The discrete weighted Weiss conjecture

Andrew Wynn [1]

[1]University College London

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Discrete linear system

Discrete Time System

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- $C \in X^*$ – **Observation Operator**.
- (y_n) – **'measurement'** from system $(x_n) = (T^n x)$.

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α-admissibility $\sim$ measurement depends continuously on initial value

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Discrete Weighted Weiss Conjecture

- Hence: **If** C **is** α**-admissible for** T then

$$(\mathbf{RC})_\alpha : \qquad \|C(I-\omega T)^{-1}\|_{X^*} \leq \frac{k}{(1-|\omega|^2)^{\frac{1-\alpha}{2}}}, \qquad \omega \in \mathbb{D}.$$

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- Exists a (more famous!) conjecture in **continuous time**.
- Similar results to those presented in this talk (work by Haak, Le Merdy, Partington, Jabob, Weiss and more!)

When is the Weiss conjecture true/false?

TRUE in the following:

(i) [Harper '06] If $\alpha = 0$ and T a **contraction** ($\|T\|_{\mathcal{L}(X)} \leq 1$).

(ii) [W '08] If $\alpha \in (0, 1)$ and T is a **normal contraction**.

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FALSE in the cases [W '09]:

(i) If $\alpha \in (-1,0)$ conjecture fails for a **normal contraction** T.

(ii) If $\alpha \in (0,1)$ fails for the **unilateral shift** on $H^2(\mathbb{D})$.

Dirichlet Spaces; Carleson measures

Defintion:

For $\beta > -1$, **weighted Dirichlet space** $\mathcal{D}_\beta(\mathbb{D})$ contains analytic $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|^2_{\mathcal{D}_\beta(\mathbb{D})} := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^\beta \, dA(z) < \infty.$$

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Definition:

A measure μ satisfying $\mathcal{D}_\beta(\mathbb{D}) \hookrightarrow L^2(\mathbb{D}, \mu)$ is called a $\mathcal{D}_\beta(\mathbb{D})$-**Carleson measure**.

Normal operators; measure conection

Lemma

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(ii) *Resolvent condition $\mathbf{(RC)}_\alpha$ holds iff*

$$\mu(S(I)) \leq c|I|^{1+\alpha}, \qquad \text{any arc } I \subset \mathbb{T}.$$

The Boxes $S(I)$

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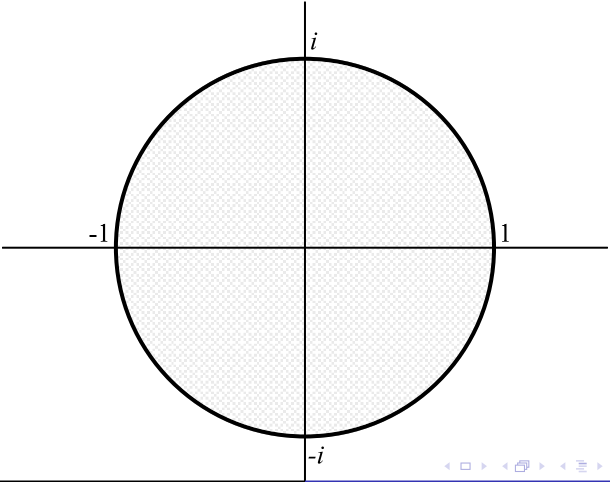

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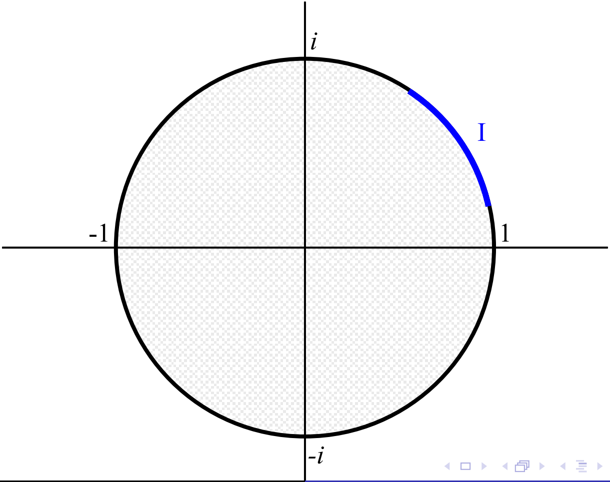

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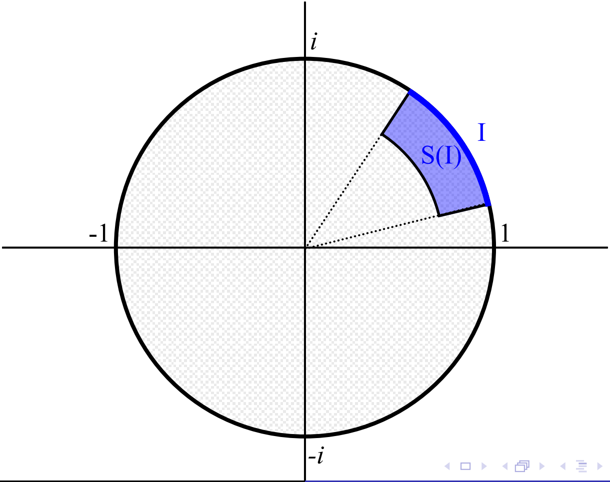

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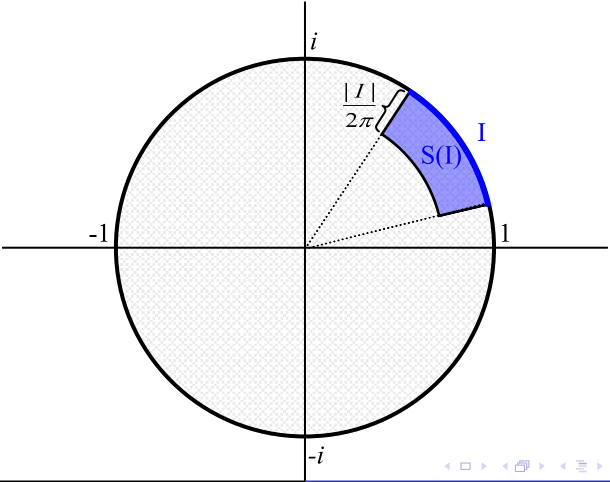

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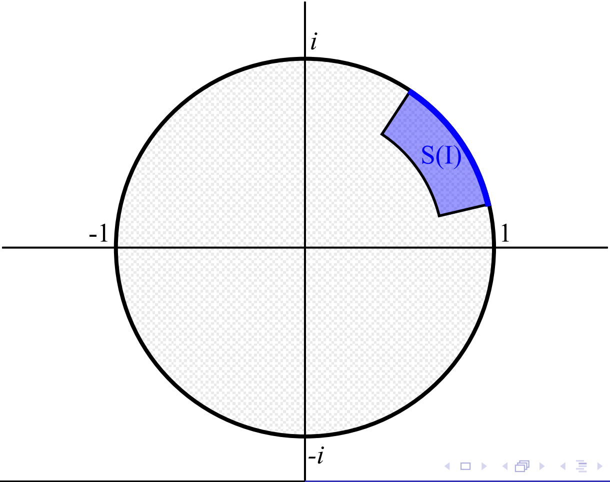

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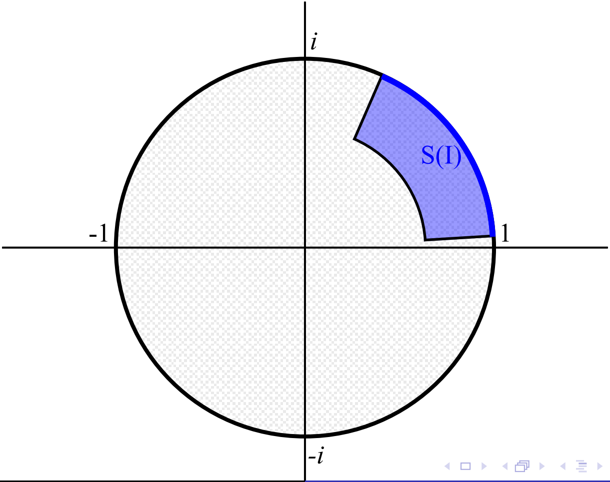

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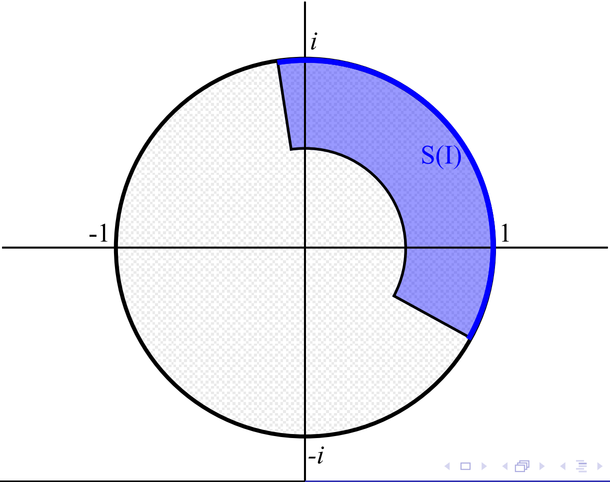

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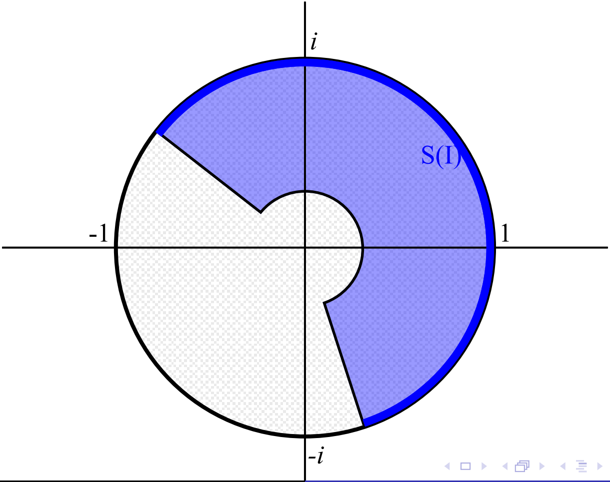

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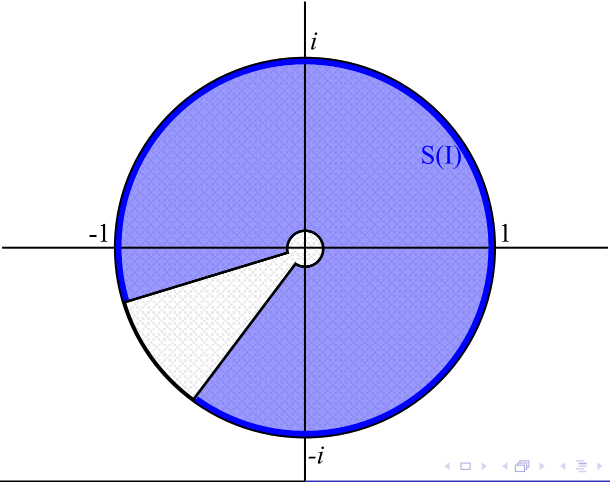

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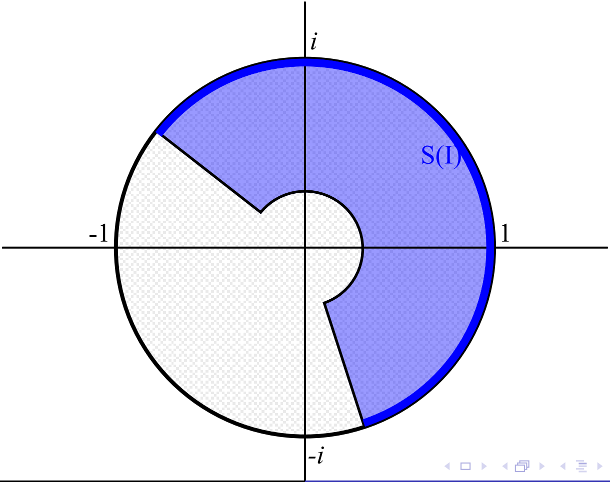

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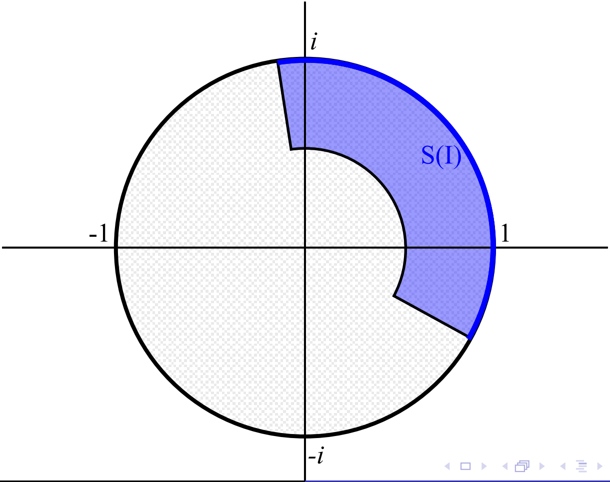

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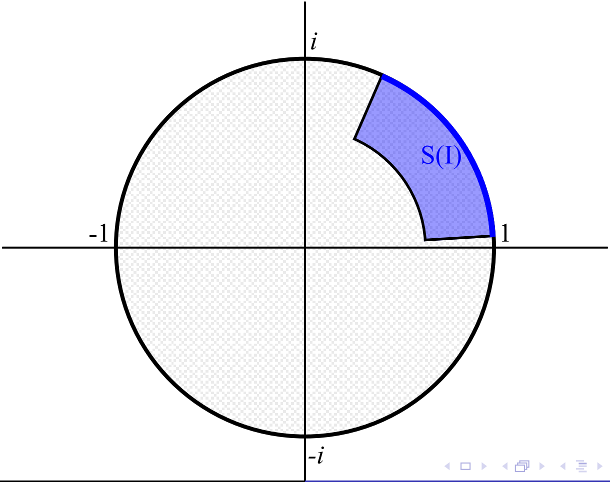

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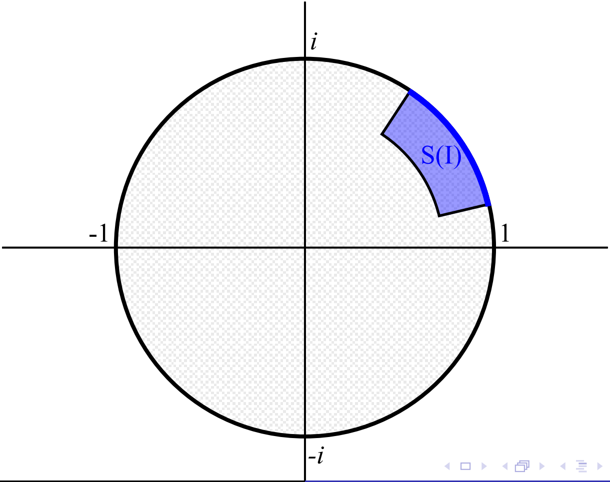

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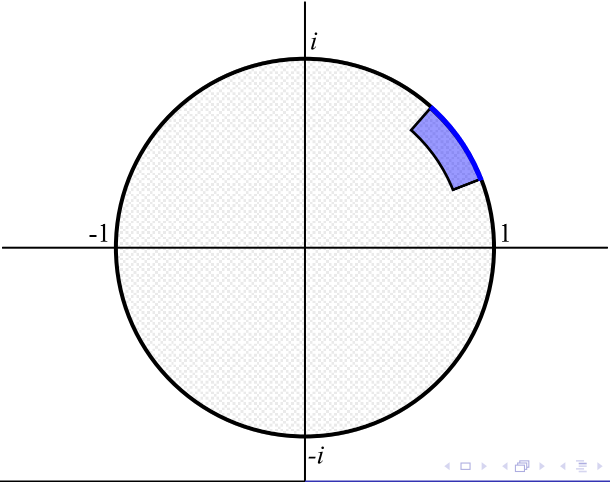

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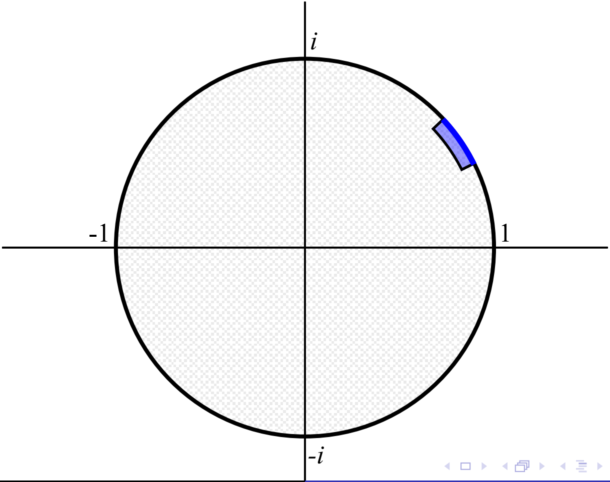

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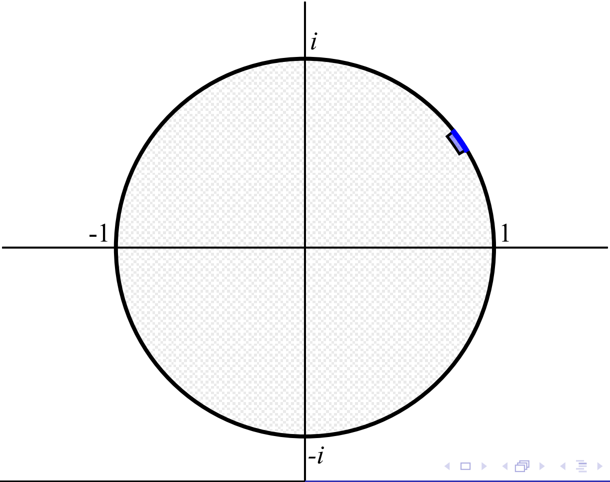

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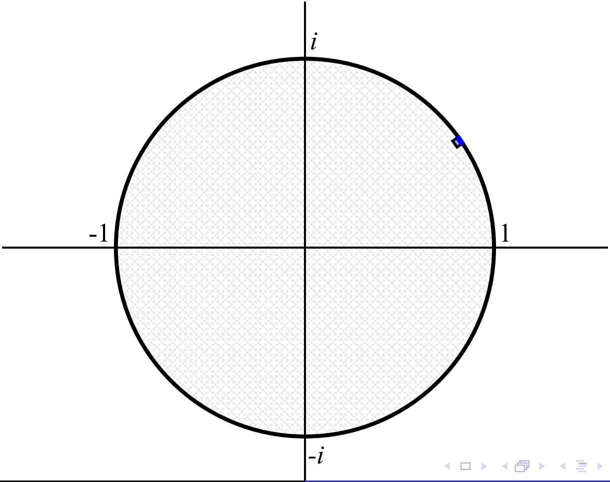

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Carleson measures vs Geometric Characterisation

Theorem (Carleson, Luecking)

Let $\alpha \in [0,1)$. A positive measure μ on $\mathbb{D}$ is $\mathcal{D}_{1+\alpha}(\mathbb{D})$-Carleson iff

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*Let $\alpha \in (-1,0)$. Exists measure μ on $\mathbb{D}$ satisfying (**SC**) which is not $\mathcal{D}_{1+\alpha}(\mathbb{D})$-Carleson measure.*

- Hence, Weighted Weiss Conjecture **false** for $\alpha \in (-1,0)$.

Weighted conjecture true for non-normal contractions?

So α-admissibility $\iff$ $\textbf{(RC)}_\alpha$ if:

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- **Answer: No**

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Definition:

(i) Hardy space $H^2(\mathbb{D})$: $f = \sum_{n=0}^{\infty} f_n z^n \in H^2(\mathbb{D})$ iff

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The Unilateral Shift on $H^2(\mathbb{D})$

Definition:

(i) Hardy space $H^2(\mathbb{D})$: $f = \sum_{n=0}^{\infty} f_n z^n \in H^2(\mathbb{D})$ iff

$$\|f\|_{H^2(\mathbb{D})}^2 := \sum_{n=0}^{\infty} |f_n|^2 < \infty.$$

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- Shift S is simple **non-normal** contraction operator.
- If Weighted Weiss conjecture true for S, very likely true for **all contraction operators**.

The Unilateral Shift on $H^2(\mathbb{D})$

Theorem (W '09)

Let $\alpha \in (0,1)$. Suppose that $C \in H^2(\mathbb{D})^*$ is given by $Cf := \langle f, c\rangle_{H^2}$, for some $c \in H^2(\mathbb{D})$. Then

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(iii) **Exists $c \in H^2(\mathbb{D})$ satisfying** (1) **but not** (2)**.**

Proof of (i): $(\mathrm{RC})_\alpha$ characterisation

- For $C \in H^2(\mathbb{D})^*$ and $\omega \in \mathbb{D}$,

$$\|C(I - \bar{\omega}S)^{-1}\|_{H^2(\mathbb{D})^*} = \left\| \frac{zc(z) - \omega c(\omega)}{z - \omega} \right\|_{H^2(\mathbb{D})}$$

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- For $\alpha \in (0,1)$, $C \in H^2(\mathbb{D})^*$ and $f = \sum f_n z^n \in H^2(\mathbb{D})$,

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- **Hankel Operator** $\Gamma_c^{\alpha} : \ell^2 \to \ell^2$ represented by matrix

$$\Gamma_c^{\alpha} \sim \begin{pmatrix} c_0 & c_1 & c_2 & \cdots \\ 2^{\frac{\alpha}{2}}c_1 & 2^{\frac{\alpha}{2}}c_2 & 2^{\frac{\alpha}{2}}c_3 & \cdots \\ 3^{\frac{\alpha}{2}}c_2 & 3^{\frac{\alpha}{2}}c_3 & 3^{\frac{\alpha}{2}}c_4 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

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- Therefore the discrete Weighted Weiss Conjecure **fails** for the shift on $H^2(\mathbb{D})$ $\square$.

Sufficient Conditions for α-admissibility

- As have seen, $\|C(I-\omega T)^{-1}\|_{X^*} \leq \frac{k}{(1-|\omega|^2)^{\frac{1-\alpha}{2}}}, \omega \in \mathbb{D}$ is not always enough for α-admissibility.

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Theorem (W '09)

Let $\alpha \in (-1,1)$. Suppose that $\phi : [0,1] \to \mathbb{R}_+$ satisfies $\int_0^1 \frac{\phi(x)}{x} dx < \infty$ and that T is **power bounded**. *Then if*

$$\textbf{(RC)}_{\phi,\alpha} \qquad \|C(I-\omega T)^{-1}\|_{X^*} \leq \frac{k \cdot \phi(1-|\omega|)}{(1-|\omega|^2)^{\frac{1-\alpha}{2}}}, \qquad \omega \in \mathbb{D}.$$

it follows that C is α-admissible for T.

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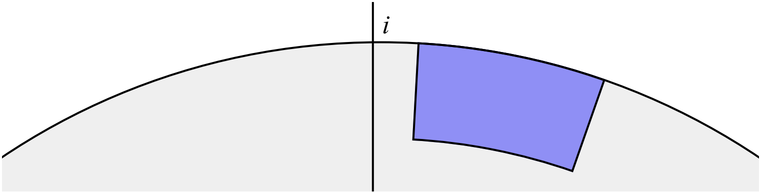

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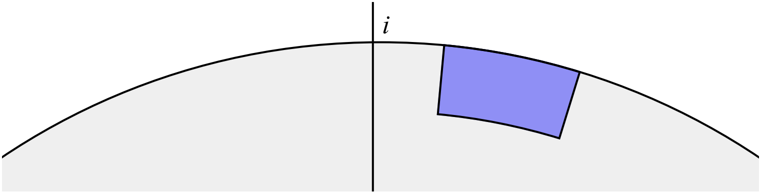

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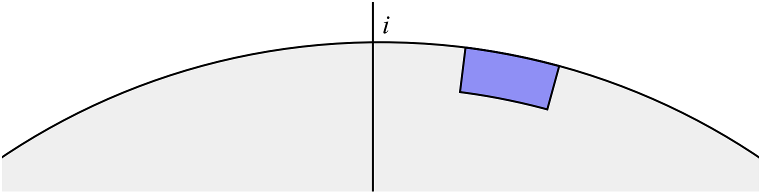

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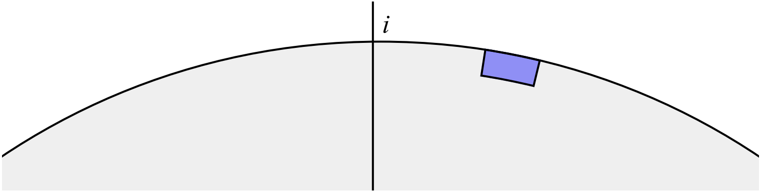

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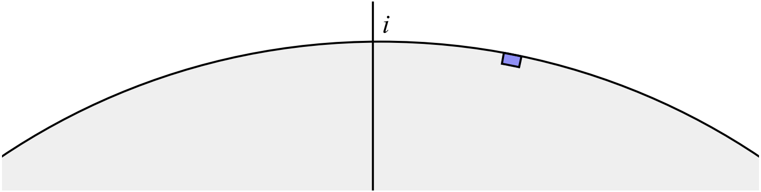

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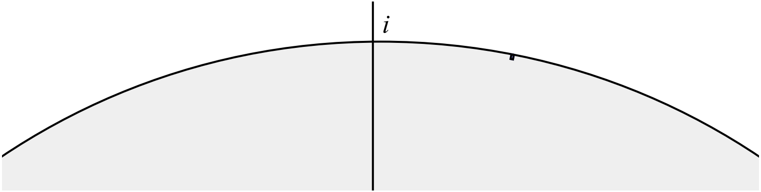

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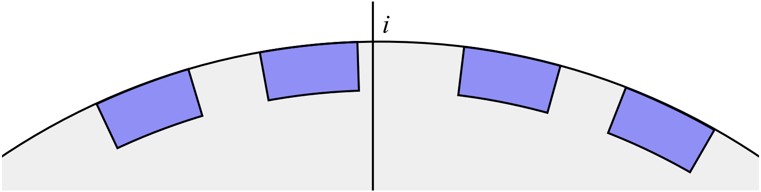

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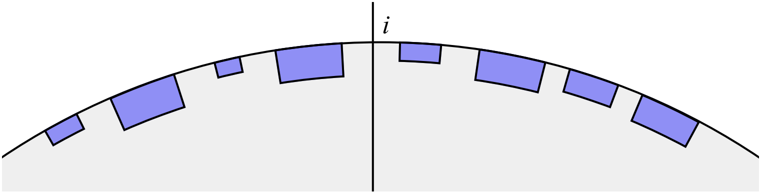

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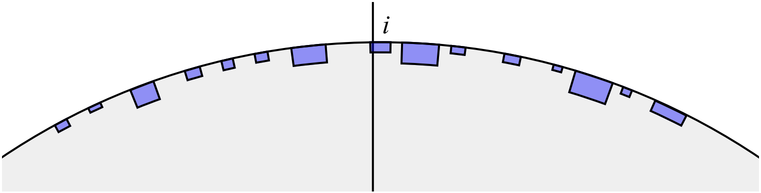

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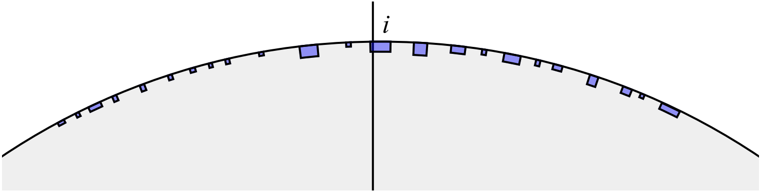

Theorem (W '09)

Let $\alpha \in (-1, 0)$ and suppose that $\phi : [0,1] \to \mathbb{R}_+$ is sufficiently regular. Then

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$$\textbf{(SC)}_{\phi,\alpha} \qquad \mu(S(I)) \leq c|I|^{1+\alpha}\phi(|I|/2\pi), \qquad I \subset \mathbb{T},$$

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- Provides much simpler **sufficient** conditions for $\mathcal{D}_{1+\alpha}(\mathbb{D})$-Carleson measures.