\omega) = \limsup_{t \to 0} \| T_t \|.$$ 

Hence $M(\omega) \geq 1$ for all $\omega > 0$. 
The Legendre Transform

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$M(\omega)$ is a monotonic decreasing function of $\omega$ which satisfies

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Hence $M(\omega) \geq 1$ for all $\omega > 0$.

$$N(t) = \inf \{ M(\omega) e^{\omega t} : 0 < \omega < \infty \} \tag{3}$$

for all $t > 0$ by the theory of the Legendre transform.
Theorem

If \( a > 0, \ b \in \mathbb{R} \) and \( a \| R_{a+ib} \| = c \geq 1 \) then

\[
M(\omega) \geq \tilde{M}(\omega) := \begin{cases} 
(a - \omega)c/a & \text{if } 0 < \omega \leq r = a(1 - 1/c) \\
1 & \text{otherwise.}
\end{cases}
\]
Theorem

If $a > 0$, $b \in \mathbb{R}$ and $\| R_{a+ib} \| = c \geq 1$ then

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Proof.

The formula

$$R_{a+ib} = \int_0^\infty T_t e^{-(a+ib)t} \, dt$$

implies that

$$c/a \leq \int_0^\infty N(t)e^{-at} \, dt \leq \int_0^\infty M(\omega)e^{\omega t-at} \, dt = M(\omega)(a-\omega)^{-1}$$

for all $\omega$ such that $0 < \omega < a$. \qed
The following theorem implies that if the resolvent norm is significantly larger than $1/a$ for some large $a$ then $N(t)$ must grow rapidly for small $t > 0$.

Theorem

If $a\|R_{a+ib}\| = c \geq 1$ and $r = a(1 - 1/c)$ then

$$N(t) \geq \min\{e^{rt}, c\}$$

for all $t \geq 0$. 
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**Theorem**

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for all $t \geq 0$.

**Proof.**

This uses

$$N(t) = \inf\{M(\omega)e^{\omega t} : \omega > 0\} \geq \inf\{\tilde{M}(\omega)e^{\omega t} : \omega > 0\}.$$
Theorem

Let $H = -\Delta + V$, acting in $L^1(\mathbb{R}^n)$, where $V \geq 0$. Then

$$(e^{-Ht}f)(x) = \int_{\mathbb{R}^n} K(t, x, y)f(y) \, dy$$

where

$$0 \leq K(t, x, y) \leq (4\pi t)^{-n/2} e^{-|x-y|^2/4t}.$$ 

This can be proved by functional integration or the Trotter product formula.
Schrödinger Operators with Potentials Bounded Below

Theorem

Let $H = -\Delta + V$, acting in $L^1(\mathbb{R}^n)$, where $V$ is continuous and bounded below, with

$$c = -\inf\{V(x) : x \in \mathbb{R}^N\}.$$  

Then

$$c = \min\{\omega : \|e^{-Ht}\| \leq e^{\omega t} \text{ for all } t \geq 0\}.$$  

Note that the situation is quite different in $L^2(\mathbb{R}^n)$. 
Polynomial Growth of $L^1$ Norms

**Theorem (Murata 1984, 1985 and Davies-Simon 1991.)**

Let $N \geq 3$. There exists a Schrödinger semigroup $e^{-Kt}$ acting in $L^1(\mathbb{R}^N)$ and positive constants $c_1$, $c_2$, $\sigma_1$ and $\sigma_2$ such that

$$c_1(1 + t)^{\sigma_1} \leq \|e^{-Kt}\| \leq c_2(1 + t)^{\sigma_2}$$

for all $t \geq 0$, even though $K$ is non-negative considered as an operator acting in $L^2(\mathbb{R}^N)$.

The constants $\sigma_1$ and $\sigma_2$ are more or less equal.

The proof involves zero energy resonances.
The Explicit Example

The potential is given by

\[ V(x) = \begin{cases} 
-c|x|^{-2} & \text{if } |x| \geq 1 \\
0 & \text{otherwise.}
\end{cases} \]

where

\[ 0 < c < \frac{(n-2)^2}{4}. \]
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The zero energy resonance is of the form

\[ 0 < \eta(x) = \begin{cases} 
|x|^{-\alpha_1} - \beta|x|^{-\alpha_2} & \text{if } |x| \geq 1 \\
1 - \beta & \text{otherwise} 
\end{cases} \]

for certain positive constants \( \alpha_1, \alpha_2 \) and \( \beta \).