Stability of operator semigroups with regular norm-behaviour

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Ciprian Foias:

"Among the other outstanding works of Béla in that time let me mention the following two: First Béla's discovery that the Banch generalized limit (which until then was only a mathematical curiosity) can be made a nonconstructive but very effective tool in Operator Theory. A lot of mathematics came out from that discovery."

(Memorial Conference for Béla Szőkefalvi-Nagy, 1999, Szeged)

$\mathcal{H}$ complex Hilbert space, $\dim \mathcal{H} = \aleph_0$

(Many of the following results can be extended to the general Banach space setting.)

$T \in \mathcal{L}(\mathcal{H})$ power bounded operator:

$$\sup\left\{\|T^n\| : n \in \mathbb{Z}_+\right\} < \infty.$$

$$\begin{aligned}\mathcal{H}_0(T) &:= \left\{x \in \mathcal{H} : \inf_n \|T^n x\| = 0\right\} \\ &= \left\{x \in \mathcal{H} : \lim_n \|T^n x\| = 0\right\}\end{aligned}$$

set of stable vectors

Classification:

$T \in C_{0\cdot}$ if $\mathcal{H}_0(T) = \mathcal{H}$, T is stable

$T \in C_{1\cdot}$ if $\mathcal{H}_0(T) = \{0\}$, T is asymp. non-vanishing

$T \in C_{\cdot j}$ if $T^* \in C_{j\cdot}$

$$C_{ij} = C_{i\cdot} \cap C_{\cdot j} \quad (i, j = 1, 2)$$

Sz.-Nagy's technique relying on Banach limits yields:

$\exists\, V \in \mathcal{L}(\mathcal{K})$ isometry, $\exists\, X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$:

(1) $(X\mathcal{H})^- = \mathcal{K}$,

(2) $\ker X = \mathcal{H}_0(T)$,

(3) $XT = VX$.

$T \in C_{1\cdot} \iff X$ quasiaffinity $\implies T \prec V$

Sz.-Nagy – Foias: If $T \in C_{11}$ then

(1) V is unitary,

(2) $T \sim V$, that is $T \prec V$ and $V \prec T$

Sz.-Nagy:

If T is invertible and T^{-1} is also power bounded, then

(1) V is unitary,

(2) $T \approx V$.

Quasisimilarity:

preserves the existence of proper hyperinv. subsp.,

does not preserve spectra,

canonical models for important classes of operators.

Similarity:

preserves the (hyper)invariant subspace lattices,

preserves spectra.

Sz.-Nagy – Foias: rich theory for contractions

FOGUEL: T power bounded $\not\Longrightarrow$ $T \approx Q$ contraction

MÜLLER – TOMILOV:

$T \in C_{11}$ power bounded $\not\Longrightarrow$ $T \approx Q$ contraction

T power bounded $\not\Longrightarrow$ $T \sim Q$ contraction

T polynomially bounded, if $\exists\, K \in \mathbb{R}_+$, $\forall\, p$ polynomial,

$$\|p(T)\| \leq K \max\{|p(z)| : |z| \leq 1\}.$$

PISIER: T pol. bounded $\not\Longrightarrow T \approx Q$ contraction

BERCOVICI – PRUNARU:

T pol. bounded $\Longrightarrow \exists\, Q_1, Q_2$ contractions, $Q_1 \prec T \prec Q_2$

OPEN: T pol. bounded $\Longrightarrow T \sim Q$ contraction?

AIM: Associate canonical isometries with non-power bounded operators extending Sz.-Nagy's method.

$\ell^\infty := \ell^\infty(\mathbb{Z}_+)$ Banach algebra

$L \in (\ell^\infty)^\#$ is a Banach limit, if

(1) $\|L\| = L(\mathbf{1}) = 1$,

(2) $L(\xi) = L(B\xi) \ \forall \xi \in \ell^\infty$.

(Here $B\xi = \eta$, where $\eta(n) := \xi(n+1)$.)

Notation: $L\text{-}\lim \xi = L(\xi)$

$\mathcal{B}$: set of all Banach limits

If $\xi \in \ell^\infty$ is real, then

$$\{L(\xi) : L \in \mathcal{B}\} = [\hat{q}(\xi), \check{q}(\xi)],$$

where

$$\check{q}(\xi) :=$$
$$\inf\{\limsup_k \tfrac{1}{r}\textstyle\sum_{j=1}^r \xi(n_j+k) : r \in \mathbb{N}, n_1, \ldots, n_r \in \mathbb{Z}_+\},$$

$$\hat{q}(\xi) :=$$
$$\sup\{\liminf_k \tfrac{1}{r}\textstyle\sum_{j=1}^r \xi(n_j+k) : r \in \mathbb{N}, n_1, \ldots, n_r \in \mathbb{Z}_+\}.$$

(Consequence of the Hahn–Banach Theorem.)

$\xi \in \ell^\infty$ (complex) almost converges to $c \in \mathbb{C}$, if

$$\big\{L(\xi) : L \in \mathcal{B}\big\} = \{c\}.$$

Notation: $\operatorname{a-lim} \xi = c$

Lorentz: $\operatorname{a-lim} \xi = c \iff$

$$\lim_k \sup_n \Big| \tfrac{1}{k} \textstyle\sum_{j=n}^{n+k-1} \xi(j) - c \Big| = 0.$$

Lemma. Given $\xi \in \ell^\infty$ and $c \in \mathbb{C}$, TFAE:

(i) $\operatorname{a-lim} |\xi - c\mathbf{1}| = 0$,

(ii) $L(\xi\eta) = cL(\eta) \;\; \forall\, \eta \in \ell^\infty, \; \forall\, L \in \mathcal{B}$.

$T \in \mathcal{L}(\mathcal{H})$ is an arbitrary operator

$p \colon \mathbb{Z}_+ \to (0, \infty)$ is a gauge function, if

$$\exists\, c \in (0, \infty), \ \ \text{a-lim} \left| \tfrac{p(n+1)}{p(n)} - c \right| = 0.$$

T has a p-regular norm-sequence, if

(1) $\|T^n\| \le p(n) \ \ \forall\, n \in \mathbb{Z}_+$ is true,

(2) $\text{a-lim}_n \|T^n\|/p(n) = 0$ fails.

Then $c = r(T)$.

The construction of the associated isometry:

$L \in \mathcal{B}$

$w(x, y) := L\text{-}\lim \langle T^n x, T^n y \rangle p(n)^{-2} \quad (x, y \in \mathcal{H})$

$$
\begin{aligned}
w(Tx, Ty) &= L\text{-}\lim \langle T^{n+1} x, T^{n+1} y \rangle p(n+1)^{-2} \cdot p(n+1)^2 p(n)^{-2} \\
&= c^2 w(x, y)
\end{aligned}
$$

$\exists!\ A \in \mathcal{L}(\mathcal{H}),\ w(x, y) = \langle Ax, y \rangle$

$\mathcal{K} := (A\mathcal{H})^-; \quad X \in \mathcal{L}(\mathcal{H}, \mathcal{K}),\ Xx := A^{1/2} x$

$T^* AT = c^2 A \implies \|XTx\| = c\|Xx\| \ \forall\, x \in \mathcal{H}$

$\exists!\ V \in \mathcal{L}(\mathcal{K})$ isometry, $XT = cVX$

$L \in \mathcal{B}$ can be chosen so that

$$\ker X = \mathcal{H}_0(T,p) := \big\{x \in \mathcal{H} : \text{a-}\lim \|T^n x\|/p(n) = 0\big\}.$$

Connection between the commutants:

$$\forall\, C \in \{T\}',\ \exists!\, D \in \{V\}',\ XC = DX;$$

$$\gamma\colon \{T\}' \to \{V\}',\ C \mapsto D \ \text{ contractive algebra-hom.};$$

$$\sigma(C) \supset \sigma(D) \ \text{ and } \ \sigma_{\mathrm{p}}(C^*) \supset \sigma_{\mathrm{p}}(D^*) \quad \forall\, C \in \{T\}'.$$

Properties of the isometry $V \in \mathcal{L}(\mathcal{K})$:

(a) $\sigma(V) = \{1\} \implies V = I$

(Gelfand's theorem in the Banach space setting);

(b) $\dim \mathcal{K} > 1 \implies V$ is not supercyclic

($\nexists\, u \in \mathcal{K},\ \{\lambda V^n u : \lambda \in \mathbb{C}, n \in \mathbb{Z}_+\}$ is dense in $\mathcal{K}$);

(c) $\{V\}''$ is semisimple

($Q \in \{V\}''$ is quasinilpotent $\implies Q = 0$).

Consequences for operators with regular norm-sequence:

$T \in \mathcal{L}(\mathcal{H})$ has p-regular norm-sequence

Theorem A. If $\sigma(T) \cap r(T)\mathbb{T}$ is countable and $\sigma_{\mathrm{p}}(T^*) \cap r(T)\mathbb{T} = \emptyset$, then $\mathcal{H}_0(T,p) = \mathcal{H}$, that is

$$\text{a-lim}\, \|T^n x\|/p(n) = 0 \quad \forall\, x \in \mathcal{H}.$$

(Extension of the Arendt–Batty–Lyubich–Vu Theorem.)

Theorem B. If T is supercyclic, then

$$\dim \mathcal{H}_0(T,p)^{\perp} \leq 1.$$

(Extension of the Ansari–Bourdon Theorem: supercyclic power bounded operators are stable.)

Theorem C. (K – Vu).

If T is cyclic and $C \in \{T\}'$ is quasinilpotent, then

$$\text{a-lim}\, \|T^n C x\|/p(n) = 0 \quad \forall\, x \in \mathcal{H}.$$

T has regular norm-sequence $\Longrightarrow$ $r(T) > 0$

$r(T) > 0$, $\{\|T^n\|^{1/n}\}$ decreasing $\Longrightarrow$ T has reg. norm-seq.

$r(T) > 0 \not\Longrightarrow$ T has regular norm-seq.

(K – Müller: complete characterization)

$T \in \mathcal{L}(\mathcal{H}) \;\hookrightarrow\; \rho \colon \mathbb{Z}_+ \to \mathcal{L}(\mathcal{H}), n \mapsto T^n$ representation

S **discrete abelian semigroup**

Assume: S is additive, cancellative,

0 is the only invertible element

Example: $S = \mathbb{Z}_+^k$

$s_1 \prec s_2$ if $\exists\, s_3,\ s_1 + s_3 = s_2$

$(S, \prec)$ directed set $\to$ limiting process is available

$\hat{q}$ and $\check{q}$ can be defined

$M \in \ell^\infty(S)^\#$ is an invariant mean, if

(1) $\|M\| = M(\mathbf{1}) = 1$,

(2) $M(\xi_t) = M(\xi) \;\; \forall\, \xi \in \ell^\infty(S), \; \forall\, t \in S$.

(Here $\xi_t(s) := \xi(s+t)$.)

$\mathcal{M}(S)$: set of invariant means

M.M. Day: $\mathcal{M}(S) \neq \emptyset$, and $\mathcal{M}(S)$ is infinite provided S has no finite ideals.

$\xi \in \ell^\infty(S)$ almost converges to $c \in \mathbb{C}$, if

$$\big\{M(\xi) : M \in \mathcal{M}(S)\big\} = \{c\}.$$

Notation: $\operatorname{a-lim} \xi = c$

$\rho\colon S \to \mathcal{L}(\mathcal{H})$ representation:

(1) $\rho(0) = I$,

(2) $\rho(s+t) = \rho(s)\rho(t) \quad \forall\, s, t \in S$.

$p\colon S \to (0, \infty)$ gauge function, if

$$\forall\, t \in S,\ \exists\, c_p(t) \in (0, \infty),\ \text{a-lim}_s \left| p(s+t)/p(s) - c_p(t) \right| = 0.$$

ρ has p-regular norm-function, if

(1) $\|\rho(s)\| \leq p(s) \quad \forall\, s \in S$ is true,

(2) $\text{a-lim}_s\, \|\rho(s)\|/p(s) = 0$ fails.

$c_\rho(s) := c_p(s)\ (s \in S)$ is independent of the choice of p,

positive character on S,

$c_\rho(s) \le r(\rho(s)) \quad \forall\, s \in S.$

$S_r(\rho) := \{s \in S : c_\rho(s) = r(\rho(s))\} \subset S$

If S is generated by $\{s_1, \dots, s_n\}$, then

$S_r(\rho) \supset (s_1 + \cdots + s_n) + S.$

If $\rho(s)$ is invertible for every $s \in S$, then $S_r(\rho) = S$.

In both cases $S_r(\rho)$ is absorbing:

$\forall\, s \in S,\ \exists\, s' \in S_r(\rho),\ s + s' \in S_r(\rho).$

$\mathcal{A}_\rho$: abelian Banach algebra, generated by $\{\rho(s) : s \in S\}$

Σ_ρ : spectrum of $\mathcal{A}_r$

Algebraic spectrum of ρ:

$$\sigma_{\mathrm{a}}(\rho) := \left\{h \circ \rho : h \in \Sigma_\rho\right\} \subset S^\# \quad \text{(characters on } S\text{)}.$$

Peripheral spectrum of ρ:

$$\sigma_{\mathrm{per}}(\rho) := \left\{\chi \in \sigma_{\mathrm{a}}(\rho) : |\chi| = c_\rho\right\}.$$

If $S_r(\rho)$ is absorbing, then $\sigma_{\mathrm{per}}(\rho) \subset \sigma_{\mathrm{ap}}(\rho)$.

Theorem. If $\sigma_{\mathrm{per}}(\rho)$ is countable and $\sigma_{\mathrm{p}}(\rho^*) \cap \sigma_{\mathrm{per}}(\rho) = \emptyset$, then

$$\text{a-lim}_s \|\rho(s)x\|/p(s) = 0 \quad \forall\, x \in \mathcal{H}.$$

(Extension of ABLV)

Locally compact abelian semigroups

L. K. – Z. Léka

G locally compact, σ-compact, Hausdorff abelian group

S closed subsemigroup of G:

$$S - S = G,\ S \cap (-S) = \{0\},\ S^\circ \neq \emptyset$$

Example: $G = \mathbb{R}^k,\ S = \mathbb{R}^k_+$

$s \prec t$ if $t - s \in S$

$(S, \prec)$ directed set $\rightarrow$ limiting process

$\widetilde{\mu}$ Haar measure on G

μ restriction of $\widetilde{\mu}$ to S

$L^\infty(S) := L^\infty(\mu),\ L^1(S) := L^1(\mu)$

$M \in L^\infty(S)^\#$ is an invariant mean, if

(1) $\|M\| = M(\mathbf{1}) = 1$,

(2) $M(f) = M(f_t) \quad \forall\, f \in L^\infty(S),\ \forall\, t \in S.$

$\mathcal{M}(S)$ set of invariant means

Markov–Kakutani Fixed Point Theorem $\implies \mathcal{M}(S) \neq \emptyset \implies$

$\exists\, \{K_n\}_{n=1}^\infty$ compact sets in S°:

(1) $K_n^\circ \neq \emptyset \quad \forall\, n$,

(2) $\lim_n \sup_{s\in K} \mu((K_n+s)\triangle K_n)/\mu(K_n) = 0$

$\forall\, K \subset S$ compact.

Folner sequence

$\mathcal{G} := \left\{ g \in L^1(S) : g \geq 0 \ \text{ and } \ \int_S g \, d\mu = 1 \right\}$

$\forall \, f \in L^\infty(S), \, \forall \, g \in \mathcal{G}, \ f * g \in L^\infty(S)$, where

$$(f * g)(t) := \int_S f(s+t) g(s) \, d\mu(s)$$

$M \in L^\infty(S)^\#$ is a topologically invariant mean, if

(1) $\|M\| = M(\mathbf{1}) = 1,$

(2) $M(f * g) = M(f) \quad \forall \, f \in L^\infty(S), \, \forall \, g \in \mathcal{G}.$

$\mathcal{M}_{\mathrm{t}}(S)$ set of topologically invariant means

$\mathcal{M}_{\mathrm{t}}(S) \subset \mathcal{M}(S)$

$\mathcal{M}_{\mathrm{t}}(S) \neq \mathcal{M}(S)$ if S is non-compact and non-discrete

If $K \subset S$ is compact and $\mu(K) > 0$, then

$$\varphi_K \in L^\infty(S)^\# \text{ and } \|\varphi_K\| = \varphi_K(\mathbf{1}) = 1, \text{ where}$$

$$\varphi_K(f) := \mu(K)^{-1} \int_K f \, d\mu.$$

$\mathcal{M}_{\mathrm{t}}(S) =$ weak-$*$ closure of the convex hull of all

weak-$*$ cluster points of the sequences

$\{\varphi_{K_n+s_n}\}_{n=1}^\infty, \quad \{s_n\}_{n=1}^\infty \subset S,$

where $\{K_n\}_n$ is any fixed Folner sequence.

$f \in L^\infty(S)$ almost converges to $c \in \mathbb{C}$, if

$$\big\{ M(f) : M \in \mathcal{M}(S) \big\} = \{c\}.$$

Notation: $\operatorname{a-lim} f = c$

$f \in L^\infty(S)$ topologically almost converges to $c \in \mathbb{C}$, if

$$\big\{ M(f) : M \in \mathcal{M}_{\mathrm{t}}(S) \big\} = \{c\}.$$

Notation: $\operatorname{ta-lim} f = c$

$\operatorname{a-lim} f = c \implies \operatorname{ta-lim} f = c \iff$

$$\lim_n \sup_{t \in S} \left| \mu(K_n)^{-1} \int_{K_n} f(t+s)\, d\mu(s) - c \right| = 0,$$

where $\{K_n\}_n$ is a Folner sequence.

$\rho\colon S \to \mathcal{L}(\mathcal{H})$ is a representation:

(1) $\rho(0) = I$,

(2) $\rho(s+t) = \rho(s)\rho(t) \quad \forall\, s, t \in S$,

(3) $\rho_x\colon S \to \mathcal{H},\ s \mapsto \rho(s)x$ is continuous $\forall\, x \in \mathcal{H}$.

$p\colon S \to [1,\infty)$ is a gauge function, if

(1) locally bounded, measurable,

(2) $\forall\, t \in S,\ p_t/p \in L^\infty(S)$,

(3) $\forall\, t \in S,\ \exists\, c_p(t) \in (0,\infty)$,

$$\text{a-lim}_s \left| p(t+s)/p(s) - c_p(t) \right| = 0.$$

(Topological gauge functions can be defined analogously.)

ρ has p-regular norm-function, if

(1) $\|\rho(s)\| \le p(s)$ holds $\forall\, s \in S$,

(2) $\text{a-lim}_s \|\rho(s)\|/p(s) = 0$ fails.

$c_\rho := c_p \colon S \to [1, \infty)$ is independent of the choice of p,

continuous,

$c_\rho(s+t) = c_\rho(s)c_\rho(t) \quad \forall\, s, t \in S$,

$c_\rho(s) \le r(\rho(s)) \quad \forall\, s \in S$.

$\chi\colon S \to \mathbb{C}$ is a character, if

(1) $\chi(0) = 1$,

(2) $\chi(s+t) = \chi(s)\chi(t) \quad \forall\, s,t \in S$,

(3) continuous.

$S^{\#}$: set of all characters

$S^{\#}_{\mathrm{b}} := \big\{\chi \in S^{\#} : \chi(s) \neq 0 \;\forall\, s \in S\big\}$ balanced characters

$C_c(S)$: continuous functions with compact support

$\forall\, f \in C_c(S)$, $\widehat{f}(\rho) \in \mathcal{L}(\mathcal{H})$, where

$\widehat{f}(\rho)x := \int_S f(s)\rho(s)x \, d\mu(s) \quad (x \in \mathcal{H})$;

$\widehat{f}(\chi) := \int_S f(s)\chi(s)\, d\mu(s) \quad (\chi \in S^{\#})$ Fourier transform

Algebraic spectrum of ρ:

$$\sigma_{\mathrm{a}}(\rho) := \left\{\chi \in S^{\#} : |\widehat{f}(\chi)| \leq \|\widehat{f}(\rho)\| \quad \forall\, f \in C_c(S)\right\}$$

Balanced spectrum of ρ:

$$\sigma_{\mathrm{b}}(\rho) := \sigma_{\mathrm{a}}(\rho) \cap S^{\#}_{\mathrm{b}}$$

Spectrum of ρ:

$$\sigma(\rho) := \left\{\chi \in \sigma_{\mathrm{a}}(\rho) : |\chi| \leq c_\rho\right\}$$

Peripheral spectrum of ρ:

$$\sigma_{\mathrm{per}}(\rho) := \left\{\chi \in \sigma(\rho) : |\chi| = c_\rho\right\}$$

$\sigma_{\mathrm{b}}(\rho) \subset \sigma(\rho)$ always holds

$S = \mathbb{R}^k_+ \implies S^{\#} = S^{\#}_{\mathrm{b}} \implies \sigma_{\mathrm{a}}(\rho) = \sigma_{\mathrm{b}}(\rho) = \sigma(\rho)$

Constructing associated isometric representation, applying results of Batty and Vu on isometric representations, we obtain the following extension of ABLV (related to a result of Batty and Yeates):

Theorem.
If $\sigma_{\mathrm{per}}(\rho)$ is countable and $\sigma_{\mathrm{p}}(\rho^*) \cap \{\chi \in S^{\#} : |\chi| = c_\rho\} = \emptyset$, then

$$\text{a-lim}_s \, \|\rho(s)x\|/p(s) = 0 \quad \forall\, x \in \mathcal{H},$$

and so

$$\lim_n \mu(K_n)^{-1} \int_{K_n} \|\rho(s)x\|/p(s)\, d\mu(s) = 0 \quad \forall\, x \in \mathcal{H},$$

where $\{K_n\}_n$ is any Folner sequence.

One-parameter semigroups: $S = \mathbb{R}_+$

$T\colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{H})$ representation (C_0-semigroup) with p-regular norm-function

Then $c_\rho(s) = r(T(s)) = e^{\omega_0 s} \quad \forall\, s \in \mathbb{R}_+$, where

$$\begin{aligned}\omega_0 &:= \lim_{s\to\infty} s^{-1} \log \|T(s)\| \\ &= \inf \big\{ \omega \in \mathbb{R} : \exists\, K \in \mathbb{R}_+,\ \forall\, s \in \mathbb{R}_+,\ \|T(s)\| \leq K e^{\omega s} \big\}.\end{aligned}$$

$\Psi\colon \mathbb{C} \to \mathbb{R}_+^{\#},\ \alpha \mapsto \chi_\alpha$ identification, where $\chi_\alpha(s) := e^{\alpha s}$

Spectrum of T:

$$\sigma(T) := \big\{ z \in \mathbb{C} : |\widehat{f}(z)| \le \|\widehat{f}(T)\| \ \ \forall\, f \in C_c(\mathbb{R}_+) \big\},$$

where $\widehat{f}(z) := \int_0^\infty f(s) e^{zs}\, ds$ Laplace transform.

Peripheral spectrum of T:

$$\sigma_{\mathrm{per}}(T) := \big\{ z \in \sigma(T) : \mathrm{Re}\ z = \omega_0 \big\}.$$

Infinitesimal generator of T:

$$A\colon \mathcal{D} \to \mathcal{H},\ \ Ax := \lim_{s\to 0} s^{-1}(T(s)x - x)$$

closed linear operator

$\sigma(A) \subset \{z \in \mathbb{C} : \mathrm{Re}\ z \leq \omega_0\}$ closed

$\rho_\infty(A)$ component of $\mathbb{C} \setminus \sigma(A)$ containing $\omega_0 + 1$

$$\sigma(T) = \mathbb{C} \setminus \rho_\infty(A) \implies \sigma_{\mathrm{per}}(T) = \{z \in \sigma(A) :\ \mathrm{Re}\ z = \omega_0\}$$

$$\sigma_{\mathrm{p}}(T^*) = \{\chi_\alpha : \alpha \in \sigma_{\mathrm{p}}(A^*)\}$$

Theorem.

If $\sigma(A) \cap \{z \in \mathbb{C} : \text{ Re } z = \omega_0\}$ is countable and

$$\sigma_{\mathrm{p}}(A^*) \cap \{z \in \mathbb{C} : \text{ Re } z = \omega_0\} = \emptyset,$$

then $\forall\, x \in \mathcal{H}$,

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}_+} \mu(K_n)^{-1} \int_{K_n} \|T(s+t)x\|/p(s+t)\, ds = 0,$$

where $\{K_n\}_n$ is any Folner sequence (e.g., $K_n = [0, n]$).

From the extension of the Ansari–Bourdon Theorem to representations of discrete semigroups we can derive:

Theorem.

If $T\colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{H})$ is a supercyclic, bounded representation, consisting of operators with dense range,

then T is stable.

G locally compact (not necessarily abelian) group

$\widetilde{\mu}$ is a (left) Haar measure on G, if

(1) regular Borel measure,

(2) $\widetilde{\mu}(x+E) = \widetilde{\mu}(E)$ $\forall\, x \in G$, $\forall\, E \subset G$ Borel set.

$L^\infty(G) := L^\infty(\widetilde{\mu})$

$M \in L^\infty(G)^\#$ is a left invariant mean, if

(1) $\|M\| = M(\mathbf{1}) = 1$,

(2) $M(f_t) = M(f)$ $\forall\, f \in L^\infty(G)$, $\forall\, t \in G$,

where $f_t(s) := f(t+s)$.

$\mathcal{M}(G)$: set of left invariant means

(A) $\quad G$ is amenable: $\mathcal{M}(G) \neq \emptyset$

(B) $\quad \forall\, \rho \colon G \to \mathcal{L}(\mathcal{H})$ bounded representation,

$\exists\, \psi \colon G \to \mathcal{L}(\mathcal{K})$ unitary representation,

$\exists\, S \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ invertible, $\rho(s) = S^{-1}\psi(s)S \;\forall\, s \in G$.

Day – Dixmier: $\quad$ (A) $\implies$ (B)

(Extension of Sz.-Nagy's theorem)

OPEN: $\quad$ (B) $\implies$ (A) ?