Asymptotic behavior of nonautonomous Cauchy problems

Roland Schnaubelt, Karlsruhe

First Meeting on Asymptotics of Operator Semigroups
Oxford, 4. September 2009

1) Given: Linear operators $A(t)$, $t \in \mathbb{R}$, on a Banach space X with domains $D(A(t))$, $f : \mathbb{R} \to X$ and $u_0 \in X$. (Mostly $f = 0$.) Solve

$$u'(t) = A(t)u(t) + f(t), \quad t \geq 0, \qquad u(0) = u_0. \tag{1}$$

1) Given: Linear operators $A(t)$, $t \in \mathbb{R}$, on a Banach space X with domains $D(A(t))$, $f : \mathbb{R} \to X$ and $u_0 \in X$. (Mostly $f = 0$.) Solve

$$u'(t) = A(t)u(t) + f(t), \quad t \geq 0, \qquad u(0) = u_0. \tag{1}$$

2) Consider a nonlinear problem with a special solution v_*:

$$v'(t) = F(v(t)), \quad t \geq 0, \qquad v(0) = v_0. \tag{2}$$

Linearization of (2) at v_* leads to pb (1) with $A(t) := F'(v_*(t))$. Exponential splittings of (1) should persist in (2) near v_* (cf. principle of linearized stab, local stable/center/unstable manifolds).

1) Given: Linear operators $A(t)$, $t \in \mathbb{R}$, on a Banach space X with domains $D(A(t))$, $f : \mathbb{R} \to X$ and $u_0 \in X$. (Mostly $f = 0$.) Solve

$$u'(t) = A(t)u(t) + f(t), \quad t \geq 0, \qquad u(0) = u_0. \tag{1}$$

3) Consider a quasilinear problem, where $x \mapsto A(v)x$ is linear:

$$v'(t) = A(v(t))v(t), \quad t \geq 0, \qquad v(0) = v_0. \tag{3}$$

Fix v, and look for solution $u = \Phi(v)$ of linear problem

$$u'(t) = A(v(t))u(t), \quad t \geq 0, \qquad u(0) = v_0. \tag{4}$$

Fixed point $\hat{v} = \Phi(\hat{v})$ then solves (3). [Kato; Sobolevskii, Amann, Acquistapace/Terreni, Yagi]

Definition

$U(t,s) \in \mathcal{B}(X)$ form an evolution family $U(\cdot,\cdot)$ if

- $U(s,s) = I$, $U(t,r)U(r,s) = U(t,s)$, and
- $(t,s) \mapsto U(t,s)$ is strongly continuous

for $t \geq r \geq s$ in $\mathbb{R}$.

Definition
$U(t,s) \in \mathcal{B}(X)$ form an evolution family $U(\cdot,\cdot)$ if

- $U(s,s) = I$, $U(t,r)U(r,s) = U(t,s)$, and
- $(t,s) \mapsto U(t,s)$ is strongly continuous

for $t \geq r \geq s$ in $\mathbb{R}$. The Cauchy problem (CP)

$$u'(t) = A(t)u(t), \quad t \geq s, \qquad u(s) = u_0, \tag{5}$$

is well posed if $\exists$ an evol. fam. $U(\cdot,\cdot)$ and dense subspaces Y_t of X s.t. $U(t,s)Y_s \subset Y_t \subset D(A(t))$ and $u = U(\cdot,s)u_0 \in C^1([s,\infty),X)$ solves (5), for all $u_0 \in Y_s$ and $t \geq s$. Then $A(\cdot)$ generates $U(\cdot,\cdot)$.

Remark. There are wellposed CP, where each $A(t)$ is a generator and $Y_t \neq D(A(t))$ for each t.

Some obstacles to nice theorems

1) $U(t,s) = q(t)q(s)^{-1}I$ for nondifferentiable $0 < q \in C(\mathbb{R})$ is an evolution family without generators.

2) There are wellposed CP where each $A(t)$ is not closable or where $\bigcap_t D(A(t)) = \{0\}$.

$A(t)$, $t \in \mathbb{R}$, are **stable** if $\exists\, M > 0, \omega \in \mathbb{R}$ with $s(A(t)) < \omega$ and

$$\|R(\lambda, A(t_n)) \cdots R(\lambda, A(t_1))\| \le M(\lambda - \omega)^{-n}$$

for all $\lambda > \omega$, $t_1 \le \cdots \le t_n$ and $n \in \mathbb{N}$.

$A(t)$, $t \in \mathbb{R}$, are **stable** if $\exists\ M > 0, \omega \in \mathbb{R}$ with $s(A(t)) < \omega$ and

$$\|R(\lambda, A(t_n)) \cdots R(\lambda, A(t_1))\| \le M(\lambda - \omega)^{-n}$$

for all $\lambda > \omega$, $t_1 \le \cdots \le t_n$ and $n \in \mathbb{N}$.

Theorem [Kato '70]
Let $A(\cdot)$ be stable and $Y \subset X$ be dense s.t. $e^{\tau A(t)} Y \subset Y \subset D(A(t))$ for all $t \in \mathbb{R}$, $\tau \ge 0$ and $A(\cdot) \in C_b(\mathbb{R}, \mathcal{B}(Y, X))$.

a) Let Y be reflexive and $A_{|Y}(\cdot)$ be stable in Y. Then $\exists$ evolution family $U(\cdot,\cdot)$ with $U(t,s)Y \subset Y$ for all $t \ge s$ and $\partial_t U(t,s)y = A(t)U(t,s)y$ for all $y \in$Y and a.e. $t \in [s, \infty)$.

b) Under a technical assumpt., $A(\cdot)$ generates $U(\cdot,\cdot)$ with $Y_t := Y$.

$A(t)$, $t \in \mathbb{R}$, are **stable** if $\exists\ M > 0, \omega \in \mathbb{R}$ with $s(A(t)) < \omega$ and

$$\|R(\lambda, A(t_n)) \cdots R(\lambda, A(t_1))\| \le M(\lambda - \omega)^{-n}$$

for all $\lambda > \omega$, $t_1 \le \cdots \le t_n$ and $n \in \mathbb{N}$.

Theorem [Kato '70]

Let $A(\cdot)$ be stable and $Y \subset X$ be dense s.t. $e^{\tau A(t)} Y \subset Y \subset D(A(t))$ for all $t \in \mathbb{R}$, $\tau \ge 0$ and $A(\cdot) \in C_b(\mathbb{R}, \mathcal{B}(Y, X))$.

a) Let Y be reflexive and $A_{|Y}(\cdot)$ be stable in Y. Then $\exists$ evolution family $U(\cdot,\cdot)$ with $U(t,s)Y \subset Y$ for all $t \ge s$ and $\partial_t U(t,s)y = A(t)U(t,s)y$ for all $y \in$Y and a.e. $t \in [s, \infty)$.

b) Under a technical assumpt., $A(\cdot)$ generates $U(\cdot,\cdot)$ with $Y_t := Y$.

Idea: $U(t,s)x = \lim_{n\to\infty} \prod_{k=1}^{n} e^{\frac{t-s}{n} A(s_k)} x; \qquad s_k = s + \frac{k(t-s)}{n}.$

Remarks 1) There are variants of Kato's thm. Simplest case: If $D(A(t)) = Y$ for all t, Y is dense, $A(\cdot)$ is stable in X and $A(\cdot) \in C_b^1(\mathbb{R}, \mathcal{B}(Y, X))$, then CP is wellposed.

Remarks 1) There are variants of Kato's thm. Simplest case: If $D(A(t)) = Y$ for all t, Y is dense, $A(\cdot)$ is stable in X and $A(\cdot) \in C_b^1(\mathbb{R}, \mathcal{B}(Y, X))$, then CP is wellposed.

2) Example in [Colombini et.al. '79]: Blow up in equation $u_{tt}(t,x) = a(t)u_{xx}(t,x)$ with $x \in \mathbb{R}$ for some $a \in \bigcap_{\alpha<1} C^{\alpha}(\mathbb{R})$.

Remarks 1) There are variants of Kato's thm. Simplest case: If $D(A(t)) = Y$ for all t, Y is dense, $A(\cdot)$ is stable in X and $A(\cdot) \in C_b^1(\mathbb{R}, \mathcal{B}(Y,X))$, then CP is wellposed.

2) Example in [Colombini et.al. '79]: Blow up in equation $u_{tt}(t,x) = a(t)u_{xx}(t,x)$ with $x \in \mathbb{R}$ for some $a \in \bigcap_{\alpha<1} C^\alpha(\mathbb{R})$.

3) Let $A(\cdot)$ generate $U(\cdot,\cdot)$ and $t \mapsto R(\lambda, A(t))$ be str.cont.
a) If $A(t)$ is a gen., $\|e^{\tau A(t)}\| \le e^{\omega\tau}$ for all $t \in \mathbb{R}$ and $\tau \ge 0$, then $\|U(t,s)\| \le e^{\omega(t-s)}$ for all $t \ge s$. [Nickel '99], [S. '99]

$$U(t,s)x = \lim_{n\to\infty} \prod_{k=1}^{n} e^{\frac{t-s}{n}A(s_k)}x$$

Remarks 1) There are variants of Kato's thm. Simplest case: If $D(A(t)) = Y$ for all t, Y is dense, $A(\cdot)$ is stable in X and $A(\cdot) \in C_b^1(\mathbb{R}, \mathcal{B}(Y,X))$, then CP is wellposed.

2) Example in [Colombini et.al. '79]: Blow up in equation $u_{tt}(t,x) = a(t)u_{xx}(t,x)$ with $x \in \mathbb{R}$ for some $a \in \bigcap_{\alpha<1} C^\alpha(\mathbb{R})$.

3) Let $A(\cdot)$ generate $U(\cdot,\cdot)$ and $t \mapsto R(\lambda, A(t))$ be str.cont.

a) If $A(t)$ is a gen., $\|e^{\tau A(t)}\| \le e^{\omega\tau}$ for all $t \in \mathbb{R}$ and $\tau \ge 0$, then $\|U(t,s)\| \le e^{\omega(t-s)}$ for all $t \ge s$. [Nickel '99], [S. '99]

b) If $A(\cdot)$ is stable in a Banach lattice X and $e^{\tau A(t)} \ge 0$, then $U(t,s) \ge 0$ for all $t \ge s$. [S. '99]

$$U(t,s)x = \lim_{n\to\infty} \prod_{k=1}^{n} e^{\frac{t-s}{n} A(s_k)} x$$

$A(t)$ satisfy the Acquistapace-Terreni condition (AT) if $\exists\ \omega \in \mathbb{R}$, $\phi \in (\pi/2, \pi)$, $K, L > 0$, $\mu, \nu > 0$ such that $\mu + \nu > 1$ and

$$\|R(\lambda, A(t))\| \leq \frac{K}{1 + |\lambda - \omega|},$$
$$\|\lambda^{\nu} A(t) R(\lambda, A(t)) \left[R(\omega, A(t)) - R(\omega, A(s))\right]\| \leq L\,|t - s|^{\mu}$$

for all $t, s \in \mathbb{R}$ and $\lambda \neq \omega$ with $|arg\,(\lambda - \omega)| \leq \phi$. For simplicity, let $D(A(t))$ be dense for all t.

$A(t)$ satisfy the Acquistapace-Terreni condition (AT) if $\exists\ \omega \in \mathbb{R}$, $\phi \in (\pi/2, \pi)$, $K, L > 0$, $\mu, \nu > 0$ such that $\mu + \nu > 1$ and

$$\|R(\lambda, A(t))\| \leq \frac{K}{1 + |\lambda - \omega|},$$
$$\|\lambda^\nu A(t) R(\lambda, A(t)) \left[R(\omega, A(t)) - R(\omega, A(s))\right]\| \leq L\,|t - s|^\mu$$

for all $t, s \in \mathbb{R}$ and $\lambda \neq \omega$ with $|arg\,(\lambda - \omega)| \leq \phi$. For simplicity, let $D(A(t))$ be dense for all t.

Theorem [AT '87-'88] Let (AT) hold. Then $A(\cdot)$ generates an evol. fam. $U(\cdot, \cdot)$ with $\|A(t)U(t,s)\| \leq c\,(t-s)^{-1} e^{d(t-s)}$ for all $t > s$.

See also Amann, Yagi; Kato-Tanabe.

Definition

An evol.fam. $U(\cdot,\cdot)$ has an exponential dichotomy (ED) on an interval J if $\exists$ projections $P(t)$, $t \in J$, and $N, \delta > 0$ such that

- $U(t,s)P(s) = P(t)U(t,s)$,
- $U(t,s) : Q(s)X \to Q(t)X$ has inverse $U_Q(s,t)$
- $\|U(t,s)P(s)\|, \|U_Q(s,t)Q(t)\| \le Ne^{-\delta(t-s)}$

for all $t \geq s$ in J. Here $Q(t) = I - P(t)$. If $P(t) = I$, then $U(\cdot,\cdot)$ is exponentially stable. (Mostly $J = \mathbb{R}$.)

Definition

An evol.fam. $U(\cdot,\cdot)$ has an exponential dichotomy (ED) on an interval J if $\exists$ projections $P(t)$, $t \in J$, and $N, \delta > 0$ such that

- $U(t,s)P(s) = P(t)U(t,s)$,
- $U(t,s) : Q(s)X \to Q(t)X$ has inverse $U_Q(s,t)$
- $\|U(t,s)P(s)\|, \|U_Q(s,t)Q(t)\| \leq Ne^{-\delta(t-s)}$

for all $t \geq s$ in J. Here $Q(t) = I - P(t)$. If $P(t) = I$, then $U(\cdot,\cdot)$ is exponentially stable. (Mostly $J = \mathbb{R}$.)

Remark: If $U(t+\tau, s+\tau) = U(t,s)$ for some $\tau > 0$ and all $t \geq s$ (i.e., $A(t) = A(t+\tau)$), then $U(\cdot,\cdot)$ has ED iff $\sigma(U(\tau,0)) \cap \mathbb{T} = \emptyset$.

More obstacles to nice theorems

1) There is a wellposed CP on $X = L^1(\mu)$ with $\|e^{\tau A(t)}\| = 2$ and $e^{\tau A(t)} \geq 0$ for all $\tau > 0$, $t \in \mathbb{R}$, such that $\sigma(U(t,s)) = \{0\}$ for all $t > s$ and $\sigma(A(t)) = \emptyset$ for all $t \in \mathbb{R}$, except for some $t_n \to \infty$ with $A(t_n) = 0$, but $t \mapsto \|U(t,s)\|$ grows more than exponentially. [Nickel/S. '96], [S. '99]

More obstacles to nice theorems

1) There is a wellposed CP on $X = L^1(\mu)$ with $\|e^{\tau A(t)}\| = 2$ and $e^{\tau A(t)} \geq 0$ for all $\tau > 0$, $t \in \mathbb{R}$, such that $\sigma(U(t,s)) = \{0\}$ for all $t > s$ and $\sigma(A(t)) = \emptyset$ for all $t \in \mathbb{R}$, except for some $t_n \to \infty$ with $A(t_n) = 0$, but $t \mapsto \|U(t,s)\|$ grows more than exponentially. [Nickel/S. '96], [S. '99]

2) Let $A_k(t) = D(-t)A_kD(t)$ with

$$D(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & -5 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $A_k(t)$ is smooth and periodic in $t \in \mathbb{R}$, $\sigma(A_1(t)) = \{-1\}$, $\sigma(A_2(t)) = \{-1, 1\}$, and $|R(\lambda, A_k(t))|_2 = |R(\lambda, A_k(0))|_2$, but $|U_1(t,s)|$ grows exp. and $U_2(\cdot,\cdot)$ has no ED. ($A_2(t)$ is even sym.)

Let $U(\cdot,\cdot)$ be an evol.fam. and exponentially bounded, i.e., $\|U(t,s)\| \le Me^{\omega(t-s)}$ for all $t \ge s$ and some $M \ge 1$, $\omega \in \mathbb{R}$.
On $E = C_0(\mathbb{R}, X)$ or $E = L^p(\mathbb{R}, X)$ with $1 \le p < \infty$, define the evolution semigroup by

$$(T(t)f)(s) = U(s, s-t)f(s-t), \qquad s \in \mathbb{R}, f \in E, t \ge 0.$$

Then $T(\cdot)$ is a C_0-semigroup, with generator G.
If $A(\cdot)$ gener. $U(\cdot,\cdot)$, we have $G = \overline{G_0}$, where $G_0 u = -u' + A(\cdot)u$ and $D(G_0) = \{u \in E : u(t) \in D(A(t))\ \forall\, t, u' \in E, A(\cdot)u \in E\}$.

Let $U(\cdot,\cdot)$ be an evol.fam. and exponentially bounded, i.e., $\|U(t,s)\| \le Me^{\omega(t-s)}$ for all $t \ge s$ and some $M \ge 1$, $\omega \in \mathbb{R}$.
On $E = C_0(\mathbb{R}, X)$ or $E = L^p(\mathbb{R}, X)$ with $1 \le p < \infty$, define the evolution semigroup by

$$(T(t)f)(s) = U(s, s-t)f(s-t), \qquad s \in \mathbb{R}, f \in E, t \ge 0.$$

Then $T(\cdot)$ is a C_0-semigroup, with generator G.
If $A(\cdot)$ gener. $U(\cdot,\cdot)$, we have $G = \overline{G_0}$, where $G_0 u = -u' + A(\cdot)u$ and $D(G_0) = \{u \in E : u(t) \in D(A(t)) \, \forall\, t, u' \in E, A(\cdot)u \in E\}$.
It holds: $\sigma(T(t)) = \mathbb{T} \cdot \sigma(T(t))$ for all $t > 0$, $\sigma(G) = \sigma(G) + i\mathbb{R}$.

Theorem [Latushkin/Montgomery-Smith '95, Levitan/Zhikov '82]
$U(\cdot,\cdot)$ has ED iff $I - T(t)$ is invertible for some $t > 0$
iff G is invertible iff for all $f \in E$ there is exactly one $u \in E$ with

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)d\tau, \qquad \forall\, t \ge s.$$

Theorem [Latushkin/Montgomery-Smith '95, Levitan/Zhikov '82]
$U(\cdot,\cdot)$ has ED iff $I - T(t)$ is invertible for some $t > 0$
iff G is invertible iff for all $f \in E$ there is exactly one $u \in E$ with

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)d\tau, \qquad \forall\, t \geq s.$$

Remarks. 1) $P(\cdot)$ (from ED) is the dich. proj. for $T(\cdot)$, it it exists.

2) G is Fredholm iff $U(\cdot,\cdot)$ has ED on $(-\infty,a]$ and $[b,\infty)$ for some $b \geq a$ and $Q(b)U(b,a)_{|} : Q(a)X \to Q(b)X$ is Fredholm.
[Latushkin/Pogan/S. '07], [Latushkin/Tomilov '05]

3) Let $U(\cdot,\cdot)$ be bounded. Then $U(t,s)x \to 0$ as $t \to \infty$ for all $x \in X$ and $s \in \mathbb{R}$ iff G has dense range for $E = L^1(\mathbb{R},X)$.
[Batty/Chill/Tomilov '02]

Theorem [Maniar/S. '03]
Let $A(\cdot)$ satisfy (AT) and generate $U(\cdot,\cdot)$ having an ED with proj. $P(\cdot)$. Let $R(\omega, A(\cdot))$ and $f \in C^\alpha(\mathbb{R}, X)$ be almost periodic. Then

$$u(t) = \int_{-\infty}^{t} U(t,s)P(s)f(s)\,ds - \int_t^\infty U_Q(t,s)Q(s)f(s)\,ds, \quad t \in \mathbb{R},$$

is the (unique) almost periodic solution of $u'(t) = A(t)u(t) + f(t)$, $t \in \mathbb{R}$.

References

- On evolution semigroups: Chicone/Latushkin, AMS '99.
- General surveys: [Nagel/Nickel '02], [S. '02].
- ED for ODE: Coppel, Springer '78.
- ED for $A(t) = A + B(t)$ with A sectorial and $B(t) \in \mathcal{B}(D((\omega - A)^{\alpha}), X)$, $\alpha < 1$: Henry, Springer '81.
- Survey in parabolic case: [S. '04], with proofs for asymptotic behavior. Many comments on related work.

Theorem [Robustness]

Let $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$ be exp. bdd. evol.fam., let $U(\cdot,\cdot)$ have ED. Then $\exists\ q > 0$ such that: If $\|U(s,s-1) - V(s,s-1)\| \le q$ for all $s \in \mathbb{R}$, then $V(\cdot,\cdot)$ has ED and proj. of U and V have same ranks.

Theorem [Robustness]

Let $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$ be exp. bdd. evol.fam., let $U(\cdot,\cdot)$ have ED. Then $\exists\ q>0$ such that: If $\|U(s,s-1)-V(s,s-1)\|\le q$ for all $s\in\mathbb{R}$, then $V(\cdot,\cdot)$ has ED and proj. of U and V have same ranks.

Proof. $\|T_U(1)-T_V(1)\| = \sup_s \|U(s,s-1)-V(s,s-1)\|$.

Theorem [Robustness]

Let $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$ be exp. bdd. evol.fam., let $U(\cdot,\cdot)$ have ED. Then $\exists\ q > 0$ such that: If $\|U(s,s-1) - V(s,s-1)\| \le q$ for all $s \in \mathbb{R}$, then $V(\cdot,\cdot)$ has ED and proj. of U and V have same ranks.

Proof. $\quad \|T_U(1) - T_V(1)\| = \sup_s \|U(s,s-1) - V(s,s-1)\|$.

Cor. Let $U(\cdot,\cdot)$ be exp. bdd. evol.fam., let $B(t) \in \mathcal{B}(X)$ be unif. bdd. and str. cont. Then $\exists!$ exp. bdd. evol.fam. $V(\cdot,\cdot)$ such that

$$V(t,s)x = U(t,s)x + \int_s^t U(t,\tau)B(\tau)V(\tau,s)x\,d\tau$$

for all $t \ge s$ and $x \in X$. If $U(\cdot,\cdot)$ has ED and $\|B(\cdot)\|_\infty$ is small enough, then $V(\cdot,\cdot)$ has ED and proj. of U and V have same rank.

Theorem [Robustness]

Let $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$ be exp. bdd. evol.fam., let $U(\cdot,\cdot)$ have ED. Then $\exists\ q > 0$ such that: If $\|U(s,s-1) - V(s,s-1)\| \leq q$ for all $s \in \mathbb{R}$, then $V(\cdot,\cdot)$ has ED and proj. of U and V have same ranks.

Cor. Let $U(\cdot,\cdot)$ be exp. bdd. evol.fam., let $B(t) \in \mathcal{B}(X)$ be unif. bdd. and str. cont. Then $\exists!$ exp. bdd. evol.fam. $V(\cdot,\cdot)$ such that

$$V(t,s)x = U(t,s)x + \int_s^t U(t,\tau)B(\tau)V(\tau,s)x\,d\tau$$

for all $t \geq s$ and $x \in X$. If $U(\cdot,\cdot)$ has ED and $\|B(\cdot)\|_\infty$ is small enough, then $V(\cdot,\cdot)$ has ED and proj. of U and V have same rank.

Remark: Even if $U(t,s) = e^{(t-s)A}$, the CP for $A + B(\cdot)$ does not need to be well-posed [Phillips '53].

Remark

There is a robustness result for symmetric hyperbolic systems on $\mathbb{R}^n$ with coefficients of the form $a(t,x) = a_0(x) + a_\varepsilon(t,x)$, where a_ε is small and the autonomous system corresp. to a_0 has an ED. [Shirikan-Volevich '02].

Remark

There is a robustness result for symmetric hyperbolic systems on $\mathbb{R}^n$ with coefficients of the form $a(t,x) = a_0(x) + a_\varepsilon(t,x)$, where a_ε is small and the autonomous system corresp. to a_0 has an ED. [Shirikan-Volevich '02].

Corollary (Parabolic case) [Batty/Chill '02, S. '02 +'04]

Let $A(\cdot)$ and $B(\cdot)$ satisfy (AT), generating $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$. Let $U(\cdot,\cdot)$ have ED. Then $\exists\ q > 0$ such that:
If $\|R(\omega, A(s)) - R(\omega, B(s)))\| \leq q$ for all $s \in \mathbb{R}$, then $V(\cdot,\cdot)$ has ED, and projections of $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$ have same ranks.

Remark

There is a robustness result for symmetric hyperbolic systems on $\mathbb{R}^n$ with coefficients of the form $a(t,x) = a_0(x) + a_\varepsilon(t,x)$, where a_ε is small and the autonomous system corresp. to a_0 has an ED. [Shirikan-Volevich '02].

Corollary (Parabolic case) [Batty/Chill '02, S. '02 +'04]

Let $A(\cdot)$ and $B(\cdot)$ satisfy (AT), generating $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$. Let $U(\cdot,\cdot)$ have ED. Then $\exists\ q > 0$ such that:
If $\|R(\omega, A(s)) - R(\omega, B(s)))\| \leq q$ for all $s \in \mathbb{R}$, then $V(\cdot,\cdot)$ has ED, and projections of $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$ have same ranks.

Proof. Idea:

$$V(t,s) - U(t,s) = \int_s^t V(t,\tau)(\omega - B(\tau))[R(\omega, A(\tau)) - R(\omega, B(\tau))] \cdot (\omega - A(\tau))U(\tau,s)\,d\tau$$

Theorem [Batty-Chill '02, S. '01+'04]
Let $A(\cdot)$ satisfy (AT) and generate $U(\cdot,\cdot)$ and let $R(\omega, A(t)) \to R(\omega, A)$ in $\mathcal{B}(X)$ as $t \to \infty$, where A is sectorial and $\sigma(A) \cap i\mathbb{R} = \emptyset$. Then $\exists\ b \in \mathbb{R}$ such that $U(\cdot,\cdot)$ has ED on $[b, \infty)$, projections of $U(\cdot,\cdot)$ and e^{tA} have same ranks, and $P(s) \to P_A$ strongly as $s \to \infty$.

Proof. Set $A_b(t) = A(t)$ for $t \geq b$ and $A_b(t) = A(b)$ for $t \leq b$ and a suff. large b. Use cor. to robustness thm. for A and $A_b(\cdot)$.

Example Consider a bdd. C^2-domain $\Omega \subset \mathbb{R}^n$ with outer normal ν,

$$\mathcal{A}(t,x,D) = \begin{pmatrix} \operatorname{div} a(t,x)\nabla + a_0(t,x) & b(t,x) \\ c(t,x) & \operatorname{div} d(t,x)\nabla + d_0(t,x) \end{pmatrix},$$

$$\mathcal{B}(t,x,D) = \begin{pmatrix} a(t,x)\nu(x)\cdot\nabla & 0 \\ 0 & d(t,x)\nu(x)\cdot\nabla \end{pmatrix},$$

$0<\delta\le a, d \in C_b^{1/2}(\mathbb{R}, C^1(\overline{\Omega},\mathbb{R}))$, $a_0, b, c, d_0 \in C_b^{1/2}(\mathbb{R}, C(\overline{\Omega},\mathbb{R}))$ tending to constants as $t\to\infty$ in $C(\overline{\Omega})$. Then $A(t)u = \mathcal{A}(t,\cdot,D)u$ with $D(A(t)) = \{u \in W_p^2(\Omega)^2 : \mathcal{B}(t,\cdot,D)u = 0 \text{ on } \partial\Omega\}$, satisfy (AT) on $L^p(\Omega)^2$, $1<p<\infty$, and $R(\omega, A(t)) \to R(\omega, A(\infty))$ in op. norm.

Example Consider a bdd. C^2-domain $\Omega \subset \mathbb{R}^n$ with outer normal ν,

$$\mathcal{A}(t,x,D) = \begin{pmatrix} \operatorname{div} a(t,x)\nabla + a_0(t,x) & b(t,x) \\ c(t,x) & \operatorname{div} d(t,x)\nabla + d_0(t,x) \end{pmatrix},$$

$$\mathcal{B}(t,x,D) = \begin{pmatrix} a(t,x)\nu(x)\cdot\nabla & 0 \\ 0 & d(t,x)\nu(x)\cdot\nabla \end{pmatrix},$$

$0<\delta\leq a, d \in C_b^{1/2}(\mathbb{R}, C^1(\overline{\Omega},\mathbb{R}))$, $a_0, b, c, d_0 \in C_b^{1/2}(\mathbb{R}, C(\overline{\Omega},\mathbb{R}))$ tending to constants as $t\to\infty$ in $C(\overline{\Omega})$. Then $A(t)u = \mathcal{A}(t,\cdot,D)u$ with $D(A(t)) = \{u \in W_p^2(\Omega)^2 : \mathcal{B}(t,\cdot,D)u = 0 \text{ on } \partial\Omega\}$, satisfy (AT) on $L^p(\Omega)^2$, $1<p<\infty$, and $R(\omega, A(t)) \to R(\omega, A(\infty))$ in op. norm. Let μ_n be EV of Neumann Laplacian and

$$M_n = \begin{pmatrix} a(\infty)\mu_n + a_0(\infty) & b(\infty) \\ c(\infty) & d(\infty)\mu_n + d_0(\infty) \end{pmatrix}.$$

Then $\sigma(A(\infty)) \cap i\mathbb{R} \neq \emptyset$ iff either $\det M_n = 0$ for some $n\in\mathbb{N}_0$, or $\operatorname{tr} M_n = 0$ and $\det M_n > 0$ for some $n \in \mathbb{N}_0$.

Theorem [S. '00 ,'04]
Let $A(\cdot)$ satisfy (AT) such that $\|R(\lambda, A(t))\| \le c$ for all $t \in \mathbb{R}$ and $|\operatorname{Re}\lambda| \le r$ and some $r, c > 0$. If the Hölder constant $L \ge 0$ in (AT) is small enough, then the evolution family $U(\cdot,\cdot)$ generated by $A(\cdot)$ has ED with proj. having the same ranks as those of $e^{\tau A(0)}$.

Proof. Using $e^{\tau A(s)}$, construct left and right inverse of G with small error.

Theorem [S. '00 ,'04]
Let $A(\cdot)$ satisfy (AT) such that $\|R(\lambda, A(t))\| \le c$ for all $t \in \mathbb{R}$ and $|\operatorname{Re}\lambda| \le r$ and some $r, c > 0$. If the Hölder constant $L \ge 0$ in (AT) is small enough, then the evolution family $U(\cdot,\cdot)$ generated by $A(\cdot)$ has ED with proj. having the same ranks as those of $e^{\tau A(0)}$.

Proof. Using $e^{\tau A(s)}$, construct left and right inverse of G with small error.

Cor. Let assumptions of thm hold except for smallness of L. Then, for suff. small $\varepsilon > 0$, $A(\varepsilon t)$ generate an evol.fam. with ED.

Theorem [Henry '81]
Let A be sectorial and $B(\cdot) \in C_b^\alpha(\mathbb{R}, \mathcal{B}(D((\omega - A)^\theta), X)$ for some $\alpha > 0$ and $\theta \in [0,1)$. Assume that

$$\exists \quad B_0 = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} B(s)\, ds$$

uniformly in t and that $\sigma(A + B_0) \cap i\mathbb{R} = \emptyset$. Then, for suff. large $\gamma > 0$, the evolution family generated by $A + B(\gamma t)$ has ED.

Remark. Similar results for parab. pde on $\mathbb{R}^n$ by [Levenshtam '93], see also Sobolevskii.

Let $A(t)u(x) = \operatorname{tr}(q(t)D^2u(x)) + b(t)x \cdot \nabla u(x)$ with $b, q \in C_b(\mathbb{R}, M_{n,n}(\mathbb{R}))$ with $q(t) = q(t)^T \geq \delta I > 0$. There are operators $U(t,s)$ solving $u'(t) = A(t)u(t)$ in $C_b(\mathbb{R}^n)$.
Assume that $b(\cdot)$ generates an exponentially stable evol.fam. on $\mathbb{R}^n$. Then $\exists$ exist probability measures μ_t, $t \in \mathbb{R}$, on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} U(t,s)\varphi \, d\mu_t = \int_{\mathbb{R}^n} \varphi \, d\mu_s \,, \qquad \forall \, \varphi \in C_b(\mathbb{R}^n), \; t \geq s.$$

Extend $U(t,s)$ to contractive $U(t,s) : L^2(\mu_s) \to L^2(\mu_t)$ and define 'evol.sgr.' $T(\cdot)$ on $L^2(\mathbb{R}^{1+n}, \nu)$ with $\nu(I \times M) = \int_I \mu_s(M)\, ds$. Let G be its generator.

Theorem [Geissert/Lunardi '08,'09]
Let $Q(t)\varphi = \int \varphi \, d\mu_t$. Then $Q(t)U(t,s) = U(t,s)Q(s)$ is the identity and $U(t,s) : N(Q(t)) \to N(Q(s))$ decays exponentially.

Proof. Use $T(\cdot)$ and, for 'regular' $u \in D(G)$,

$$\int_{\mathbb{R}^{1+n}} uGu \, d\nu = -\int_{\mathbb{R}^{1+n}} (q\nabla u) \cdot \nabla u \, d\nu.$$

Remarks. Periodic case in [Da Prato/Lunardi '07].
For more general coeff., see [Kunze/Lorenzi/Lunardi] for existence of μ_t and [Lorenzi/Lunardi/Zamboni] for asymptotic behavior in periodic case.