An index theorem, the spectral flow, and the spectral shift function for relatively trace class perturbations

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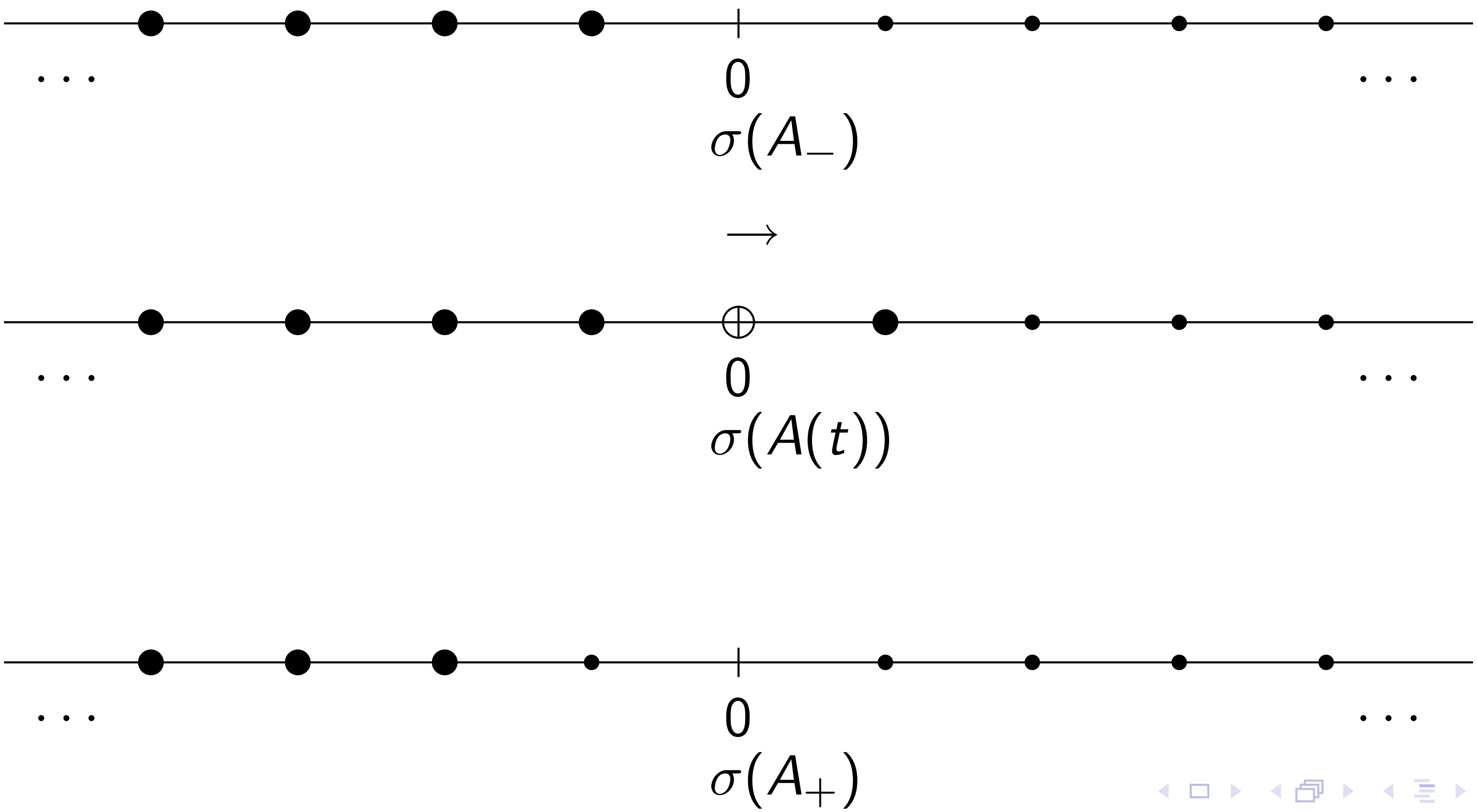
Prehistory

UNC - Chapel Hill Winter 2006 Maslov index
Chris Jones / Deng: Morse theory for PDEs
Chris Jones and Richard Rimanyi
Tomilov Torun 2006, 2007 – A. Pushnitski
J. Robbin and D. Salamon, *The spectral flow and the Maslov index*, Bull. London Math. Soc. **27**, 1–33 (1995)
"Atiyah, Patodi and Singer studied operators of the form $D_A = \frac{d}{dt} + A(t)$"
Here and below $\big(D_A u\big)(t) = u'(t) + A(t)u(t)$, $u \in L^2(\mathbb{R};\mathcal{H})$, $t \in \mathbb{R}$
The Fredholm index of D_A, $\operatorname{Ind} D_A = \dim\ker D_A - \dim\ker D_A^*$
is equal to the spectral flow of $\{A(t)\}_{t=-\infty}^{+\infty}$
$A(t)$ are unbounded selfadjoint operators on a Hilbert space $\mathcal{H}$ with compact resolvent (thus discrete spectrum) and constant domains
$A_\pm = \lim_{t\to\pm\infty} A(t)$ exist and are invertible
A_+ **and** A_- **are invertible throughout the talk**

What is the spectral flow?

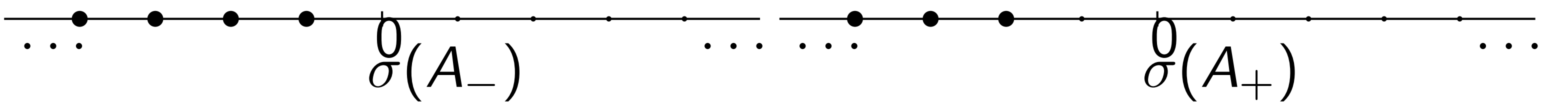
Spectral flow= (the number of eigenvalues of $A(t)$ that cross 0 rightward)
minus (the number of eigenvalues of $A(t)$ that cross 0 leftward)
as t runs from $-\infty$ to $+\infty$



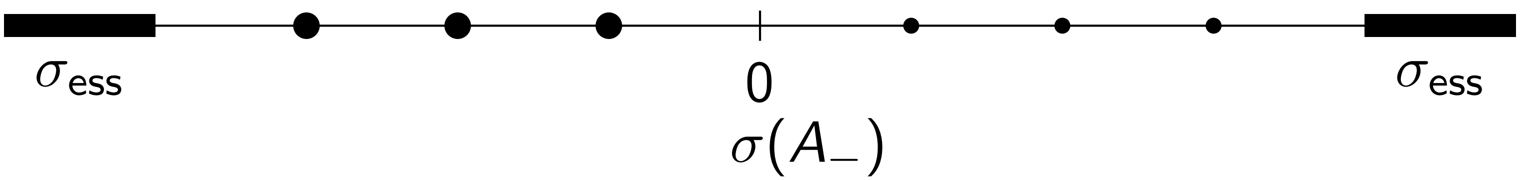
Before and after

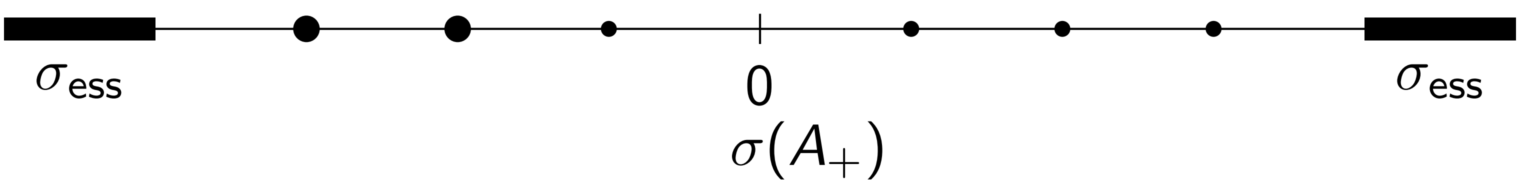
We prove: Fredholm index = spectral shift function at 0 = spectral flow
This allows one to handle more general families of operators than before

Before:



After:





Literature guide

We generalize recent results regarding $D_A = \frac{d}{dt} + A(t)$ on $L^2(\mathbb{R}; \mathcal{H})$ in
A. Pushnitski, *The spectral flow, the Fredholm index, and the spectral shift function*, in *Spectral Theory of Differential Operators: M. Sh. Birman 80th Anniversary Collection*, 2008, pp. 141–155.
Pushnitski studied the case of *trace class* perturbations $A(t) - A_-$
We study the case of **relatively** *trace class* perturbations $A(t) - A_-$
E. g., when $A_- = -\partial_x^2$ and $A(t) = -\partial_x^2 + V(t, x)$ in $\mathcal{H} = L^2(\mathbb{R})$,
the Schrödinger operator with sufficiently fast decaying potential

Earlier related papers:
C. Callias, *Axial anomalies and index theorems on open spaces*, Commun. Math. Phys. **62**, 213–234 (1978)
(famous Callias' formula for $\mathcal{H} = \mathbb{R}^d$ gives the index of the Dirac operator for odd dimensions)

Literature guide continued

D. Bolle, F. Gesztesy, H. Grosse, W. Schweiger, and B. Simon, *Witten index, axial anomaly, and Krein's spectral shift function in supersymmetric quantum mechanics*, J. Math. Phys. **28**, 1512–1525 (1987) (the scalar case when $A(t)$ is a function)

The literature on the spectral flow is endless (M. Atiyah, V. Patodi, I. Singer; B. Boos-Bavbnek, K. Furutani, P. Kirk, M. Lesch, N. Nicolaescu, J. Phillips)

There are noncommutative versions of the theory (N. Azamov, M.-T. Benameur, A. Carey, P. Dodds, A. Rennie, F. Sukochev, K. P. Wojciechowski)

A standing assumption: the resolvents of A_+ and A_- are compact (or the operators $(A_+^2 + I)^{-1/2}$ and $(A_-^2 + I)^{-1/2}$ are compact)

We can handle: $A_+ - A_-$ is of trace class relatively to A_- (e.g., this implies that the *difference* of the resolvents of A_+ and A_- or the *difference* $A_+(A_+^2 + I)^{-1/2} - A_-(A_-^2 + I)^{-1/2}$ is of trace class)

Third Main Theorem:

Assume that $A(t) = A_- + B(t)$ where $B(t)$ is an appropriate relatively trace class perturbation, $B(t)(A_- - z)^{-1} \in \mathcal{B}_1(\mathcal{H})$

Theorem.

Under appropriate (relatively trace class perturbation) assumptions, the Fredholm index of the operator $D_A = \frac{d}{dt} + A(t)$ on $L^2(\mathbb{R};\mathcal{H})$ is related to the **Krein's spectral shift function (ξ-function)** of the operators $A_- = \lim_{t\to-\infty} A(t)$ and $A_+ = \lim_{t\to+\infty} A(t)$ on the Hilbert space $\mathcal{H}$:

$$\operatorname{Ind} D_A = \xi_{\mathcal{H}}(0; A_+, A_-)$$

The spectral shift function $\xi_{\mathcal{H}}(\lambda; A_+, A_-)$, $\lambda \in \mathbb{R}$, is (assumptions!) $\xi_{\mathcal{H}}(\lambda; A_+, A_-) =$ (the number of eigenvalues of $A(t)$ that cross λ rightward) minus (the number of eigenvalues of $A(t)$ that cross λ leftward)

Corollary

Fredholm index of D_A= Spectral flow

Second Main Theorem

Assume that $A(t) = A_- + B(t)$ where $B(t)$ is an appropriate relatively trace class perturbation, $B(t)(A_- - z)^{-1} \in \mathcal{B}_1(\mathcal{H})$
Given $D_A = \frac{d}{dt} + A(t)$, we introduce on $L^2(\mathbb{R};\mathcal{H})$ the operators

$$H_1 = D_A^* D_A = -\frac{d^2}{dt^2} + A(t)^2 - A'(t)$$

$$H_2 = D_A D_A^* = -\frac{d^2}{dt^2} + A(t)^2 + A'(t)$$

Theorem.

Under appropriate (relatively trace class perturbation) assumptions, the following **Pushnitski formula** holds:

$$\xi_{L^2(\mathbb{R};\mathcal{H})}(\lambda; H_2, H_1) = \frac{1}{\pi}\int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi_{\mathcal{H}}(s; A_+, A_-)\, ds}{(\lambda - s^2)^{1/2}}, \text{ a.e. } \lambda \in \mathbb{R}$$

The index via Pushnitski's formula

Since A_- and A_+ are invertible:

- $\xi(\lambda; A_+, A_-)$ is constant for a. e. λ near 0
- H_1 and H_2 have no essential spectrum near zero
- therefore, $\xi(\lambda; H_2, H_1)$ is constant for λ near 0

Then by Pushnitski's formula, $\xi(\lambda; A_+, A_-) = \xi(\lambda; H_2, H_1)$ for λ near 0

But $H_1 = D_A^* D_A$ and $H_2 = D_A D_A^*$ imply:

- $\operatorname{Ind} D_A = \dim\ker D_A - \dim\ker D_A^* = \dim\ker H_1 - \dim\ker H_2$

On the other hand, properties of the ξ-function imply:

- $\xi(\lambda; H_2, H_1) = \dim\ker H_1 - \dim\ker H_2$ for λ near 0

Putting all this together,

- $\operatorname{Ind} D_A = \xi(0; H_2, H_1) = \xi(0; A_+, A_-)$

First Main Theorem:

Assume that $A(t) = A_- + B(t)$ where $B(t)$ is an appropriate relatively trace class perturbation, $B(t)(A_- - z)^{-1} \in \mathcal{B}_1(\mathcal{H})$

Recall notations

$H_1 = D_A^* D_A = -\frac{d^2}{dt^2} + A(t)^2 - A'(t), H_2 = D_A D_A^* = -\frac{d^2}{dt^2} + A(t)^2 + A'(t)$

Introduce the "smoothed out" signum functions

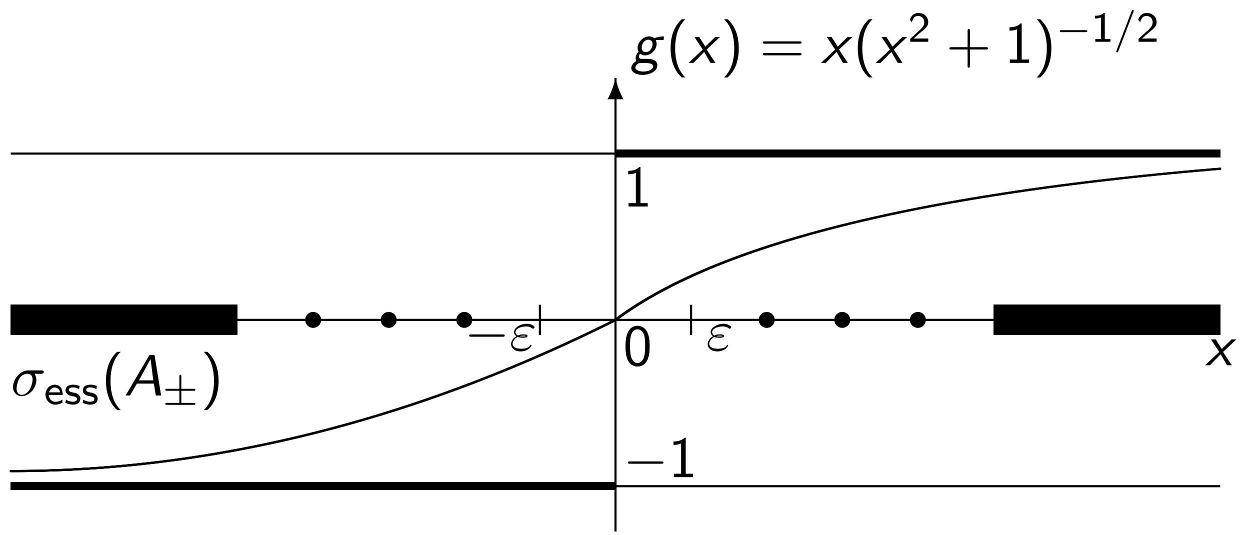
$$g_z(x) = \frac{x}{\sqrt{x^2 - z}}, \quad z \in \mathbb{C} \setminus [0, \infty), \quad g(x) = \frac{x}{\sqrt{x^2 + 1}}, \quad x \in \mathbb{R}$$

Theorem.

Under appropriate (relatively trace class perturbation) assumptions,

- $g_z(A_+) - g_z(A_-)$ is of trace class in $\mathcal{H}$,
- $(H_2 - z)^{-1} - (H_1 - z)^{-1}$ is of trace class in $L^2(\mathbb{R}; \mathcal{H})$, and
- $\operatorname{tr}_{L^2(\mathbb{R};\mathcal{H})}((H_2 - z)^{-1} - (H_1 - z)^{-1}) = \frac{1}{2z} \operatorname{tr}_{\mathcal{H}}(g_z(A_+) - g_z(A_-))$

Consequences of the First Main Theorem



Define the Morse spectral projections $E_{A_+} = E_{A_+}(0)$ and $E_{A_-} = E_{A_-}(0)$ corresponding to the *negative* spectrum of A_+ and A_-, and $S_\pm = \operatorname{rg} E_{A_\pm}$

Corollary Under appropriate (relatively trace class perturbation) assumptions $E_{A_+} - E_{A_-}$ is of trace class in $\mathcal{H}$ and therefore

$$\xi(0; A_+, A_-) = \operatorname{tr}(E_{A_-} - E_{A_+}) = \operatorname{Ind}(E_{A_-}, E_{A_+})$$
$$:= \dim(S_- \cap S_+^\perp) - \dim(S_-^\perp \cap S_+)$$
$$(= \dim S_+ - \dim S_- \text{ provided } \dim S_\pm < \infty)$$

The Riesz metric and the spectral flow

The graph metric $\|(A_+ + i)^{-1} - (A_- + i)^{-1}\|_{\mathcal{B}(\mathcal{H})}$ induces a weaker topology than the Riesz metric $\|g(A_+) - g(A_-)\|_{\mathcal{B}(\mathcal{H})}$ defined by $g(x) = x(x^2+1)^{-1/2}$ (M. Lesch 2005). Under our assumptions the path $\{A(t)\}_{t=-\infty}^{\infty}$ is (absolutely) continuous in the Riesz metric.

The spectral flow of the continuous path $\{A(t)\}_{t=-\infty}^{\infty}$ of selfadjoint Fredholm operators (J. Phillips 1996): Choose a subdivision $-\infty = t_0 < t_1 < \cdots < t_{n-1} < t_n = +\infty$ such that there exist $\varepsilon_j > 0$ with $\pm\varepsilon_j \notin \sigma(A(t))$ and $[-\varepsilon_j, \varepsilon_j] \cap \sigma_{\text{ess}}(A(t)) = \emptyset$ for $t_{j-1} \leqslant t \leqslant t_j$, $j = 1, \ldots, n$, and let

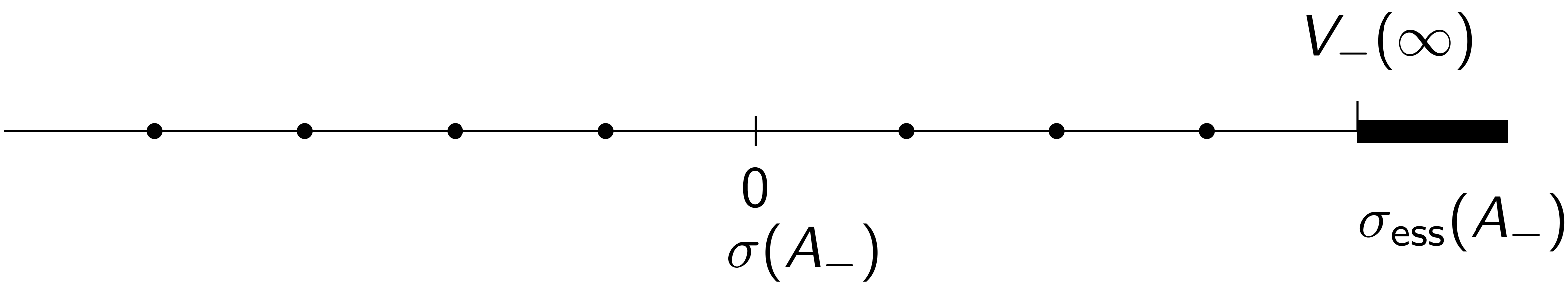
$$SpFlow(A) = \sum_{j=1}^{n} \left(\mathrm{rank}\, E_{A(t_{j-1})}([0, \varepsilon_j)) - \mathrm{rank}\, E_{A(t_j)}([0, \varepsilon_j))\right)$$

Corollary

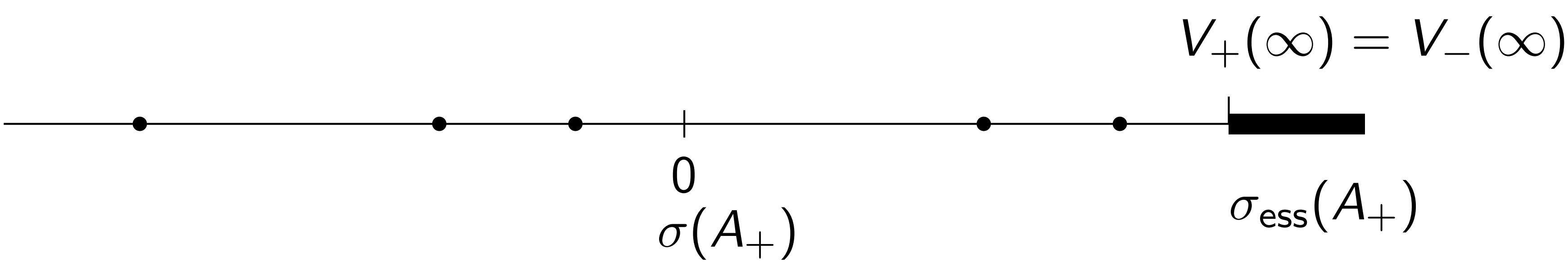
Under appropriate (relatively trace class perturbation) assumptions $SpFlow(A) = \mathrm{Ind}(E_{A_-}, E_{A_+}) = \mathrm{tr}(E_{A_-} - E_{A_+}) = \xi(0; A_+, A_-)$

A semibounded example

$A(t) = -\partial_x^2 + V(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}$ – Schrödinger operators on $L^2(\mathbb{R})$
Assume that $V_-(\cdot) \in L^\infty(\mathbb{R})$, $V_-(\cdot) - V_-(\infty) \in L^1(\mathbb{R})$
Assume that $V_+(\cdot) \in L^\infty(\mathbb{R})$, $V_+(\cdot) - V_+(\infty) \in L^1(\mathbb{R})$
Denote $A_- = -\partial_x^2 + V_-(x)$ and assume that A_- is invertible



Denote $A_+ = -\partial_x^2 + V_+(x)$ and assume that A_+ is invertible



A semibounded example continued

$A(t) = -\partial_x^2 + V(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}$ – Schrödinger operators on $L^2(\mathbb{R})$,
$A_- = -\partial_x^2 + V_-(x)$ with $V_-(\cdot) \in L^\infty(\mathbb{R})$, $V_-(\cdot) - V_-(\infty) \in L^1(\mathbb{R})$
$A_+ = -\partial_x^2 + V_+(x)$ with $V_+(\cdot) \in L^\infty(\mathbb{R})$, $V_+(\cdot) - V_+(\infty) \in L^1(\mathbb{R})$

Finally, assume that $V(t, \cdot) - V_-(\cdot) \in \ell^1(L^2(\mathbb{R}))$ for all $t \in \mathbb{R}$, and also $V_+(\cdot) - V_-(\cdot) \in \ell^1(L^2(\mathbb{R}))$ where $\ell^1(L^2(\mathbb{R}))$ is the Birman-Solomyak class

Then the perturbation $B(t) = V(t, \cdot) - V_-(\cdot)$ is of relatively trace class
To complete the set of assumptions, we need: $\partial_t V(t,\cdot) \in \ell^1(L^2(\mathbb{R}))$ and $\int_{\mathbb{R}} \|\partial_t V(t,\cdot)(-\partial_x^2+1)^{-1}\|_{\mathcal{B}_1(L^2(\mathbb{R}))}\, dt < \infty$

The Birman-Solomyak class:
$L^1(\mathbb{R}; (1+|x|)^\delta) \subsetneq \ell^1(L^2(\mathbb{R})) \subsetneq L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $\delta \geqslant 1/2$
$\ell^1(L^2(\mathbb{R})) = \Big\{ v \in L^2_{\text{loc}}(\mathbb{R}) : \sum_{n\in\mathbb{Z}} \big(\int_{Q_n} |v(x)|^2\, dx\big)^{1/2} < \infty \Big\}$
$v \in \ell^1(L^2(\mathbb{R}))$ if and only if $v(-\partial_x^2 - z)^{-1}$ is of trace class
$v \in L^2(\mathbb{R})$ if and only if $v(-\partial_x^2 - z)^{-1}$ is of Hilbert-Schmidt class

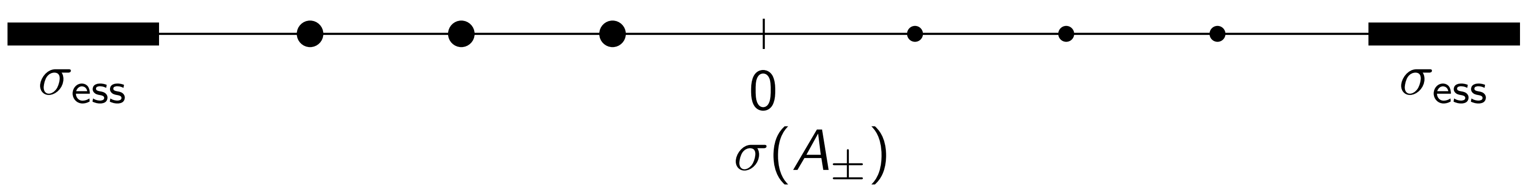
Two Schrödingers

$$A(t) = \begin{bmatrix} -\partial_x^2 + \alpha & 0 \\ 0 & \partial^2 - \alpha \end{bmatrix} + V(t,x),\ \alpha > 0,\ V(t,x) \text{ is a } 2 \times 2 \text{ matrix}$$

$$V(t,\cdot) - V_\pm(\cdot) \in \ell^1(L^2(\mathbb{R})),\ V_\pm(\cdot) - V_\pm(\infty) \in L^1(\mathbb{R})$$

$$A_\pm = \begin{bmatrix} -\partial_x^2 + \alpha & 0 \\ 0 & \partial^2 - \alpha \end{bmatrix} + V_\pm(x)$$

Then $B(t) = V(t,x) - V_\pm(x)$ is a relative trace class perturbation



Dirac

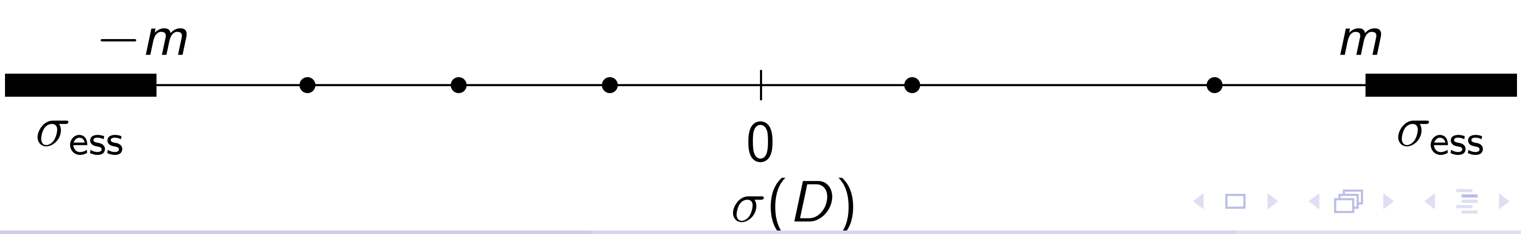
$$D_0 = \begin{bmatrix} 1 & -\partial_x \\ \partial_x & -1 \end{bmatrix} \text{ Fourier } \begin{bmatrix} 1 & -i\xi \\ i\xi & -1 \end{bmatrix} \text{ is similar to } \begin{bmatrix} -\sqrt{1+\xi^2} & 0 \\ 0 & \sqrt{1+\xi^2} \end{bmatrix}$$

$$\sigma(D_0) = (-\infty, -1] \cup [1, \infty)$$

Dirac operator is

$$D = \begin{bmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{bmatrix} + \begin{bmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{bmatrix}, \; A(t) = \begin{bmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{bmatrix} + \begin{bmatrix} p(t,x) & q(t,x) \\ q(t,x) & -p(t,x) \end{bmatrix}$$

In particular, if $q(x) = 0, p(x) = V(x) + m$ then D is one-dimensional stationary Dirac's operator of quantum relativity theory.
If $V(\pm\infty) = 0$ then $\sigma_{\text{ess}}(D) = (-\infty, -m] \cup [m, \infty)$



A 5-min course on the ξ-function

Given two **selfadjoint** operators H, H_0 we may assume that the perturbation $V = H - H_0$ satisfies:

- trace class assumption: $V = H - H_0$ is of trace class
- *relative* trace class: $V(H_0 - z)^{-1}$ is of trace class
- resolvent comparable: $(H - z)^{-1} - (H_0 - z)^{-1}$ is of trace class

The Krein spectral shift function $\xi = \xi(\lambda; H, H_0)$ is a function on $\mathbb{R}$ that satisfies the trace formula

$$\mathrm{tr}(f(H) - f(H_0)) = \int_{\mathbb{R}} f'(\lambda)\xi(\lambda; H, H_0)\, d\lambda$$

say, for $f \in C^1_{\mathrm{loc}}$ with $f'(\lambda) = \int_{\mathbb{R}} e^{-i\lambda t}\sigma(dt)$ and a finite measure σ

A 5-min course on the ξ-function continued Examples

- $H, H_0 \in \mathbb{R}$ (Newton-Leibnitz trace formula)

$f(H) - f(H_0) = \int_{H_0}^{H} f'(\lambda)\, d\lambda$ implies
$\xi(\lambda; H, H_0) =$ characteristic function of the segment $[H_0, H]$

- H, H_0 are symmetric matrices (I. M. Lifshitz trace formula)

$$\begin{aligned}\operatorname{tr}(f(H) - f(H_0)) &= \operatorname{tr}\Big(\int_{\mathbb{R}} f(\lambda) dE_H(\lambda) - \int_{\mathbb{R}} f(\lambda) dE_{H_0}(\lambda)\Big)\\ &= \operatorname{tr}\int_{\mathbb{R}} f(\lambda) d(E_H(\lambda) - E_{H_0}(\lambda)) = \int_{\mathbb{R}} f(\lambda) d\operatorname{tr}(E_H(\lambda) - E_{H_0}(\lambda))\\ &= -\int_{\mathbb{R}} f'(\lambda) \operatorname{tr}(E_H(\lambda) - E_{H_0}(\lambda)) d\lambda \text{ implies}\end{aligned}$$

$\xi(\lambda; H, H_0) = \operatorname{tr}\big(E_{H_0}(\lambda) - E_H(\lambda)\big) =$
$\#$ e. v. of $H(t) = tH + (1-t)H_0$ that cross λ rightward
minus $\#$ e. v. of $H(t) = tH + (1-t)H_0$ that cross λ leftward

A 5-min course on the ξ-function continued

Generally $\xi(\lambda; H, H_0) = \operatorname{tr}\big(E_{H_0}(\lambda) - E_H(\lambda)\big)$ is not correct
$E_{H_0}(\lambda) - E_H(\lambda)$ is NOT of trace class even for one-dimensional $H - H_0$

The perturbation determinant
$\Delta(z) = \det\big((H - z)(H_0 - z)^{-1}\big) = \det\big(I + V(H_0 - z)^{-1}\big)$
For example, if $H_0 = -\partial_x^2$, $H = -\partial_x^2 + V(x)$ then
$\Delta(z) =$ Jost function $=$ Evans function
General Krein's formula for the trace class $H - H_0$:

$$\xi(\lambda; H, H_0) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \arg \Delta(\lambda + i\varepsilon) \quad \text{a.e. } \lambda \in \mathbb{R}$$

Then $\int |\xi(\lambda)|\, d\lambda \leqslant \|V\|_{\mathcal{B}_1}$ and $\int \xi(\lambda)\, d\lambda = \operatorname{tr} V$
If $\mathrm{rank}\, E_{H_0}(a - \epsilon, b + \epsilon) < \infty$ then
$\xi(b - 0) - \xi(a + 0) = \mathrm{rank}\, E_{H_0}(a, b) - \mathrm{rank}\, E_H(a, b)$

Index = Spectral Flow via Dichotomy Theorem

Completely different approach to $D_A = d/dt + A(t)$
Palmer's Theorem – infinite dimensional versions are due to Sandstede, Scheel, Rabier, Lunardi, Schnaubelt, Pogan, Lin, Tomilov, Latushkin
Suppose $y' = A(t)y$ is a well-posed equation. In our case $A_\pm$ are selfadjoint *bounded from below*, assume $\operatorname{Dom} A_- = \operatorname{Dom} A_+$ and
$A(t) = B(t) + \begin{cases} A_-, & t \leqslant 0 \\ A_+, & t > 0 \end{cases}$. Assume that $B(t)$ is compact for each $t \in \mathbb{R}$, and $B(\cdot)$ is continuous, $\|B(t)\|_{\mathcal{B}} \to 0$ as $|t| \to \infty$

Dichotomy Theorem

The operator $D_A = d/dt + A(t)$ is Fredholm **if and only if** the operators $A_\pm$ are invertible and the pair of subspaces $(S_-, S_+^\perp)$ is Fredholm; the index formula holds: $\operatorname{Ind} D_A = \dim(S_- \cap S_+^\perp) - \dim(S_-^\perp \cap S_+)$.

$S_\pm = \operatorname{rg} E_{A_\pm}(0)$, the Morse projection for the negative spectrum of $A_\pm$

Assumptions

Main assumptions:

- A_- – selfadjoint on $\operatorname{Dom} A_- \subseteq \mathcal{H}$, complex separable Hilbert
- $B(t), t \in \mathbb{R}$ – closed symmetric, $\operatorname{Dom} B(t) \supseteq \operatorname{Dom} A_-$
- $B'(t)$, $t \in \mathbb{R}$ – closed symmetric, $\operatorname{Dom} B'(t) \supseteq \operatorname{Dom} A_-$, such that $B(t)(|A_-| + I)^{-1}$, $t \in \mathbb{R}$, is weakly differentiable and $\frac{d}{dt}(B(t)(|A_-| + I)^{-1}g, h)_{\mathcal{H}} = (B'(t)(|A_-| + I)^{-1}g, h)_{\mathcal{H}}$, $g, h \in \mathcal{H}$
- $B'(\cdot)(|A_-| + I)^{-1} \in L^1(\mathbb{R}; \mathcal{B}_1(\mathcal{H}))$, in particular,

$$\int_{\mathbb{R}} \left\| B'(t)(|A_-| + I)^{-1} \right\|_{\mathcal{B}_1(\mathcal{H})} dt < \infty$$

Consequences

Let $A(t) = A_- + B(t)$, $\operatorname{Dom} A(t) = \operatorname{Dom} A_-$, $\mathcal{H}_1 := (\operatorname{Dom} A_-, \|\cdot\|_{A_-})$
Norms of $A(t) : \mathcal{H}_1 \to \mathcal{H}$ are bounded uniformly for $t \in \mathbb{R}$
There exists $A_+ = A(+\infty) = A_- + B(+\infty)$, $\operatorname{Dom} A_+ = \operatorname{Dom} A_-$,
$B(+\infty)(|A_-| + I)^{-1} = \int_{-\infty}^{+\infty} B'(t)(|A_-| + I)^{-1}\, dt$

$\sup_{t\in\mathbb{R}} \|B(t)(|A_-| - z)^{-1}\|_{\mathcal{B}_1(\mathcal{H})} = o(1)$, $z \to -\infty$

Lemma.

The operator $D_A = d/dt + A(t)$, $\operatorname{Dom} D_A = \operatorname{Dom}(d/dt) \cap \operatorname{Dom} \mathbb{A}_-$, is closed; its graph norm is equivalent to the norm in $W_1^2(\mathbb{R}; \mathcal{H}) \cap L^2(\mathbb{R}; \mathcal{H}_1)$

The proof of the trace formula

$$\mathrm{tr}_{L^2(\mathbb{R};\mathcal{H})}((H_2 - z)^{-1} - (H_1 - z)^{-1}) = \frac{1}{2z}\,\mathrm{tr}_{\mathcal{H}}(g(A_+) - g(A_-))$$
$$H_1 = -\frac{d^2}{dt^2} + A(t)^2 - B'(t),\ H_2 = -\frac{d^2}{dt^2} + A(t)^2 + B'(t),\ g(x) = \frac{x}{\sqrt{x^2+1}}$$

Approximation: $P_n = E_{A_-}(-n, n)$, $A_n(t) = P_n A(t) P_n$, $B_n(t) = P_n B(t) P_n$

Pushnitski's Theorem

$$\mathrm{tr}_{L^2(\mathbb{R};\mathcal{H})}((H_{2,n} - z)^{-1} - (H_{1,n} - z)^{-1}) = \frac{1}{2z}\,\mathrm{tr}_{\mathcal{H}}(g(A_{+,n}) - g(A_{-,n}))$$

Left Hand Side Proposition

$$\left\|((H_2 - z)^{-1} - (H_1 - z)^{-1}) - ((H_{2,n} - z)^{-1} - (H_{1,n} - z)^{-1})\right\|_{\mathcal{B}_1(L^2(\mathbb{R};\mathcal{H}))} \to 0$$

Right Hand Side Proposition

$$\left\|(g(A_+) - g(A_-)) - (g(A_{+,n}) - g(A_{-,n}))\right\|_{\mathcal{B}_1(\mathcal{H})} \to 0 \text{ as } n \to \infty$$

Main issues with the propositions: the LHS

The trace of $(H_2 - z)^{-1} - (H_1 - z)^{-1}$: a major step is, as $z \to -\infty$,

$$\big\| R_0^{1/2}(z)\mathbb{B}' R_0^{1/2}(z) \big\|_{\mathcal{B}_1(L^2(\mathbb{R};\mathcal{H}))} = o(1) \int_{\mathbb{R}} \| B'(t)(|A_-| + I)^{-1} \|_{\mathcal{B}_1(\mathcal{H})}$$

Here $R_0(z) = (H_0 - z)^{-1}$, $H_0 = D_{A_-}^* D_{A_-} = -\frac{d^2}{dt^2} + \mathbb{A}_-^2$
Then the resolvent identity for $R_j = (H_j - z)^{-1}$ does the job:
$R_1(z) - R_2(z) = R_1(z)(V_2 - V_1)R_2(z) = \mathbb{M} R_0^{1/2}(z)\mathbb{B}' R_0^{1/2}(z)\mathbb{N}$

Main issues with the propositions: the RHS

The trace of $g(A_+) - g(A_-)$: a major step is to see that $g(A_+) - g(A_-) \in \mathcal{B}_1(\mathcal{H})$ since $g(+\infty) = 1$ and $g(-\infty) = -1$ for $g(x) = \frac{x}{\sqrt{x^2+1}}$.
Need new technique: double operator integrals to show

Main Lemma

$g(A_+) - g(A_-) = T(K)$ where $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ is a bounded operator and $K =$ closure $(|A_+| + I)^{-1/2}(A_+ - A_-)(|A_-| + I)^{-1/2}$ is a trace class operator in $\mathcal{H}$

A 5 min course on double operator integrals

Daletskij and S. G. Krein (1960'), Birman and Solomyak (1960-70'), Peller, dePagter, Suckochev (1990-05), and many others

Our main goal: Given selfadjoint operators A_- and A_+ and a Borel function f, we represent $f(A_+) - f(A_-)$ as a double Stieltjes integral with respect to the spectral measures $dE_{A_+}(\lambda)$ and $dE_{A_-}(\mu)$. If $A_\pm$ are matrices then $A_+ = \sum_{j=1}^n \lambda_j E_{A_+}(\lambda_j)$ and $A_- = \sum_{k=1}^n \mu_k E_{A_-}(\mu_k)$ imply:

$$f(A_+) - f(A_-) = \sum_{j=1}^n \sum_{k=1}^n (f(\lambda_j) - f(\mu_k)) E_{A_+}(\lambda_j) E_{A_-}(\mu_k)$$

$$= \sum_{j=1}^n \sum_{k=1}^n \frac{f(\lambda_j)-f(\mu_k)}{\lambda_j-\mu_k} E_{A_+}(\lambda_j)(\lambda_j - \mu_k) E_{A_-}(\mu_k)$$

$$= \sum_{j=1}^n \sum_{k=1}^n \frac{f(\lambda_j)-f(\mu_k)}{\lambda_j-\mu_k}$$

$$\times E_{A_+}(\lambda_j)\Big(\sum_{j'=1}^n \lambda_{j'} E_{A_+}(\lambda_{j'}) - \sum_{k'=1}^n \mu_{k'} E_{A_-}(\mu_{k'})\Big) E_{A_-}(\mu_k)$$

$$= \sum_{j=1}^n \sum_{k=1}^n \frac{f(\lambda_j)-f(\mu_k)}{\lambda_j-\mu_k} E_{A_+}(\lambda_j)(A_+ - A_-) E_{A_-}(\mu_k)$$

A 5 min course on DOIs continued

Birman-Solomyak formula:

$$f(A_+) - f(A_-) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} dE_{A_+}(\lambda)(A_+ - A_-) dE_{A_-}(\mu)$$

More general: for a bounded Borel function $\phi(\lambda, \mu)$ we would like to define a *bounded transformator* $T_\phi : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ so that

$$T_\phi(K) = \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(\lambda, \mu) dE_{A_+}(\lambda) K dE_{A_-}(\mu), \quad K \in \mathcal{B}_1(\mathcal{H}).$$

$T_\phi(K) = \int_{\mathbb{R}} \alpha(\lambda) dE_{A_+}(\lambda) \cdot K \cdot \int_{\mathbb{R}} \beta(\mu) dE_{A_-}(\mu)$ for $\phi(\lambda, \mu) = \alpha(\lambda)\beta(\mu)$

$T_\phi(K) = \int_{\mathbb{R}} \alpha_s(A_+) K \beta_s(A_-)\, \nu(s) ds$ for $\phi(\lambda, \mu) = \int_{\mathbb{R}} \alpha_s(\lambda)\beta_s(\mu)\, \nu(s) ds$,
where α_s, β_s are bounded Borel functions, $\int_{\mathbb{R}} \|\alpha_s\|_\infty \|\beta_s\|_\infty \nu(s)\, ds < \infty$.
The (Wiener) class of such ϕ's is denoted by $\mathfrak{A}_0$

Back to our business

Recall that $(A_+ - A_-)(A_-^2 + I)^{-1/2} \in \mathcal{B}_1(\mathcal{H})$ by assumptions

Interpolation lemma

$\overline{K} \in \mathcal{B}_1(\mathcal{H})$, $K = (A_+^2 + I)^{-1/4}(A_+ - A_-)(A_-^2 + I)^{-1/4}$, $\operatorname{Dom} K = \operatorname{Dom} A_-$

Consider the function

$$\phi(\lambda, \mu) = (1 + \lambda^2)^{1/4} \frac{g(\lambda) - g(\mu)}{\lambda - \mu} (1 + \mu^2)^{1/4}, \quad g(x) = x(1 + x^2)^{-1/2}$$

Double operator integral lemma

$\phi(\lambda, \mu) \in \mathfrak{A}_0$ and thus $T_\phi : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ is bounded. Also, $g(A_+) - g(A_-) = T_\phi(\overline{K})$ and thus $g(A_+) - g(A_-) \in \mathcal{B}_1(\mathcal{H})$

Proof of the lemma

Formally: $T_\phi(K) = \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(\lambda,\mu) dE_{A_+}(\lambda) K dE_{A_-}(\mu) =$
$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{g(\lambda)-g(\mu)}{\lambda-\mu} dE_{A_+}(\lambda)(A_+ - A_-) dE_{A_-}(\mu) = g(A_+) - g(A_-)$
To see that $\phi \in \mathfrak{A}_0$ we split: $\phi(\lambda,\mu) = (1+\lambda^2)^{1/4} \frac{g(\lambda)-g(\mu)}{\lambda-\mu}(1+\mu^2)^{1/4}$
$= \psi(\lambda,\mu) + \frac{\psi(\lambda,\mu)}{(1+\lambda^2)^{1/2}(1+\mu^2)^{1/2}} + \frac{\lambda\psi(\lambda,\mu)\mu}{(1+\lambda^2)^{1/2}(1+\mu^2)^{1/2}}$ where
$\psi(\lambda,\mu) := \frac{(1+\lambda^2)^{1/4}(1+\mu^2)^{1/4}}{(1+\lambda^2)^{1/2}+(1+\mu^2)^{1/2}} = \zeta\big(\log(1+\lambda^2)^{1/2} - \log(1+\mu^2)^{1/2}\big)$,
$\zeta(\lambda-\mu) := \big(e^{(\lambda-\mu)/2} + e^{-(\lambda-\mu)/2}\big)^{-1} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{is\lambda} e^{-is\mu}\widehat{\zeta}(s)\, ds$
Since $\widehat{\zeta} \in L^1(\mathbb{R})$, $\psi \in \mathfrak{A}_0$ due to
$\psi(\lambda,\mu) = \frac{1}{2\pi}\int_{\mathbb{R}}(1+\lambda^2)^{is/2}(1+\mu^2)^{-is/2}\widehat{\zeta}(s)\, ds$

Trace class, Hilbert-Schmidt, Fredholm determinants

$K \in \mathcal{B}_2(\mathcal{H})$, Hilbert-Schmidt: K is compact and

$$\sum_{j=1}^{\infty} \Big(\lambda_j[(K^*K)^{1/2}]\Big)^2 < \infty$$

$K \in \mathcal{B}_1(\mathcal{H})$, of trace class: K is compact and

$$\sum_{j=1}^{\infty} \lambda_j[(K^*K)^{1/2}] < \infty$$

If K is Hilbert-Schmidt then

$$\det{}_2[I - K] = \det[(I - K)e^K] = \prod_{\lambda \in \sigma(K)} (1 - \lambda)e^{\lambda}$$

If K is of trace class then

$$\det{}_2[I - K] = e^{\mathrm{tr}(K)} \det(I - K) = e^{\mathrm{tr}(K)} \prod_{\lambda \in \sigma(K)} (1 - \lambda)$$